The Variational Principle for Entropy of Countable State Shift Spaces with Specification

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Abstract - We generalize the well-studied specification property to countable-state shift spaces. We present an infinite class of examples of such spaces. Using a generalized notion of Gurevich entropy, we prove the variational principle in the setting of locally finite countable-state shift spaces with specification. This lays the foundation for developing the theory of equilibrium states in this setting.

Keywords : countable-state shift spaces; variational principle; Gurevich entropy; specification property; thermodynamic formalism

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1 Introduction

A *topological dynamical system* (*TDS*) is a pair (X, f), where X is a (usually compact) metric space and $f : X \to X$ is a homeomorphism. One of the most well-studied classes of TDS is the class of shift spaces. Given a (usually finite) set $\mathcal{A}$, the set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *subshift* over the *alphabet* $\mathcal{A}$ if it is closed in the product topology and invariant under the *shift map* $\sigma : X \to X$ defined by $\sigma(\cdots x_{-1}.x_0x_1\cdots) = \cdots x_{-1}x_0.x_1\cdots$. We often refer to the *shift space* (X, σ) as X, since any subshift is endowed with the action of the homeomorphism σ.

We denote by $\mathcal{L}_n(X)$ the set of n-length *words* which appear in *points* in X and let $\mathcal{L}(X) = \bigcup_{n\geq 0} \mathcal{L}_n(X)$. When the underlying shift space X is clear from context, we frequently write $\mathcal{L}_n$ and $\mathcal{L}$ instead. For a finite-state shift space (that is, the alphabet $\mathcal{A}$ is finite), the notion of *topological entropy* for a general TDS can be defined as follows: $h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$.

Thermodynamic formalism studies the relationship between topological entropy and *measure-theoretic entropy*, which is denoted by $h_\mu(f)$ for a measure μ in the set $\mathcal{M}_f(X)$ of f-invariant Borel probability measures on X. The variational principle [21, Theorem 11.40] is a classical result that relates the topological and measure-theoretic entropy of a compact dynamical system (X, f):

$$h_{\text{top}}(X) = \sup_{\mu\in\mathcal{M}_f(X)} h_\mu(f) \tag{1}$$

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One prominent question in thermodynamic formalism is the existence and uniqueness of measures which attain the supremum on the RHS of (1), called *measures of maximal entropy* or *MMEs* for short. For finite-state shift spaces, there always exist MMEs by the weak* compactness of $M_\sigma(X)$, which comes from the compactness and separability of X. For general TDS, Bowen [1] showed that the specification property guarantees the existence of a unique MME. A finite-state shift space X has the *specification property* if there exists a $\tau \in \mathbb{N}_0$ such that, for all $u, w \in \mathcal{L}$, there exists $v \in \mathcal{L}_\tau$ so that $uvw \in \mathcal{L}$. Recent progress in extending Bowen's results, including in the symbolic setting, is surveyed in [3] and [12].

The situation in the non-compact setting is much more complex. First, there is no general notion of topological entropy for non-compact TDS. In the symbolic setting, Gurevich [7] defined a notion of entropy for irreducible topological Markov shifts. Let $\mathcal{A}$ be a countable (possibly finite) set and let $T = (t_{ab})_{\mathcal{A}\times\mathcal{A}}$ be a matrix of zeros and ones with no rows or columns which consist only of zeros. The *topological Markov shift* (*TMS*) with alphabet $\mathcal{A}$ and *transition matrix* T is the shift space

$$X := \{x \in \mathcal{A}^{\mathbb{Z}} : T_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

We say that a TMS X is irreducible if its transition matrix T is irreducible; that is, if, for all $a, b \in \mathcal{A}$, there exists $n \in \mathbb{N}$ so that $(T^n)_{ab} \geq 1$.

Given a TMS X, let $\mathrm{Per}_n^a(X)$ denote the set of n-length *periodic points* in X which start at a; that is, $\mathrm{Per}_n^a(X) = \{x \in X : x_0 = a, \sigma^n = x\}$. The *Gurevich* entropy of X is defined to be

$$h_G(X) := \limsup_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_n^a(X)| \tag{2}$$

for an arbitrary $a \in \mathcal{A}$. Two natural questions arise: is the definition of $h_G(X)$ independent of choice of $a \in \mathcal{A}$ and is the limsup on the RHS of (2) a limit? As shown by Gurevich [7] and later Sarig [18], the answer to the first question is yes if X is irreducible, and the answer to the second question is yes if X is *mixing*; that is, if, for all $a, b \in \mathcal{A}$, there exists $N \in \mathbb{N}$ so that $n \geq N$ implies $(T^n)_{ab} \geq 1$. Further, if X is mixing, the variational principle holds. [17] is a comprehensive account of modern thermodynamic formalism for topological Markov shifts.

In the finite-state case, TMS are known as *shifts of finite type* or *SFTs*, and are the simplest and best-understood example of a finite-state shift space. A SFT has the specification property if and only if it is mixing. Non-SFT shift spaces with specification have been known for decades: examples include β-shifts [14, 10, 15, 20] and cocyclic subshifts [11]. The natural question about the existence of non-TMS countable-state shift spaces with specification and their associated thermodynamic properties arises.

Fix a countable-state shift space X. We generalize the finite-state specification property to the countable-state setting. Our generalized property is weaker than the one considered in [8]: it is local rather than global. We prove the following about this property.

Theorem A

1. *A TMS X has the specification property if and only if it is mixing.*

2. *There exist infinitely many non-TMS countable-state shift spaces with the specification property.*

It is natural to assume that our shift spaces are *locally finite*; that is, for all $a \in \mathcal{A}$, the sets $B_a(X) := \{b \in \mathcal{A} : ba \in \mathcal{L}\}$ and $F_a(X) := \{b \in \mathcal{A} : ab \in \mathcal{L}\}$ ("B" for "before" and "F" for "follow") are finite. Local finiteness is a working assumption in, for example, the theory of infinite volume limits of the Ising model [6, 5] and certain results for TMS [4]. A TMS is locally finite if and only if it is locally compact, which is the working assumption in upcoming work by Climenhaga, Thompson, and Wang.

Let us denote by $\mathcal{L}_n^a(X)$ the set of n-length words which start at a and can be followed by a; that is, $\mathcal{L}_n^a(X) = \{\omega \in \mathcal{L}(X) : \omega_0 = a, \omega a \in \mathcal{L}(X)\}$. If X is a TMS, then one sees that a word $\omega \in \mathcal{L}_n^a(X)$ induces a periodic point $x = \omega\omega\cdots \in \mathrm{Per}_n^a(X)$ and vice versa. In general, there are more words in $\mathcal{L}_n^a(X)$ than points in $\mathrm{Per}_n^a(X)$ since $\mathcal{L}_n^a(X)$ does not satisfy the *free concatenation property* for arbitrary shift spaces. We define the *Gurevich entropy* of X to be

$$h_G(X) := \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n^a(X)|.$$

This definition is novel and extends the definition of Gurevich to any countable-state shift space. We show the following.

Theorem B *If X is a locally finite countable-state shift with specification, then the definition of $h_G(X)$ is independent of choice of $a \in \mathcal{A}$ and the limsup in the definition of $h_G(X)$ is a limit.*

Theorem C *If X is a locally finite countable-state shift space with specification, then the variational principle holds; that is,*

$$h_G(X) = \sup_{\mu \in \mathcal{M}_\sigma(X)} h_\mu(\sigma).$$

These theorems indicate that our setting is apt for developing the theory of equilibrium states. We hope to do so in future work. TMS can be used to code more complicated dynamical systems and gain information about their thermodynamic properties, for example, [2]. We hope that developing thermodynamic formalism in our setting will allow us to mimic previously-used codings into shift spaces with specification in the countable-state setting, for example, [16].

The layout of our paper is as follows. In Section 2we introduce background material from symbolic dynamics. In Section 3 we define and discuss known properties of TMS. In Section 4 we define countable-state specification properties, classify TMS with specification, and provide an infinite class of non-TMS examples, proving Theorem A. In Section 5, we define Gurevich entropy for countable-state shift spaces and prove Theorem B. In Section 6 we prove Theorem C, the variational principle.

2 Preliminaries

Let $\mathcal{A}$ be a countable (possibly finite) set. We call $\mathcal{A}$ the set of *states* or the *alphabet*. A *shift space* X over $\mathcal{A}$ is a subset of the *full shift* $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some *forbidden word set* $\mathcal{F}$. The notion of isomorphism for shift spaces is known as *conjugacy* (see, e.g. [19, Definition 3.17]).

The elements $x \in X$ are called *points*, and finite-length substrings of x are called *words*. For $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, we let $\mathcal{L}_n(X)$ be the set of n-length words in X and then define the *language* of X to be

$$\mathcal{L}(X) = \bigcup_{n \geq 0} \mathcal{L}_n(X).$$

For a word $u \in \mathcal{L}(X)$ with $u = u_0 \cdots u_{n-1}$, we denote u_t the last letter of u; i.e., u_{n-1}. For $m, n \in \mathbb{N}_0$ and $a_{-m}, \dots, a_n \in \mathcal{A}$, *cylinder sets* are subsets of X defined to be

$$[a_{-m}, \dots, a_n] = \{x \in X : x_i = a_i \text{ for all } -m \leq i \leq n\}.$$

Given $x, y \in X$, we define a metric on X:

$$d(x, y) = \begin{cases} 2^{-\min\{i : x_i \neq y_i\}} & x \neq y \\ 0 & x = y. \end{cases}$$

The metric space (X, d) is complete, separable, and is compact if and only if $|\mathcal{A}| < \infty$.

3 Topological Markov Shifts

Definition 3.1 *Let $\mathcal{A}$ be countable and $T = (t_{ij})_{\mathcal{A}\times\mathcal{A}}$ a matrix of zeroes and ones. We call T a transition matrix if for all $a \in \mathcal{A}$ there exist $i, j \in \mathcal{A}$ with $T_{ai} = T_{ja} = 1$. Given such a T, we define the topological Markov shift (TMS) over $\mathcal{A}$ generated by T to be*

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

We note in passing that a TMS generated by $T = (t_{ij})_{\mathcal{A}\times\mathcal{A}}$ can be thought of as being generated by bi-infinite walks on the directed graph $G_T = (V, \mathcal{E})$ where $V = \mathcal{A}$ and $\mathcal{E} = \{(i, j) \in \mathcal{A} \times \mathcal{A} : t_{ij} = 1\}$. In fact, any countably infinite shift space can be represented as bi-infinite walks on a directed graph, as proven by Sobottka ([19], Theorem 6.1), so graphs provide good intuition for how countable state shift spaces behave. This notably does not hold in the finite state setting (for example, non-sofic cocyclic subshifts presented in [11]).

Proposition 3.2 *A shift space X is conjugate to a TMS if and only if there exists a forbidden word set $\mathcal{F}$ generating X whose words have bounded length.*

Proof. ($\Longrightarrow$) If X is conjugate to a TMS, then it is generated by a countable matrix T which is indexed by $\mathcal{A}^M \times \mathcal{A}^M$ for some M, for if not, it'd be indexed by an uncountable set. Then T gives rise to a forbidden word set containing only words of length M.

($\Longleftarrow$) Assume that $X = X_{\mathcal{F}}$ where the words in $\mathcal{F}$ have bounded length. Put $M = \max\{|\omega| : \omega \in \mathcal{F}\}$ and note that $\mathcal{F}$ can be recoded to only contain words of length $M+1$ (the proof of Proposition 2.1.7 in [13] generalizes). Then we can easily create an adjacency matrix $T = (t_{ij})_{\mathcal{A}^{M+1} \times \mathcal{A}^{M+1}}$ which generates X. □

We will compare the following properties to the specification property in the next section.

Definition 3.3 *A shift space X is* irreducible *if, for all $u, w \in \mathcal{L}(X)$, there exists a $v \in \mathcal{L}(X)$ such that $uvw \in \mathcal{L}(X)$.*

Definition 3.4 *A matrix $T = (t_{ij})_{\mathcal{A}\times\mathcal{A}}$ is* irreducible *if for all pairs $i, j \in \mathcal{A}$, there exists an $\ell > 0$ such that $(T^{\ell})_{ij} > 0$.*

Lemma 3.5 *A TMS X is irreducible if and only if the transition matrix that generates it is irreducible.*

Proof. ($\Longrightarrow$) X is irreducible, so, in particular, for all $i, j \in \mathcal{A}$, there exists a $v \in \mathcal{L}(X)$ such that $ivj \in \mathcal{L}(X)$. Then, $(T^{|v|})_{ij} > 0$. ($\Longleftarrow$) For all $u, w \in \mathcal{L}(X)$, there exists an ℓ-length word v such that $u_t v w_0 \in \mathcal{L}(X)$ and thus $uvw \in \mathcal{L}(X)$ since X is a TMS. □

Definition 3.6 *A shift space X is* mixing *if, for all $u, w \in \mathcal{L}(X)$, there exists an $N \in \mathbb{N}_0$ such that, for all $n \geq N$, there exists an n-length word v such that $uvw \in \mathcal{L}(X)$.*

Lemma 3.7 *A TMS X over $\mathcal{A}$ is mixing if and only if, for all $a, b \in \mathcal{A}$, there exists an $N_{ab} \in \mathbb{N}_0$ such that, for all $n \geq N_{ab}$, there exists an n-length word v such that $avb \in \mathcal{L}(X)$.*

Proof. The forward direction is clear. For the reverse direction, note that since X is a TMS, for any $u, w \in \mathcal{L}(X)$, $u_t v w_0 \in \mathcal{L}(X)$ implies $uvw \in \mathcal{L}(X)$ for any $v \in \mathcal{L}(X)$. □

4 The Generalized Specification Property

The following is a well-studied property of finite state shift spaces:

Definition 4.1 *A shift space $X \subseteq \{0, 1, \ldots, n-1\}^{\mathbb{Z}}$ satisfies the specification property if there exists a $\tau \in \mathbb{N}_0$ such that for all $u, w \in \mathcal{L}(X)$ there exists a $v \in \mathcal{L}_{\tau}(X)$ such that $uvw \in \mathcal{L}(X)$.*

In the more general countable state setting, we introduce the following definition, which is the symbolic version of a property of complete separable metric spaces due to Climenhaga, Thompson, and Wang.

Definition 4.2 *Letting $I \subseteq \mathcal{A}$ be finite, we define $\mathcal{L}^I(X) = \{\omega \in \mathcal{L}(X) : \omega_0 \in I, \omega a \in \mathcal{L}(X)$ for some $a \in I\}$. A countable state shift space X satisfies the (countable-state) specification property if for all finite $I \subseteq \mathcal{A}$ there exists a $\tau = \tau(I) \in \mathbb{N}_0$ such that for all $u, w \in \mathcal{L}^I(X)$ there exists a $v \in \mathcal{L}_\tau(X)$ such that $uvw \in \mathcal{L}^I(X)$.*

Remark 4.3 This *(countable-state) specification property*, though for simplicity we will call it *specification* throughout, is a symbolic analogue of Bowen's specification property [1]. Bowen's specification property can be applied to any metrizable topological dynamical system. For finite alphabets, countable-state specification coincides with the usual symbolic version of Bowen's specification property (Definition 4.1), while for countable alphabets it provides a natural extension of this symbolic version to the non-compact, countable-state setting.

Definition 4.4 *A countable state shift space X satisfies the weak specification property if for all finite $I \subseteq \mathcal{A}$ there exists a $\tau = \tau(I) \in \mathbb{N}_0$ such that for all $u, w \in \mathcal{L}^I(X)$ there exists a $v \in \mathcal{L}(X)$ with $|v| \leq \tau$ such that $uvw \in \mathcal{L}^I(X)$.*

Proposition 4.5 *If a shift space X satisfies the weak specification property, then it is irreducible.*

Proof. For any $u, w \in \mathcal{L}(X)$, there exists a finite $I \subseteq \mathcal{A}$ such that $u, w \in \mathcal{L}^I(X)$. Then, there exists a $v \in \mathcal{L}(X)$ (with $|v| \leq \tau(I)$) such that $uvw \in \mathcal{L}(X)$. □

We get as an immediate result of Proposition 4.5 that all shift spaces satisfying the specification property are irreducible.

We now characterize these specification properties in the class of TMSs.

Proposition 4.6 *A TMS satisfies the weak specification property if and only if it is irreducible.*

Proof. Let X be a TMS over $\mathcal{A}$ generated by $T = (t_{ij})_{\mathcal{A}\times\mathcal{A}}$. ($\implies$) If X satisfies the weak specification property, then it is irreducible by Proposition 4.5.

($\impliedby$) Let $I \subseteq \mathcal{A}$ be finite. For each pair $a, b \in I$, there exists a ℓ_{ab} such that $(T^{\ell_{ab}})_{ab} > 0$. Putting $\tau = \max\{\ell_{ab} : a, b \in I\}$, we see that X satisfies the weak specification property. □

Proposition 4.7 *A TMS X satisfies the specification property if and only if it is mixing.*

Proof. ($\implies$) That X is irreducible follows from Proposition 4.6. To show that X is mixing, let $I = \{a, b\} \subseteq \mathcal{A}$. Since X satisfies the specification property, there exists a $\tau = \tau(I) \in \mathbb{N}_0$ such that, for all $u, w \in \mathcal{L}^I(X)$ with $|u| = n$, $u_0 = a$, and $w_0 = b$, there exists a $v \in \mathcal{L}_{\tau+n-1}(X)$ such that $uvw \in \mathcal{L}(X)$. Thus, by varying the length of u, we see that $N_{ab} = \tau$ works.

($\impliedby$) X is irreducible and thus satisfies the weak specification property by Proposition 4.6. Fix $I \subseteq \mathcal{A}$ finite. Since X is mixing, for any $a, b \in I$, there exists an N_{ab} such that for all $n \geq N_{ab}$, there exists a $v \in \mathcal{L}_n(X)$ such that $avb \in \mathcal{L}(X)$ by Lemma 3.7. Let $\tau = \max\{N_{ab} : a, b \in I\}$ to finish since X is a TMS. □

The following is an infinite class of non-TMS shift spaces which satisfy the specification property.

Theorem 4.8 *Fix $r > 0$. Let $X \subseteq \{0, 1, \dots, r-1\}^{\mathbb{Z}}$ be a shift space with specification such that any forbidden word set generating X contains words of unbounded length. Let $Y \subseteq \{r, r+1, \dots\}^{\mathbb{Z}}$ be a TMS satisfying the specification property with $Y = Y_{\mathcal{F}'}$. Then the shift $Z = Z_{\mathcal{F}\cup\mathcal{F}'} \subseteq \{0, 1, 2, \dots\}^{\mathbb{Z}}$ satisies the specification property and is not a TMS.*

Proof. That Z isn't a TMS follows from Proposition 3.2. To show that Z satisfies the specification property, put $\mathcal{A} = \mathbb{N}_0$, $\mathcal{A}_X = \{0, 1, \dots, r-1\}$, and $\mathcal{A}_Y = \{r, r+1, \dots\}$, and let $a \in \mathcal{A}_X$, $b \in \mathcal{A}_Y$, and $I \subseteq \mathcal{A}$ be finite. For $u, w \in \mathcal{L}^I(X)$, we have

$$\begin{aligned} uaw \in \mathcal{L}^I(X) &\text{ if } u_t, w_0 \in \mathcal{A}_Y, \\ ubw \in \mathcal{L}^I(X) &\text{ if } u_t, w_0 \in \mathcal{A}_X, \\ ucw \in \mathcal{L}^I(X) \text{ for some } c \in \mathcal{A}_X &\text{ if } u_t \in \mathcal{A}_X, w_0 \in \mathcal{A}_Y, \\ udw \in \mathcal{L}^I(X) \text{ for some } d \in \mathcal{A}_Y &\text{ if } u_t \in \mathcal{A}_Y, w_0 \in \mathcal{A}_X, \end{aligned}$$

so $\tau = \tau(I) = 1$ works as the specification constant. $\square$

5 Gurevich Entropy

Gurevich proved that the following definition is well-defined in the case of TMSs (see [7] and [8] in [18]).

Definition 5.1 *Let X be a shift space over a countable alphabet $\mathcal{A}$. For any finite $I \subseteq \mathcal{A}$, we define*

$$h_G^I(X) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n^I(X),$$

which takes values in $[0, \infty]$. Then we define the Gurevich entropy of X to be $h_G(X) = h_G^I(X)$ for an arbitrary finite $I \subseteq \mathcal{A}$.

We impose the following property on our shift spaces to ensure that their Gurevich entropies do not blow up in finite time.

Definition 5.2 *Let X be a shift space over a countable alphabet $\mathcal{A}$. For any $a \in \mathcal{A}$, we define the letter follower set of a to be $F_a = \{a\ell \in \mathcal{L}(X) : \ell \in \mathcal{A}\}$. We say that X is locally finite if for all $a \in \mathcal{A}$, $\#F_a < \infty$.*

Proposition 5.3 *Define*

$$\widetilde{\mathcal{L}}^I(X) = \{\omega \in \mathcal{L}(X) : \omega_0, \omega_t \in I\}.$$

X is a locally finite shift space with specification if and only if, for all finite $I \subseteq \mathcal{A}$, there exists a $\tau = \tau(I) \in \mathbb{N}$ such that, for all $u, w \in \widetilde{\mathcal{L}}^I(X)$, there exists a $v \in \mathcal{L}_\tau(X)$ such that $uvw \in \widetilde{\mathcal{L}}^I(X)$.

Proof. Fix a finite $I \subseteq \mathcal{A}$. ($\implies$) Since X is locally finite, $\tilde{I} = I \cup \bigcup_{a\in I} F_a$ is finite, so apply X's specification property to $\tilde{I}$.

($\impliedby$) For any $\tilde{u}, \tilde{w} \in \widetilde{\mathcal{L}}^I(X)$ and $\tilde{v} \in \mathcal{L}_\tau(X)$ such that $\tilde{u}\tilde{v}\tilde{w} \in \widetilde{\mathcal{L}}^I(X)$, let u (resp. w) be $\tilde{u}$ (resp. $\tilde{w}$) without its last (resp. first) letter, and let $v = \tilde{u}_t\tilde{v}\tilde{w}_0$. Then, $u, w \in \mathcal{L}^I(X)$ and $v \in \mathcal{L}_{\tau+2}(X)$ with $uvw \in \mathcal{L}^I(X)$. □

Remark 5.4 Moving forwards, we adopt $\widetilde{\mathcal{L}}^I(X)$ as the definition of $\mathcal{L}^I(X)$, since we'll be working exclusively with locally finite shift spaces.

Lemma 5.5 *Let X be a locally finite shift space over a countable alphabet $\mathcal{A}$. Then for all $n \in \mathbb{N}_0, a \in \mathcal{A}$, $\#\mathcal{L}_n^{\{a\}}(X) < \infty$.*

Proof. Fix $n \in \mathbb{N}_0, a \in \mathcal{A}$. We denote $S_n^{\{a\}}(X) = \{a\omega \in \mathcal{L}_n(X) : \omega \in \mathcal{L}_{n-1}(X)\}$ (S for "start"). That $\#\mathcal{L}_n^{\{a\}}(X) < \#S_n^{\{a\}}(X)$ is clear, so it suffices to prove that $\#S_n^{\{a\}}(X) < \infty$ for all $n \in \mathbb{N}_0$. We do this by induction. First, we have that $\#S_0(X) = \#S_1(x) = 1$. Now if $\#S_n(X) < \infty$, the set $E = \{\ell \in \mathcal{A} : \ell = \omega_t \text{ for some } \omega \in S_n^{\{a\}}(X)\}$ is finite, so we can let $M = \max_{\ell\in E}\{\#F_\ell\}$ with $M < \infty$ because X is locally compact. Then we have $\#S_{n+1}(X) \leq M\#E < \infty$, so we are finished. □

Theorem 5.6 *Let X be a locally finite shift space over a countable alphabet $\mathcal{A}$ which satisfies the specification property. Then for any finite $I \subseteq \mathcal{A}$, $h_G^I(X)$ exists, and $h_G(X)$ is well-defined.*

Proof. ($h_G^I(X)$ exists) Let $m, n \in \mathbb{N}_0$. Since X satisfies the specification property, there exists a $\tau = \tau(I) \in \mathbb{N}_0$ such that, for any $u \in \mathcal{L}_m^I(X), w \in \mathcal{L}_n^I(X)$, there exists a $v \in \mathcal{L}_\tau(X)$ such that $uvw \in \mathcal{L}_{m+n+\tau}(X)$. Thus, there exists an injection between $\mathcal{L}_m^I(X) \times \mathcal{L}_n^I(X)$ and $\mathcal{L}_{m+n+\tau}(X)$, showing that $\#\mathcal{L}_m^I(X)\#\mathcal{L}_n^I(X) \leq \#\mathcal{L}_{m+n+\tau}^I(X)$ (here we use Lemma 5.5). Taking the log of both sides and multiplying by -1, we get

$$-\log\#\mathcal{L}_m^I(X) - \log\#\mathcal{L}_n^I(X) \geq -\log\#\mathcal{L}_{m+n+\tau}^I(X). \quad (3)$$

Now we put $x_n = -\log\#\mathcal{L}_n^I(X)$ and $y_{n+\tau} = x_n$ so that (3) becomes $y_{m_\tau} + y_{n+\tau} \geq y_{m+n+2\tau}$, to which we apply Fekete's lemma to see that $\lim_{n\to\infty} \frac{y_n}{n}$ exists; that is, $h_G^I(X)$ exists.

($h_G(X)$ is well-defined) Fix $I, J \subseteq \mathcal{A}$ finite. It suffices to find an injection between $\mathcal{L}_n^{I\cup J}(X)$ and $\mathcal{L}_{n+N}^J(X)$ for some $N \in \mathbb{N}_0$ since this would give

$$\#\mathcal{L}_n^I(X) \leq \#\mathcal{L}_n^{I\cup J}(X) \leq \#\mathcal{L}_{n+N}^J(X),$$

implying the result since I and J were chosen arbitrarily. By Proposition 4.5, we know that there exists $\omega \in \mathcal{L}_\ell^J(X)$ for some $\ell \in \mathbb{N}_0$. Fix $w \in \mathcal{L}_n^{I\cup J}(X)$ for some $n \in \mathbb{N}_0$ by the same reasoning. Using the specification property, there exists a $\tau = \tau(I \cup J)$ so that we can find $v, v' \in \mathcal{L}_\tau(X)$ with $\omega v w v' \omega \in \mathcal{L}_{n+2\tau+2\ell}^J(X)$, providing the desired injection. □

Thanks to Theorem 5.6, we can do the preceding calculations of $h_G(X)$ on $h_G^I(X)$ for any finite $I \subseteq \mathcal{A}$.

The following is a property of shift spaces which guarantees finite Gurevich entropy.

Definition 5.7 *A shift space X is* uniformly locally finite *if there exists a constant $C \in \mathbb{N}_0$ such that, for all $a \in \mathcal{A}$, $\#F_a \leq C$.*

Proposition 5.8 *If X is a uniformly locally finite shift space, then $h_G(X) < \infty$.*

Proof. Pick a constant $C \in \mathbb{N}_0$ such that, for all $a \in \mathcal{A}$, $\#F_a \leq C$. Recalling the definition of $S_n^{\{a\}}(X)$ from the proof of Lemma 5.5, we have

$$h_G(X) \leq \lim_{n\to\infty} \frac{1}{n} \log \#S_n^{\{a\}}(X) \leq \lim_{n\to\infty} \frac{1}{n} \log C^n = \log C < \infty.$$

□

6 The Variational Principle

For a probability measure μ on X, we denote by $h_\mu(\sigma)$ the *measure-theoretic entropy* of X. The following is a well-known result for finite state shift spaces, known as the variational principle.

Theorem 6.1 *Let $X \subseteq \{0, 1, \ldots, n-1\}^{\mathbb{Z}}$ be shift space and denote by $\mathcal{M}_\sigma(X)$ the set of ergodic σ-invariant probability measures on X. Then,*

$$h(X) = \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(X)\}.$$

Lemma 6.2 *Let $X = X_{\mathcal{F}}$ be a shift space and define $\mathcal{K} = \{K \subseteq X : K = I^{\mathbb{Z}} \cap X \text{ for some finite } I \subseteq \mathcal{A}\}$. Then for all $K \in \mathcal{K}$, K (which may be empty) is a subshift of X.*

Proof. It suffices to show that each element of $\mathcal{K}$ is a subshift over some finite alphabet. For $K \in \mathcal{K}$ with $K = I^{\mathbb{Z}} \cap X$ where $I \subseteq \mathcal{A}$ is finite, we see that K is a subshift over the alphabet I with forbidden word set $\mathcal{F} \cup I^c$. □

Proposition 6.3 *Let X be a locally finite countable state shift space with specification over $\mathcal{A} \cong \mathbb{N}_0$. Then $h_G(X) = \sup\{h(K) : K \in \mathcal{K}\}$.*

Proof. ($\geq$) For any $I \subseteq \mathcal{A}$, it is immediate that $\mathcal{L}_n(K) \subseteq \mathcal{L}_n^I(X)$ for big enough n, implying $h(K) \leq h_G(X)$ and thus the result.

($\leq$) For $k, L \in \mathbb{N}_0$, let $I_k = \{0, 1, \ldots, k-1\}$ and define

$$X_L^k = \{\omega \in X : \omega = \cdots w_{-1}v_{-1}w_0v_0w_1v_1\cdots : w_i \in \mathcal{L}_L^{I_k}(X), |v_i| = \tau(I_k) \text{ for all } i\}.$$

X_L^k is clearly σ-invariant and thus a shift space, and since X is locally finite, it is compact. Thanks to the specification property, $X_L^k \neq \emptyset$, so we have

$$\#\mathcal{L}_{mL+(m-1)\tau(I_k)}(X_L^k) \geq \#\mathcal{L}_L^{I_k}(X)^m$$

for any $m \in \mathbb{N}_0$. Dividing both sides by $mL + (m-1)\tau(I_k)$ and taking the log, we get

$$\begin{aligned}\frac{1}{mL + (m-1)\tau(I_k)} \log \#\mathcal{L}_{mL+(m-1)\tau(I_k)}(X_L^k) &\geq \frac{m}{mL + (m-1)\tau(I_k)} \log \#\mathcal{L}_L^{I_k}(X) \\ &> \frac{1}{L + \tau(I_k)} \log \#\mathcal{L}_L^{I_k}(X).\end{aligned}$$

Sending $m \to \infty$ and then $L \to \infty$, we see that $\liminf_{L\to\infty} h_G(X_L^k) > h_G(X)$; i.e., for all $\epsilon > 0$ and $k \in \mathbb{N}_0$, there exists an $L \in \mathbb{N}_0$ such that $h_G(X) - \epsilon < h_G(X_L^k)$. □

The following is our main result, towards which we will now start working:

Theorem 6.4 *Let X be a locally finite countable state shift space satisfying the specification property. Then the variational principle holds; i.e.,*

$$h_G(X) = \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(X)\}. \tag{4}$$

Proving one direction of Theorem 6.4 is easy:

Proposition 6.5 *Let X be a locally finite countable state shift space satisfying the specification property. Then,*

$$h_G(X) \leq \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(X)\}.$$

Proof. Fix a compact subshift $K \subseteq X$. By Theorem 6.1 and Proposition 6.3, we have

$$\begin{aligned}h_G(X) &= \sup_K h(K) \\ &= \sup_K \sup_{\mu\in\mathcal{M}_\sigma(K)} h_\mu(\sigma) \\ &\leq \sup_K \sup_{\mu\in\mathcal{M}_\sigma(X)} h_\mu(\sigma) \\ &= \sup_{\mu\in\mathcal{M}_\sigma(X)} h_\mu(\sigma),\end{aligned}$$

where the sup is taken over compact subshifts and we use that $\mathcal{M}_\sigma(K)$ injects naturally into $\mathcal{M}_\sigma(X)$. □

Definition 6.6 *For $x \in X$, $n \in \mathbb{N}_0$, and $\epsilon > 0$, we define the Bowen ball around x of radius ϵ and and order n by*

$$B_n(x, \epsilon) = \{y \in X : d(\sigma^j(y), \sigma^j(x)) < \epsilon \text{ for all } 0 \leq j \leq n\}.$$

If $E \subseteq X$, we define $B_n(E, \epsilon) = \bigcup_{x\in E} B_n(x, \epsilon)$.

To prove the other direction of Theorem 6.4, we use results from Chapter 3 of [9]. Part of their discussion holds in our setting, so that we have the following.

Theorem 6.7 *Let X be a locally finite countable state shift space over $\mathcal{A}$ and μ an ergodic σ-invariant probability measure on X. Then, for every $\delta \in (0,1)$, we have*

$$h_\mu(\sigma) \le \liminf_{n\to\infty} \frac{1}{n} \log \min_{E\subseteq X}\{\#E : \mu(B_n(E,1)) > 1-\delta\}.$$

Given $I \subseteq \mathcal{A}$ finite, let $[I] = \bigcup_{a\in I}[a]$. Given a $\mu \in \mathcal{M}_\sigma(X)$, there exists a finite $I \subseteq \mathcal{A}$ so that $\mu([I]) > 1 - \frac{\delta}{2}$. Then $\mu(\sigma^{-n}[I]) > 1 - \frac{\delta}{2}$ and we have

$$\mu([I] \cap \sigma^{-n}([I])) > 1-\delta$$

for all n. This gives the following.

Proposition 6.8 *Let X be a locally finite countable state shift space which satisfies the specification property. Then,*

$$h_G(X) \ge \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(X)\}.$$

Proof. Fix $0 < \delta < 1$ and $\mu \in \mathcal{M}_\sigma(X)$ and pick a finite $I \subseteq \mathcal{A}$ as above. We calculate,

$$\begin{aligned}
h_\mu(\sigma) &\le \liminf_{n\to\infty} \frac{1}{n} \log \min_{E\subseteq X}\{\#E : \mu(B_n(E,1)) > 1-\delta\} \\
&\le \liminf_{n\to\infty} \frac{1}{n} \log \min_{E\subseteq [I]\cap\sigma^{-n}([I])}\{\#E : \mu(B_n(E,1)) > 1-\delta\} \\
&= \liminf_{n\to\infty} \frac{1}{n} \log \min_{E'\subseteq \mathcal{L}_n^I(X)} \left\{\#E' : \sum_{\omega\in E'} \mu([\omega]) > 1-\delta\right\} \\
&\le \lim_{n\to\infty} \frac{1}{n} \log \min_{E'\subseteq L_n^I(X)} \left\{\#E' : \sum_{\omega\in E'} \mu([\omega]) > 1-\delta\right\} \\
&\le h_G^I(X) \\
&= h_G(X).
\end{aligned}$$

□

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