Hex Mosaic Diagrams are More Tile Efficient Than Square Mosaic Diagrams

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Abstract - There are multiple types of tiles that can be used to represent projections of knots onto small grids, namely square tiles and hexagonal tiles. In this paper we show that given any knot on a square grid, we can project it onto a hexagon grid using fewer tiles. Efficiency can sometimes be improved using simple shifts of crossings.

Keywords : knot theory; mosaics; hexagonal mosaics

Mathematics Subject Classification (2020) **:** 57M25

1 Introduction

The study of knots is inherently a topological field, but there are many ways to represent knots in combinatorial ways that can help break these complex objects into smaller pieces. In 2008, Kauffman and Lomonaco introduced mosaic tile diagrams as a tool to study quantum knots [13]. They conjectured that the study of mosaic knots was equivalent to the study of classical knots, which was later proven by Kuriya and Shehab [9]. In the meantime, others began work on studying mosaic knots for their own sake. An interesting question that arises is the efficiency of these diagrams. One measurement of efficiency is the *mosaic number* of a knot (the smallest square grid size of a mosaic diagram for the knot) [4], [6], [10], [11]. On the other hand, one can examine the *tile number* of a knot (the smallest number of tiles needed for a mosaic diagram of the knot) [5].

More recently, Jennifer McLoud-Mann introduced hexagonal tiled knots [2], [3]. Work has been done by Howards, Li, and Liu on examining hex mosaics and answering similar questions about the size of diagrams to those considered for square tiles [7], [8]. They primarily considered the hexagonal mosaic number, a radial measure of grid size of a hexagonal tiling. One can also consider the hexagonal tile number, defined similarly to the square tile number [2].

In this paper, we consider the question of efficiency for these two mosaic types in relation to each other. Since it is more directly comparable, we use the tile number as a measurement of efficiency. Our main result, Theorem 5.2, shows that for any knot, the hexagonal tile number is strictly less than the square tile number.

Section 2 gives a more detailed background and necessary definitions. In Section 3, we give an algorithm for converting a square tiling into a hexagonal tiling. This conversion lays the foundation for our main theorem. Section 4 gives a detailed example of this conversion process. At the end of this process we have a hexagonal tiling, but may have used some extra blank square tiles to finish the tiling. In Section 5 we address these *remainder tiles* and show that while there must be remainder tiles, there cannot be as many remainder tiles as filled tiles in a diagram. This allows us to prove Theorem 5.2, verifying that for any knot, a hexagonal tiling can be made with fewer tiles than the most tile efficient square tiling of the knot. Interestingly, in many cases hexagons are approximately twice as efficient as square tiles. We conclude the paper in Section 6 with noting this and other questions for future research.

2 Background

A *knot* is an embedded closed curve in three dimensions. A *link* is a collection of potentially intertwined knots. Links may have multiple components, while knots only have one continuous component. A *diagram* of a knot or link is a two dimensional projection of the knot with over and under crossings marked by breaks in the strand. *Reidemeister moves* may be used to manipulate a diagram, without changing the knot. Any two diagrams of the same knot can be related by a sequence of Reidemeister moves. This and other basics of knot theory can be found in detail in Adams [1] or Livingston [12].

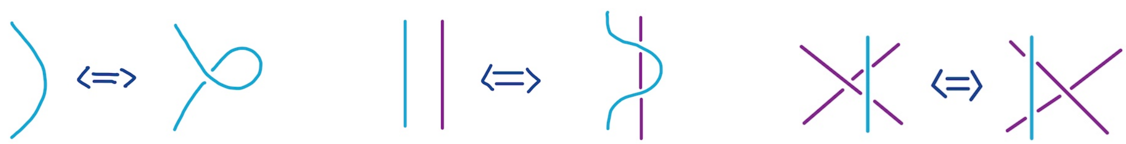

Figure 1: Reidemeister moves

We can combine the tiles shown in Figure 2 to form a mosaic for a specific knot. There are 10 possible tiles and the blank tile, and any knot can be made by arranging these tiles [9]. Note that with square mosaics, there can be a maximum of one crossing per tile. The number of non-empty square mosaic tiles used to create a diagram of a knot is referred to as the diagram's *tile number*, denoted $t_s(D)$.

Definition 2.1 *The* ***(square) tile number*** *for a specific knot,* $t_s(k)$ *is the minimum amount of non-empty tiles necessary to create that knot.*

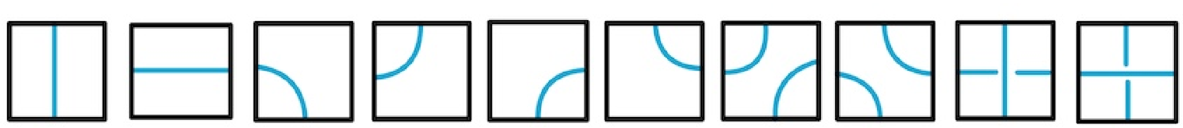

Figure 2: Tiles that can be used in a square mosaic, excluding the empty tile.

Hexagon tiles, shown in Figure 3, provide an alternative method for representing knots in a plane, offering more tile options and accommodating a greater number of crossings on a single tile. Up to rotation, there are 25 possible hexagon tiles and the blank tile. A single hexagon tile has the capacity to contain up to three crossings.

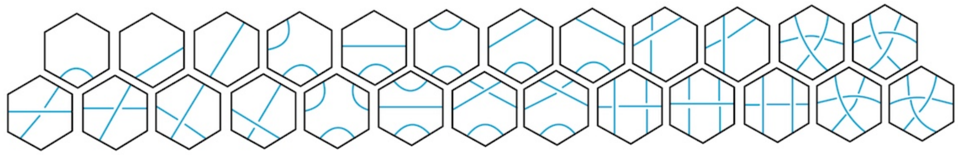

Figure 3: Tiles that can be used in a hexagon mosaic, excluding the empty tile. We can also use any rotations of these tiles.

Similarly to the square mosaic tile number, the hexagon tile number of a diagram is the number of non-empty tiles used in that diagram, $t_h(D)$.

Definition 2.2 *Given a knot k, the **hexagon tile number** is the minimum number of tiles over all possible hexagon tilings of k. We will denote this with $t_h(k)$.*

It should be noted that in both square and hexagon tile numbers, blank tiles are not included in the count.

The endpoints of arcs on our tiles are referred to as *connection points*.

Definition 2.3 *A mosaic diagram is **suitably connected** if each connection point is touching a connection point on the tile adjacent to it.*

If there are connection points that don't touch other connection points, the diagram is *unsuitably connected*.

3 Converting Squares into Hexagons

Our main goal in this paper is to show that hexagonal tilings are more tile efficient than square tilings. As a first step to seeing this, we show an algorithm for converting a square mosaic tiling to a hexagonal mosaic tiling.

Lemma 3.1 *Let k be a knot. Given a square mosaic tiling of k, D_k, we can combine neighboring pairs of square mosaic tiles and stretch them to form hexagonal tiles, making a hexagonal mosaic tiling of k, D'_k.*

Proof. We can convert a knot mosaic diagram, D_k, from squares to hexagons by grouping our square tiles into pairs. Each pair is referred to as a brick. When we combine pairs to form a brick, we are left with six edges of the two squares on the border of our bricks. This is why we can easily convert and stretch our bricks into hexagons. As we convert the bricks to hexagons, any arcs and crossings on the tiles are joined and warped as the

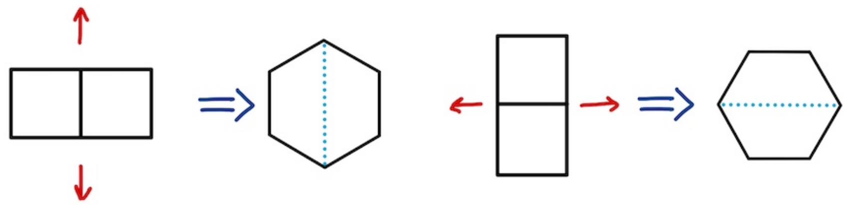

Figure 4: There are two ways that we can create hexagons from square tiles.

hexagons form, with the connection points along the outside remaining the same. This produces a suitably connected hexagonal tiling (as in Definition 2.3).

By dividing our square grids into pairs, we are setting up our diagram to fit two square tiles onto a single hexagon tile once we convert it. When our bricks are properly aligned, we are able to stretch them into each other, producing a new grid of hexagons from the original squares. Using the brick pattern to convert a square diagram to a hexagon diagram gives us a reliable way to produce a knot mosaic, D'_k, on hexagon tiles when using a given a diagram, D_k, of square tiles.

□

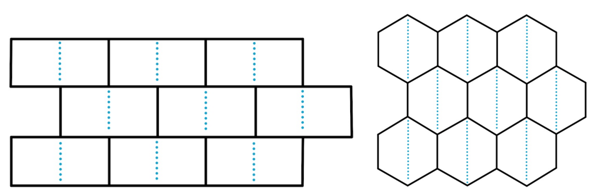

Figure 5: A square grid is divided into bricks and then converted to hexagons.

As shown in Figure 4, we can stretch to form vertical or horizontal bricks, but all of the bricks must have the same orientation in order for all of the hexagons to fit together properly. Similarly, to have the hexagons form a plane tiling, we need the bricks to alternate in a brick wall pattern to allow the bricks to stretch together in the right way. A proper brick laying is shown in Figure 5. If we have two bricks directly on top of each other, they will no longer be connected when we stretch them into hexagons. Instead, they will only be touching at the corner where the center of the two bricks were connected, resulting in unsuitably connected tiles. We are left with a similar problem when we try to use both horizontal and vertical bricks in the same diagram. The results of these incorrect brick patterns are shown in Figure 6. While all of the diagrams that we show with the bricks are using their horizontal orientation, either orientation can be used as long as it remains consistent throughout the diagram.

An interesting result of this transformation of square tiles is that it consistently produces a hexagonal mosaic diagram that uses fewer tiles than the original square diagram. This is because we are fitting the contents of two square tiles onto one hexagon tile.

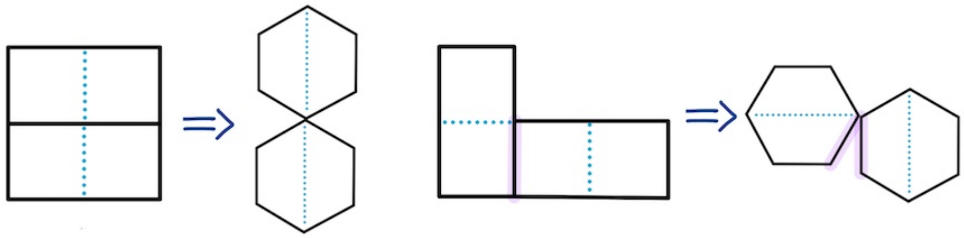

Figure 6: Results of incorrect brick patterns

Corollary 3.2 *For any knot, k, the hexagonal tile number is less than or equal to the square tile number:*

$$t_h(k) \leq t_s(k).$$

Proof. Following the process laid out in Lemma 3.1, when we convert our square mosaic diagram to a hexagon mosaic diagram, each brick that we create will have at least one occupied square tile within it. This means that we will never convert a square diagram to a hexagon diagram and end up with more hexagon tiles than we had square tiles. If we start with a tile minimal square mosaic diagram we see that the tile minimal hexagonal diagram must use fewer tiles. □

We will continue to explore this in Section 5, in particular considering the number of blank tiles that must be incorporated in the brick design to implement the transformation. We will refine this result to show that the inequality is strict in Theorem 5.2.

4 Example of a Transformation

As an example of the transformation from square tiles to hexagon tiles, we will look at the trefoil. The first step in this process is to draw our knot in its square grid form and divide it into bricks of two squares each.

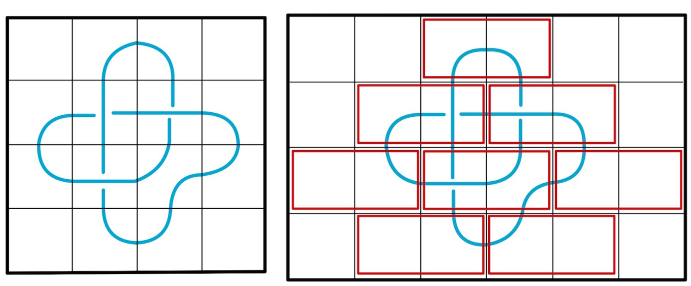

Figure 7: The trefoil drawn on a square grid and the trefoil separated into bricks.

The next step is to follow the divisions of our square diagram to copy our knot onto a hexagonal grid, noting that, if you use the brick orientation in Figure 7, vertical pieces of the knot will remain vertical, but horizontal pieces of our tiles and curves will often end up being angled.

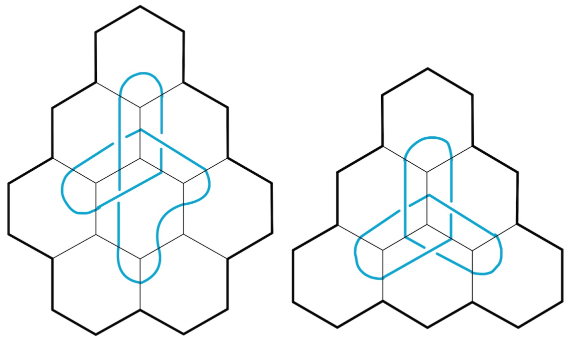

Figure 8: The trefoil on a hexagon grid directly after transforming and after shifting it to further reduce it.

At this point, we have successfully converted our diagram from squares to hexagons. The diagram can sometimes be made smaller by shifting our crossings or corners. In our trefoil example, we can see that the bottom of the diagram can be shifted to remove two tiles without changing the knot (see Figure 8). In Section 6 we'll further explore these post-conversion efficiencies.

5 Remainder Tiles

A result of the conversion process is the *remainder tile*. As described in Section 3, bricks consist of a pair of square tiles and, to ensure that the bricking produces a planar tiling of hexagons, the pattern of bricking is determined as soon as a first brick is chosen. Consequently, one might sometimes pair an occupied tile with an empty tile to complete a brick (in fact, we will show in Section 6 that all brickings require at least one such tile). Remainder tiles are any empty square tiles included in a brick. The total number of remainder tiles in a knot diagram is denoted R.

Lemma 5.1 *Given a square mosaic diagram, D_k, the number of hexagonal tiles resulting from the conversion to a hexagon mosaic diagram, D'_k, is given by the following equation:*

$$t_h(D'_k) = \frac{t_s(D_k) + R}{2}.$$

Proof. In our transformation process, each hexagon tile corresponds to a pair of square tiles known as a brick. The number of bricks is equal to the number of hexagons, $t_h(D'_k)$, in the resulting tiling. Note that there are two possible brick types. A brick can be made up of two occupied square tiles or it can have once occupied tile and one remainder tile, giving us the following relationship:

$$2t_h(D'_k) = t_s(D_k) + R$$
$$t_h(D'_k) = \frac{t_s(D_k) + R}{2}.$$

□

Now we can prove the main theorem. In Corollary 3.2 we established that $t_h(k) \leq t_s(k)$. In this theorem we refine this to a strict inequality.

Theorem 5.2 *For any knot k, $t_h(k) < t_s(k)$.*

Proof. Let D_k be a tile-minimal square mosaic for k. Using the equation from Lemma 5.1, we can see that if $R < t_s(D_k)$, then $t_h(D'_k) = \frac{t_s(D_k)+R}{2} < t_s(D_k)$ will also be true. Each brick must contain at least one occupied tile, as in Corollary 3.2, so it is immediately evident that $R \leq t_s(D_k)$.

Now, suppose that $R = t_s(D_k)$. Since a brick cannot contain two empty tiles, this implies that, when the diagram is divided into bricks, each brick must contain an empty tile. Therefore, no more than two adjacent tiles in a row can be occupied. To show that this is not possible, we will start with the leftmost occupied tile in the top row of the diagram. As shown in Figure 9, the tile to the right of this tile must also be occupied. If we choose this tile to connect further through the same row, we are left with three occupied tiles in a row. This leads to at least one brick in the row containing only occupied tiles, giving us a contradiction of our assumption that every brick contains an empty tile.

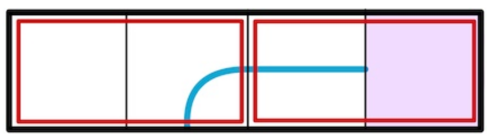

Figure 9: When the top row of the diagram has three consecutive occupied tiles $R < t_s(D_k)$.

On the other hand we could have only the first two tiles occupied, and a blank tile in the third position of the top row. From here, we know that the two tiles directly below those in the first row must be occupied, as shown in Figure 10. As we choose a bricking, in order to follow the rules of our transformation process from Section 3, we must stagger the rows of bricks. In either case, one of these two rows contains a brick with two occupied tiles. This results in a contradiction of our initial assumption.

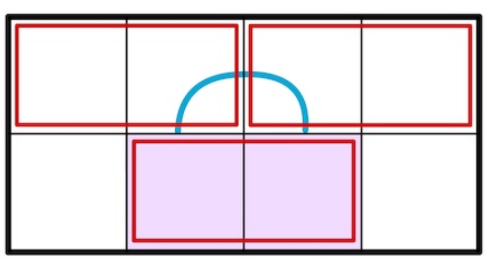

Figure 10: When the first row starts with only two consecutive occupied tiles, we still find a brick with two occupied tiles in the first two rows.

Therefore, it is not possible to have the same number of occupied tiles as remainder tiles and $R < t_s(D_k)$, so $t_h(D'_k) < t_s(D_k)$. By choosing a minimal square diagram to begin, we can verify that $t_h(k) \leq t_h(D'_k) < t_s(D_k) = t_s(k)$. In particular, $t_h(k) < t_s(k)$. □

In Lemma 5.1 we noted that $t_h(D'_k) = \frac{t_s(D_k)+R}{2}$. This suggests that the best efficiency would be obtained in cases where there are no remainder tiles. However, any diagram must have at least one remainder tile, so it is always true that $t_h(D'_k) > \frac{t_s(D_k)}{2}$.

Lemma 5.3 *Every suitably connected mosaic diagram has at least one remainder tile, when separated into bricks. In particular,*

$$t_h(D'_k) > \frac{t_s(D_k)}{2}.$$

Proof. In order to show that there must be at least one remainder tile, we will try to construct a knot diagram that has no remainder tiles. To do this, we will start with the leftmost occupied tile of the top row of our diagram and choose the first brick so that this tile occupies the left half of the brick. After making this choice, we now know where the rest of our bricks will be on our diagram. To maintain a diagram that has no remainder tiles, we will again make sure that the left half of the brick below the left side of the first one is occupied. This leads to a cycle of stair stepping down to the left of our diagram to continue avoiding remainder tiles, as in Figure 11.

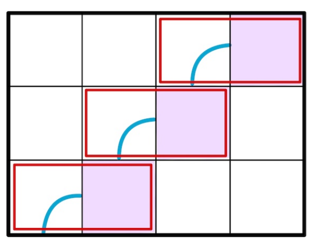

Figure 11: The stair-stepping pattern avoids remainder tiles.

Figure 12: There are two possible options for the end of the stair-stepping shape.

When we create a knot diagram, we can't have an infinite number of rows, so we know that the diagram must end at some point. Now, in order for the diagram to be suitably connected, it can't continue to move left using the upper left corner arc. Instead, the diagram must move down or to the right, as in Figure 12. In either case, the left half of the brick in this row must be an empty tile to maintain a suitably connected mosaic. Therefore, our diagram must have at least one remainder tile.

□

This result does not imply that $t_h(k) > \frac{t_s(k)}{2}$ in all cases, since $t_h(k)$ is the minimal tile number over all hexagonal tilings, but D'_k is only a particular diagram obtained from converting a square tiling. We will further discuss this and other questions related to the efficiency diagrams in Section 6.

6 Efficiency: Future Work and Questions

In Theorem 5.2 we observe that hexagon tiling is more tile efficient than square tiling, but there is a range to how much more efficient it can be. First, we observe that the efficiency depends on several choices made in the conversion process.

When deciding where to begin the brick pattern on a square grid, there are two different options. The first brick can either contain one empty square tile and one occupied square tile, or the first brick can contain two occupied square tiles. In the majority of the small cases that we have looked at, the choice of which way to start our brick pattern doesn't change the tile number after converting it to hexagons. However, in some cases, like the 9_{24} knot, the hexagon tile number may be slightly different depending on our starting choice, as shown in Figure 13.

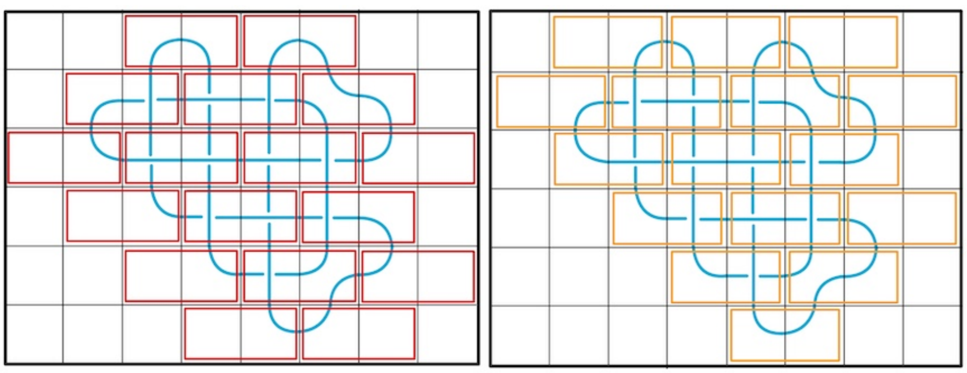

Figure 13: 9_{24} is shown with the two ways to begin the brick pattern. The left configuration uses 17 bricks, whereas the right configuration uses only 16.

This choice seems to matter in cases where the mosaic number and the square tile number of the diagram have different parities. This leads to questions about how mosaic number can tell us something about the efficiency of our diagram.

Question 6.1 How is efficiency related to the mosaic number of a diagram? What is the relationship between the tile number and mosaic number of a diagram when converting to hexagons?

With Theorem 5.2 and Lemma 5.3, we now know the number of remainder tiles is between 1 and $t_s(D_k)$. In most examples we find that R is approximately the height of the diagram (in a minimal diagram, R is approximately the mosaic number of the knot). Further work might refine these estimates. On the one hand, as mentioned in question 6.1, to consider a possible relationship between the mosaic number and the number of remainder tiles. On the other hand, to examine the bounds of this inequality for possible refinement.

Question 6.2 We know $1 \leq R < t_s(D_k)$. Can all values of this range be achieved? Can either bound be refined?

To add complication to our estimation of efficiency, we observe that not all remainder tiles necessarily occur on the periphery of the diagram. We may have gaps in the middle of our diagram that necessitate *holes* (blank tiles surrounded on all sides by occupied tiles). These holes add to the number of remainder tiles and decrease the efficiency of the transformation.

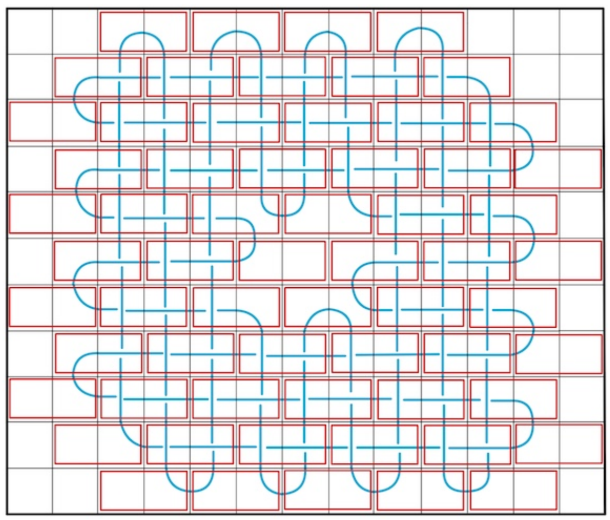

Figure 14: A knot diagram with a hole in the middle

Question 6.3 How can we estimate the number of holes in a diagram?

When we have finished our initial transformation from squares to hexagons, we can sometimes find ways to squish our diagram into fewer tiles. The easiest way that we can squish our diagram is to move long stretches of arcs off tiles that don't contain crossings so that they use fewer tiles. For example, the bottom crossing in our initial trefoil transformation can be rotated slightly so that both pieces of the crossing are angled and the string without crossings can now be connected in a much smaller space (Figure 15, left).

We can also shift the crossings of our knot slightly in order to fit more crossings onto one tile. This more complicated shift that we can do is moving three crossings onto a single tile rather than two. In order for this to work, we need to have three crossings that form a triangle pattern, as shown with the highlighted pieces of the knot in Figure 15, right.

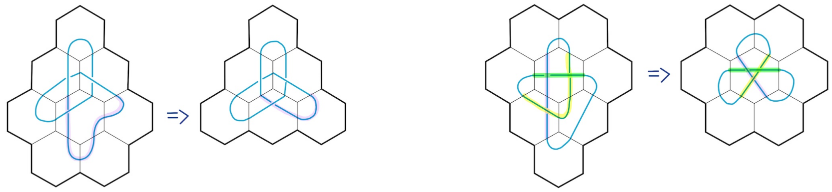

Figure 15: Left: Arcs can be contracted to reduce the number of occupied tiles. Right: Crossings can be squished onto a single tile.

While we have already seen that hexagon mosaic diagrams can be made even more efficient after the initial transformation, we don't have a consistent method or understanding of how to easily make a diagram more tile efficient. We are left with some questions that could further clarify and improve the algorithm to create efficient hexagon mosaic diagrams.

Question 6.4 Is there a method to easily identify and make improvements to a given hexagon mosaic diagram using R-moves?

Question 6.5 Can every hexagon mosaic diagram be made more efficient after the initial transformation?

Question 6.6 How does using three-crossing tiles impact the tile number of our diagram compared to two crossing tiles? Is it always better to fit more crossings on a single tile?

Question 6.7 How much more tile efficient can a hexagon mosaic diagram be made to be? Twice as efficient? Three times?

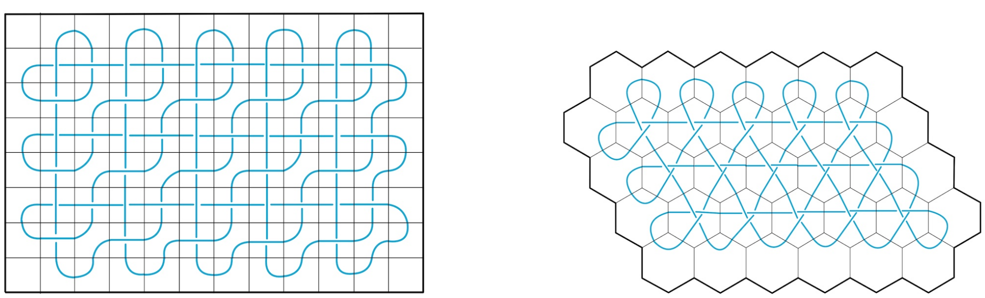

Figure 16: This knot is almost three times as efficient on hexagon tiles as on square tiles.

If we start with an efficient hexagon mosaic diagram and work backwards to create a square mosaic diagram of the same knot, we can find cases where the hexagon mosaic diagram approaches three times the efficiency of a square mosaic diagram. We can see an example of this in Figure 16, where the square tile number is 92 and the hexagon tile number is 33.

Acknowledgments

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