

A Sign-Reversing Involution on Genus-One Boundary Strata

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Abstract - We give an elementary calculation of the Euler characteristic of the boundary complex of the moduli space of genus-one stable curves. Our result agrees with the work of Chan–Galatius–Payne, who proved that this boundary complex is homotopy equivalent to a particular wedge of spheres, but the proof technique in our case is purely combinatorial: we construct a sign-reversing involution on genus-one boundary strata modeled on a similar calculation in genus zero due to Blankers–Gillespie–Levinson.

Keywords : moduli space of curves

Mathematics Subject Classification (2020) : 14H10

1 Introduction

Within the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of n -pointed, genus- g stable curves, the boundary strata are the closures of the loci of curves with a fixed topological type and distribution of marked points across components. These strata play a key role in the geometric and combinatorial study of the moduli space; for example, their cohomology classes are among the generators of the tautological subring $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$, which is a widely-studied and more tractable substitute for the full cohomology ring.

Boundary strata can be specified by the combinatorial information of a dual graph, in which the size of the edge set $E(G)$ corresponds to the codimension in $\overline{\mathcal{M}}_{g,n}$ of the associated stratum. We consider the quantity

$$\chi_{g,n} := \sum_{\substack{\text{dual graphs} \\ G \text{ for } \overline{\mathcal{M}}_{g,n}}} (-1)^{|E(G)|}. \quad (1)$$

As we explain in Section 2, this is essentially equivalent to the Euler characteristic of the boundary complex $\Delta_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$, a Δ -complex that encodes the containment relations between boundary strata.

*Both authors were partially supported by NSF CAREER Grant 2137060, and the second author also received support from NSF Grant HRD-1826490.



In genus zero, the topology of the boundary complex is well-understood: $\Delta_{0,n}$ is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$ by [3], from which the formula

$$\chi_{0,n} = (-1)^{n-3}(n-2)! \tag{2}$$

readily follows. Recent work of Blankers–Gillespie–Levinson [1, Theorem 3.1] gives an elementary proof of (2) by constructing a sign-reversing involution on the set of boundary strata for $\overline{\mathcal{M}}_{0,n}$ with precisely $(n-2)!$ fixed points, each of which is an $(n-3)$ -edge graph.¹

A similar story holds in genus one, in which case work of Chan–Galatius–Payne [2] shows that, whenever $n \geq 3$, the boundary complex $\Delta_{1,n}$ is homotopy equivalent to a wedge of $(n-1)!/2$ spheres of dimension $n-1$. This similarly yields a formula for $\chi_{1,n}$, and in this note, we give a purely combinatorial proof of this formula by way of a sign-reversing involution inspired by [1]. The resulting formula is the following:

Theorem 1.1 *For any $n \geq 3$, one has*

$$\chi_{1,n} = (-1)^n \frac{(n-1)!}{2}.$$

In Section 2, we develop the requisite background on boundary strata and their associated dual graphs, and in Section 3, we describe the sign-reversing involution that proves Theorem 1.1.

2 Background

The points of the moduli space $\overline{\mathcal{M}}_{g,n}$ are in bijection with tuples $(C; x_1, \dots, x_n)$, where C is an algebraic curve over \mathbb{C} with arithmetic genus g and at worst nodal singularities, $x_1, \dots, x_n \in C$ are distinct smooth points referred to as the **marked points**, and the tuple is **stable** in the sense that it has finitely many automorphisms. Equivalently, the condition of stability requires that every irreducible component of C of geometric genus zero has at least three special points (marked points or half-nodes) and every irreducible component of geometric genus one has at least one special point.

Each element $(C; x_1, \dots, x_n)$ of $\overline{\mathcal{M}}_{g,n}$ has an associated **dual graph**, which consists of the following data:

- a vertex v for each irreducible component of C , labeled with the geometric genus $g(v)$ of the irreducible component;
- an edge between (not necessarily distinct) vertices for each node joining the corresponding irreducible components;
- a half-edge (or “leg”) for each marked point, labeled by the index in $\{1, 2, \dots, n\}$ of the marked point.

¹The results of [1] also apply to more general moduli spaces, including Hassett spaces of weighted stable curves and Fry’s graphically stable moduli spaces.



For instance, the dual graphs for $\overline{\mathcal{M}}_{1,2}$ are shown in Figure 1. We refer to both the halves of the edges and the legs as **half-edges**, and we use the term **valence of v** , denoted $\text{val}(v)$, for the total number of incident half-edges to a vertex v . Note that stability corresponds to the condition that every vertex v for which $g(v) = 0$ has $\text{val}(v) \geq 3$, and every vertex v for which $g(v) = 1$ has $\text{val}(v) \geq 1$.

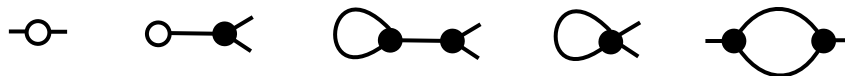


Figure 1: The dual graphs for $\overline{\mathcal{M}}_{1,2}$. We omit the labels on legs, and we indicate genus-zero vertices by a filled circle and genus-one vertices by an open circle.

For any dual graph G , we denote by S_G the closure of the locus in $\overline{\mathcal{M}}_{g,n}$ of elements with dual graph G . Such loci are referred to as **boundary strata**, and it is a standard fact that

$$\text{codim}_{\overline{\mathcal{M}}_{g,n}}(S_G) = |E(G)|,$$

where $E(G)$ denotes the set of edges of G . In particular, the maximum number of edges of a dual graph G for $\overline{\mathcal{M}}_{g,n}$ is equal to $\dim(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

Inclusions among boundary strata can be described in terms of edge-contractions on the associated dual graph. To do so, for a dual graph G and an edge e between distinct vertices v and w , define the **contraction** of G along e to be the dual graph obtained by replacing v and w by a single vertex of genus $g(v) + g(w)$ containing all of the incident half-edges from both v and w . Similarly, if e is a self-edge at a vertex v , define the contraction of G along e to be the dual graph obtained by removing e and increasing $g(v)$ to $g(v) + 1$. Geometrically, these operations correspond to smoothing nodes, and $S_G \subseteq S_H$ if and only if H can be obtained from G by contracting a subset of edges.

This idea allows one to combinatorially encode the inclusions of boundary strata into a Δ -complex $\Delta_{g,n}$ known as the **boundary complex**. The precise definition of $\Delta_{g,n}$ is not relevant for this work (see, for example, [2, Section 3]), but roughly, it can be viewed as the result of taking a k -simplex for each codimension- $(k + 1)$ boundary stratum and gluing the boundaries of these simplices according to edge contraction. Alternatively, $\Delta_{g,n}$ can be defined as the subspace of the moduli space $M_{g,n}^{\text{trop}}$ parameterizing tropical curves of volume 1, or in other words, as the link of $M_{g,n}^{\text{trop}}$ at its cone point.

The Euler characteristic of $\Delta_{g,n}$ is computed as the signed sum of the number of simplices in each dimension:

$$\chi(\Delta_{g,n}) = \sum_{i=0}^{3g-4+n} (-1)^i \cdot |\{(i + 1)\text{-edge dual graphs for } \overline{\mathcal{M}}_{g,n}\}|.$$

From here, one readily sees that $\chi(\Delta_{g,n})$ is related to the quantity $\chi_{g,n}$ of (1) by

$$\chi_{g,n} = -\chi(\Delta_{g,n}) + 1.$$

We now turn to computing this quantity in the case when $g = 1$.

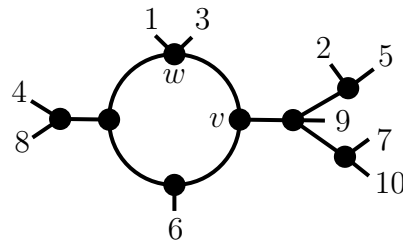


3 The Sign-Reversing Involution

Fix $n \geq 3$, and let $\mathcal{S}_{1,n}$ denote the set of dual graphs for $\overline{\mathcal{M}}_{1,n}$. Note that, for $G \in \mathcal{S}_{1,n}$, it must either be the case that G has a unique genus-one vertex, or that G has a unique cycle C as a subgraph. In the latter case, for each vertex v of C , we denote by

$$L_v \subseteq \{1, 2, \dots, n\}$$

the labels of the vertices that map to v under the contraction of all edges in $E(G) \setminus E(C)$. For example, let G be the following graph:



Then we have

$$L_v = \{2, 5, 7, 9, 10\} \quad \text{and} \quad L_w = \{1, 3\}.$$

Equipped with this notation, we define an involution

$$i : \mathcal{S}_{1,n} \rightarrow \mathcal{S}_{1,n} \tag{3}$$

as follows, where the five cases are illustrated in Figure 2:

1. If $G \in \mathcal{S}_{1,n}$ has a genus-one vertex v and $\text{val}(v) = 1$, then v has a unique incident edge e , and we set $i(G)$ to be the contraction of G along e .
2. If $G \in \mathcal{S}_{1,n}$ has a genus-one vertex v and $\text{val}(v) > 1$, set $i(G)$ to be the graph obtained from G by changing the genus of v to zero and attaching a new genus-one vertex (with no marked points) to v .
3. If $G \in \mathcal{S}_{1,n}$ has a cycle C as a subgraph and $|L_v| = 1$ for all v , set $i(G) = G$.
4. If $G \in \mathcal{S}_{1,n}$ has a cycle C as a subgraph and $|L_v| > 1$ for at least one v , then among all v with $|L_v| > 1$, choose the one for which the minimum element of L_v is smallest and suppose $\text{val}(v) = 3$. Then v has a unique incident half-edge not in C , and the fact that $|L_v| > 1$ implies that this half-edge must come from an edge e as opposed to a leg. Set $i(G)$ to be the contraction of G along e .
5. If $G \in \mathcal{S}_{1,n}$ has a cycle C as a subgraph and $|L_v| > 1$ for at least one v , then among all v with $|L_v| > 1$, choose the one for which the minimum element of L_v is smallest and suppose $\text{val}(v) > 3$. Set $i(G)$ to be the graph obtained from G by replacing v by a pair of genus-zero vertices v_1 and v_2 joined by an edge e , where v_1 is incident only to e and the two adjacent edges of C , and v_2 inherits all other half-edges of v .



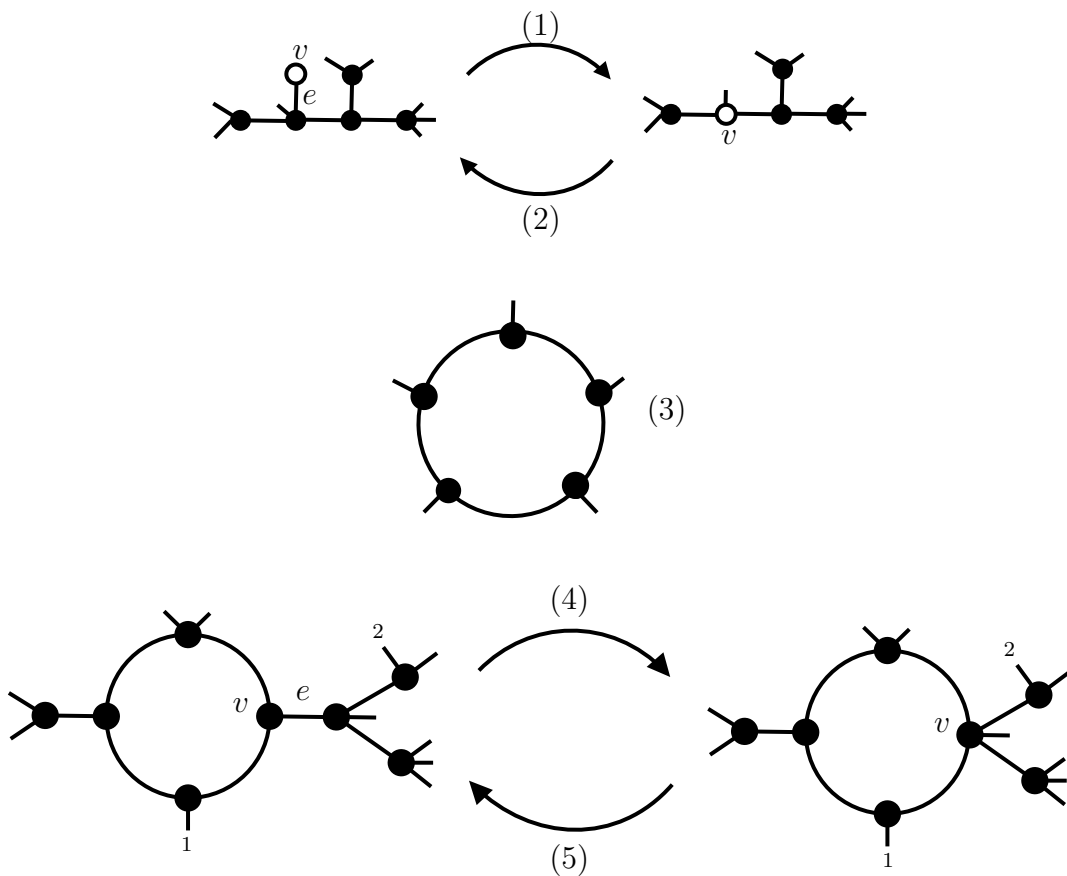


Figure 2: An illustration of cases (1)–(5) in the definition of the involution i , with genus-zero vertices indicated by a filled circle and genus-one vertices by an open circle, and with the labels on most legs omitted.

Using this definition of i , the proof of Theorem 1.1 is nearly immediate.

Proof. It is straightforward to see that i is an involution: if G satisfies case (1) in the definition of i , then $i(G)$ satisfies case (2) and vice versa, whereas if G satisfies case (4), then $i(G)$ satisfies case (5) and vice versa.

Moreover, i is sign-reversing in the sense that, if

$$\sigma(G) := (-1)^{|E(G)|},$$

then for any G not fixed by i , we have

$$\sigma(i(G)) = -\sigma(G),$$

since $i(G)$ either adds or removes a single edge from G . Thus, all terms in the sum

$$\chi_{1,n} = \sum_{G \in \mathcal{S}_{1,n}} \sigma(G)$$



cancel aside from those corresponding to graphs fixed by i , which are graphs with a cycle in which every vertex v has $|L_v| = 1$. In other words, such graphs have $G = C$, as shown in the middle picture of Figure 2. The number of such graphs is the number of cyclic orderings of $\{1, 2, \dots, n\}$, which is $(n-1)!/2$. Since each has exactly n edges, each contributes $(-1)^n$ to $\chi_{1,n}$, and the formula of Theorem 1.1 follows. \square

Acknowledgments

The authors are grateful to Maria Gillespie for giving the talk (at the conference “Combinatorics of Moduli of Curves” at the Banff International Research Station) that inspired this work, to Jake Levinson for related conversations, and to the anonymous referee for helpful comments and corrections.

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Received: December 9, 2025 **Accepted:** March 21, 2026
Communicated by John M. Dusel

