Conditions for Building Generalized Action Graphs From Sequences

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Abstract - This paper explores the properties of directed graphs, termed *generalized action graphs*, which exhibit a strong connection to certain number sequences. Focusing on the structural and combinatorial aspects, we investigate a condition under which specific sequences can generate generalized action graphs. Building upon prior research in this field, we analyze specific features of these graphs and how they correspond to patterns and properties in their sequences. These findings support a broader conclusion that establishes framework for identifying which sequences can produce generalized action graphs.

Keywords : Catalan number; directed graph; generalized action graph

Mathematics Subject Classification (2020) **:** 05A19; 05C05

1 Introduction

Action graphs are a class of directed graphs that were first constructed in work studying rooted category actions by Bergner-Hackney. Alvarez et al. later showed that action graphs can be defined inductively and are related to the Catalan numbers, a well known sequence of natural numbers that have many ties to combinatorics [1]. Caldwell et al. also observed that each action graph has subgraphs that are isomorphic to previous graphs. These graphs were later generalized by Cressman-Lin-Nguyen-Wiljanen [5] and Caldwell et al. [3], showing a different set of directed graphs are similarly related to the Fuss-Catalan numbers and the super Catalan numbers, respectively.

Caldwell et al. gave the following definition of generalized action graphs.

Definition 1.1 **[3]** *The sequence G_n of generalized action graphs for a particular sequence s_n of positive integers is a sequence of directed, labeled, rooted trees such that:*

Axiom 1 *Define G_0 as the graph with s_0 vertices labeled 0 and no edges. Construct G_n from G_{n-1} by adding s_n new vertices, which are each labeled n.*

Axiom 2 *For vertex v in G_n, the subtree of G_n with root v is isomorphic to some G_k such that $k \leq n$.*

Axiom 3 *All leaves in the graph G_n have label n.*

While previous research has established properties of generalized action graphs, a general framework for determining which sequences generate such graphs remained elusive. In this paper, we establish criteria for identifying sequences that yield generalized action graphs. This result unifies and extends previous findings in this area [1, 3, 5]. Formally, we will prove the following theorem:

Theorem 1.1 *If two sequences of positive integers $\{s_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ satisfy (i) $s_0 = 1$ and (ii) for all $n \geq 1$,*

$$s_n = \sum_{i=1}^{n} z_i s_{n-i},$$

then there are generalized action graphs G_n for all $n \geq 0$ such that z_i is the number of vertices labeled i adjacent to the root, and s_i is the total number of vertices labeled i for all $0 \leq i \leq n$. Moreover, for a vertex v labeled i adjacent to the root in G_n, the subtree with root v is isomorphic G_{n-i} with vertex labels shifted up by i.

This theorem provides a method for determining which sequences will yield generalized action graphs. Furthermore, the proof of the theorem describes a method for constructing the graphs.

In Section 2, we summarize previous work constructing graphs relating to the Catalan, Fuss-Catalan, and super Catalan number sequences and compare those sequences with our theorem. Then, in Section 3, we consider the properties that all generalized action graphs satisfy. Section 3.2 provides the proof of Theorem 1.1.

2 Motivation for the Main Theorem

Generalized action graphs have been constructed to model the Catalan numbers, the Fuss-Catalan numbers, and the super Catalan numbers. The process for generating such directed graphs in each of these settings was unique to the sequence that it represents. We will summarize the constructions for each of these sequences and compare them to Theorem 1.1. Note that Definition 1.1 was written in [3] as a generalization of the existing examples for Catalan numbers and Fuss-Catalan numbers.

Alvarez, Bergner, and Lopez were the first to relate action graphs to the Catalan numbers [1].

2.1 Catalan Numbers

Definition 2.1 [8] *The sequence of* Catalan numbers*, denoted by $\{C_n\}_{n=0}^{\infty}$, is a sequence of natural numbers given by*

$$C_0 = 1, C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} = \binom{2n}{n} \frac{1}{n+1}.$$

Observe that the first few numbers of this sequence are:

$$[1, 1, 2, 5, 14 \ldots].$$

In [1], the authors showed that the Catalan numbers count the number of leaves added to each action graph.

Definition 2.2 **[1]** *Define the action graph A_{n+1} inductively, starting with one vertex labeled 0 for A_0. Given A_n, for each leaf in A_n construct an edge to a new vertex labeled $n+1$ and for each non-trivial path to vertex labeled n in A_n construct an edge from the source of the path to a new vertex labeled $n+1$.*

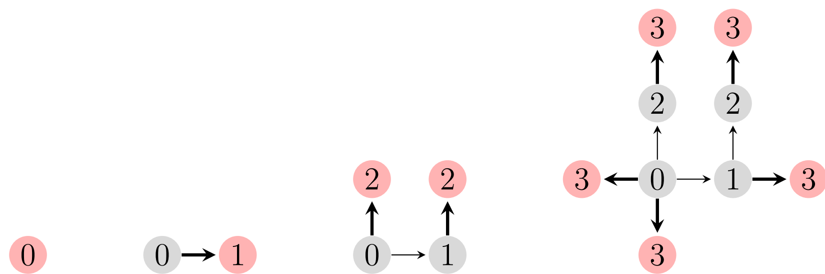


Figure 1: Action graphs A_0 through A_3.

The vertices are labeled starting at 0, and every vertex added to each graph will be labeled with the same number, corresponding to the step in the inductive definition. For example, new vertices in action graph A_2 are labeled 2, and there will be a total of C_2 vertices added to construct that graph.

Since the Catalan numbers have been shown to yield action graphs, and in fact are encoded in the original example of action graphs, we will consider how this result compares to Theorem 1.1 in the following example. More specifically, we examine A_3, given in the figure below.

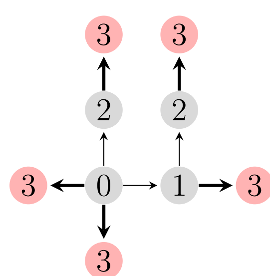


Figure 2: Action graph A_3.

Example 2.3 From Definition 2.1, $C_n = [1, 1, 2, 5, 14 \ldots]$. Taking a look at Figure 2, we can determine the z_n values of Theorem 1.1 by counting the number of vertices of each label adjacent to the root vertex. For action graph A_3, $z_n = [1, 1, 2 \ldots]$. We see that these values satisfy the formula of Theorem 1.1:

$$\begin{aligned}C_3 &= z_1 \cdot C_2 + z_2 \cdot C_1 + z_3 \cdot C_0 \\ &= 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 \\ &= 5.\end{aligned}$$

Since $C_3 = 5$, Theorem 1.1 asserts that Catalan numbers can be modeled using action graphs, at least up to $n = 3$, which is consistent with the work of [1]. By construction, we can determine the z_n values for later action graphs, $z_n = [1, 1, 2, 5 \ldots]$.

We see that these values satisfy the formula of Theorem 1.1 in the case where $n = 4$:

$$\begin{aligned}C_4 &= z_1 \cdot C_3 + z_2 \cdot C_2 + z_3 \cdot C_1 + z_4 \cdot C_0 \\ &= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 \\ &= 14.\end{aligned}$$

Notice that so far the values of the z_n have been Catalan numbers. We will prove that this pattern continues for all n.

Theorem 2.4 *Sequences $s_n = C_n$ and $z_n = C_{n-1}$ satisfy the hypotheses of Theorem 1.1.*

Proof. Let $s_n = C_n$ and $z_n = C_{n-1}$. Note that these are both positive integer sequences.

By Definition 2.1 we know that

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}.$$

Observe that by unpacking this sum, we have

$$\sum_{i=0}^{n-1} C_i C_{n-1-i} = C_0 C_{n-1} + C_1 C_{n-2} \cdots + C_{n-2} C_1 + C_{n-1} C_0.$$

Since $s_n = C_n$ and $z_n = C_{n-1}$, this can be written as:

$$\begin{aligned}\sum_{i=0}^{n-1} C_i C_{n-1-i} &= z_1 s_{n-1} + z_2 s_{n-2} + \cdots + z_{n-1} s_1 + z_n s_0 \\ &= \sum_{i=1}^{n} z_i s_{n-i}.\end{aligned}$$

Therefore, Theorem 1.1 applies to the sequence of Catalan numbers for $z_n = C_{n-1}$ and corresponding generalized action graphs exist. □

2.2 Path Length

Two important aspects of constructing generalized action graphs are *paths* and *path length*. For our purposes, throughout this paper *paths* will refer to directed paths that start at any vertex and end at a leaf. Note that the longest paths in the graph A_{n+1} will start at the vertex labeled 0 and end at a vertex labeled $n+1$. We will denote path length by ℓ, defined as the number of edges traveled from the initial vertex to the final vertex.

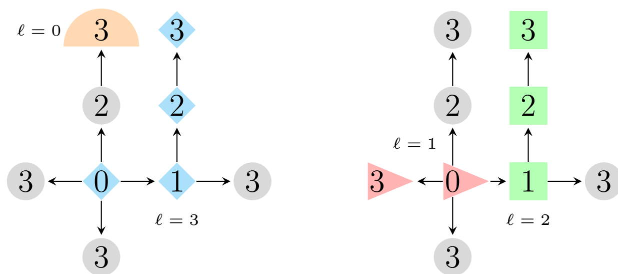


Figure 3: Different paths of length ℓ on action graph A_3.

Figure 3 shows two versions of action graph A_3, colored in different ways to highlight different paths. As seen above, there are four colored paths on graph A_3, of varying path length, ℓ. The graph on the left shows a path of length 3, starting from the 0 vertex, highlighted by blue diamonds. A trivial path length of 0, starting and ending at a vertex labeled 3 is highlighted by an orange semicircle. The graph on the right shows a path length of 1 in red triangles, starting at the vertex labeled 0. There is also a path length of 2, of green rectangles, starting from the 1 vertex. Path length does not impact the construction of the action graphs related to the Catalan numbers; however, it plays a large role in constructing the generalized action graphs related to other number sequences.

2.3 Fuss-Catalan Numbers

The *Fuss-Catalan* numbers are related to the Catalan numbers, and they can also be modeled using a generalization of the action graph construction.

Definition 2.5 [8] *The* Fuss-Catalan *numbers are a generalization of the Catalan numbers with two arguments, n and k. There are both recursive and explicit formulas for the Fuss-Catalan numbers:*

$$C_{n,k} = \sum_{n_1+n_2+\ldots+n_{k+1}=n-1} \prod_{i=1}^{k+1} C_{n_i,k} = \frac{\binom{n(k+1)}{n}}{kn+1}.$$

Unlike the Catalan numbers, the Fuss-Catalan numbers are dependent upon two variables. When $k=2$, the values of the Fuss-Catalan numbers are:

$$C_{n,2} = [1, 1, 3, 12, 55 \ldots].$$

When $k = 1$:

$$C_{n,1} = C_n.$$

The authors in [5] expanded on the work done previously by Alvarez, Bergner, and Lopez by developing new action graphs for the Fuss-Catalan numbers, which they called *generalized action graphs.*

Definition 2.6 [5] *The* generalized action graph $T_{n+1,k}$ *is generated inductively from* $T_{n,k}$ *by adding* $\binom{\ell+k-1}{\ell}$ *new leaves labeled* $n+1$ *adjacent to each vertex* v *for each path of length* ℓ *from* v *to a vertex labeled* n *in* $T_{n,k}$*. The first graph,* A_0 *consists of one vertex labeled 0. The total number of new vertices in* $T_{n,k}$ *is equal to the Fuss-Catalan number* $C_{n,k}$*.*

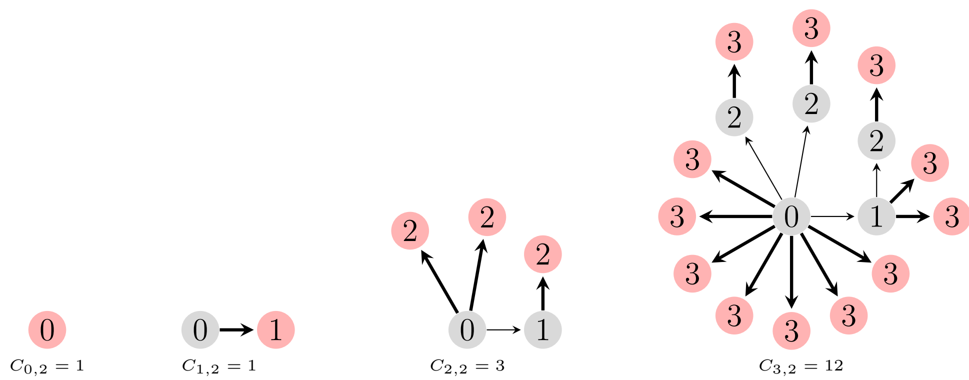


Figure 4: Generalized action graphs $T_{0,2}$ through $T_{3,2}$ for the Fuss-Catalan numbers.

Unlike the action graphs that are related to the Catalan numbers, the number of new vertices added to each vertex for each path from that vertex to a leaf for the *generalized action graphs* is $\binom{\ell+k-1}{\ell}$, which is dependent on path length. Note that by definition

$$\binom{\ell+k-1}{\ell} = \frac{(\ell+k-1)!}{\ell!(k-1)!}.$$

When $k = 2$ and $\ell = 1$, this expression simplifies to

$$\frac{(\ell+k-1)!}{\ell!(k-1)!} = \frac{(1+2-1)!}{1!(2-1)!} = \frac{2!}{1!(1!)} = 2.$$

So, when $k = 2$, the number of vertices added to the start of each path when $\ell = 0, 1, 2, 3\ldots$ is $[1, 2, 3, 4\ldots]$.

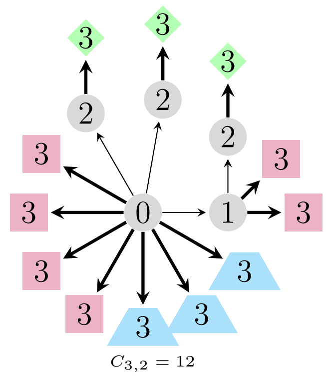


Figure 5: Generalized action graph $T_{3,2}$ colored based on path length.

Figure 5 illustrates how each vertex is added to the start of each path, based on path length. The green diamond vertices are added based on path lengths of 0 in $T_{2,2}$, where one vertex is added per path. The purple rectangle vertices are added for path lengths of $\ell = 1$ in $T_{2,2}$. These vertices are added to the initial vertex of the path, with two vertices being added to the vertex labeled 1, and four vertices added to the 0 vertex, for the two paths of $\ell = 1$ ending at the vertices labeled 2. The blue trapezoid vertices represent the vertices added when $\ell = 2$. Since there is only one path in this graph of length two, three vertices are added to the 0 vertex.

Generalized action graphs were constructed for the Fuss-Catalan numbers in [5]. The following example shows that the Fuss-Catalan numbers also satisfy the hypotheses of Theorem 1.1, demonstrated using the generalized action graph for $C_{3,2}$.

Example 2.7 Consider the case where $k = 2$. Observe from Figure 5, $z_n = [1, 2, 7 \ldots]$, and from Definition 2.5, our sequence values are $s_n = C_{n,2} = [1, 1, 3, 12 \ldots]$. Then to satisfy the hypothesis of Theorem 1.1, we must check that

$$C_{n,2} = \sum_{i=1}^{n} z_i \cdot C_{(n-i),2}.$$

Consider $n = 3$:

$$\begin{aligned} C_{3,2} &= z_1 \cdot C_{2,2} + z_2 \cdot C_{1,2} + z_3 \cdot C_{0,2} \\ &= 1 \cdot 3 + 2 \cdot 1 + 7 \cdot 1 \\ &= 12. \end{aligned}$$

Since $C_{3,2} = 12$, Theorem 1.1 asserts that the Fuss-Catalan numbers yield generalized action graphs for $k = 2$ and $n \leq 3$.

Next, we wish to prove that Theorem 1.1 applies to the sequence of Fuss-Catalan numbers, $C_{n,k}$, for any fixed k. In order to do that, we will introduce notation for a generalization of the Fuss-Catalan sequence and some results about that sequence.

Definition 2.8 ([9]) *The* two-parameter Fuss-Catalan numbers *are given by*

$$A_m(p,r) = \frac{r}{m!}\prod_{i=1}^{m-1}(mp+r-i) = \frac{r}{mp+r}\binom{mp+r}{m}.$$

From the following algebra, we can see that the Fuss-Catalan numbers of Definition 2.5 are a special case of the two-parameter Fuss-Catalan numbers where $m = n$, $p = k+1$, and $r = 1$:

$$\begin{aligned} C_{n,k} &= \binom{n(k+1)}{n}\frac{1}{nk+1} \\ &= \frac{(n(k+1))(n(k+1)-1)\cdots(nk+2)(nk+1)}{n!}\cdot\frac{1}{nk+1} \\ &= \frac{n(k+1)+1}{n(k+1)+1}\cdot\frac{(n(k+1))(n(k+1)-1)\cdots(nk+2)}{n!} \\ &= \binom{n(k+1)+1}{n}\frac{1}{n(k+1)+1} \\ &= A_n(k+1,1) \end{aligned}$$

There are two previously known results about the two-parameter Fuss-Catalan numbers that will help us.

Lemma 2.9 [9]

$$\sum_{k=0}^{m} A_k(p,r)A_{m-k}(p,s) = A_m(p,r+s).$$

Lemma 2.10 [9]

$$A_m(p,p) = A_{m+1}(p,1).$$

We now have the tools needed to prove our theorem.

Theorem 2.11 *Let k be a fixed positive integer. If $s_n = C_{n,k}$ and*

$$z_n = \frac{k}{(k+1)(n-1)+k}\binom{(k+1)(n-1)+k}{n-1},$$

then

$$s_n = \sum_{i=1}^{n} z_i s_{n-i}$$

for all $n \geq 1$. Therefore, the hypotheses of Theorem 1.1 are satisfied.

Proof. In order to make the indices easier to work with, note that, by replacing n with $n+1$, proving the desired equation is equivalent to proving

$$s_{n+1} = \sum_{i=1}^{n+1} z_i s_{n+1-i}$$

for all $n \geq 0$. Furthermore, we could re-index the sum

$$\sum_{i=1}^{n+1} z_i s_{n+1-i} = \sum_{i=0}^{n} z_{i+1} s_{n-i}.$$

We observe that $z_{i+1} = A_i(k+1,k)$ and $s_{n-i} = A_{n-i}(k+1,1)$. Substituting into our sum and applying Lemma 2.9 and 2.10 we have:

$$\begin{aligned}\sum_{i=0}^{n} z_{i+1} s_{n-i} &= \sum_{i=0}^{n} A_i(k+1,k)A_{n-i}(k+1,1)\\ &= A_n(k+1,k+1)\\ &= A_{n+1}(k+1,1).\end{aligned}$$

Finally, by observing that $A_{n+1}(k+1,1) = C_{n+1,k} = s_{n+1}$, we have proven that $s_n = \sum_{i=1}^{n} z_i s_{n-i}$ for all $n \geq 1$ as desired. □

2.4 Super Catalan Numbers

The *super Catalan* numbers are another generalization of the sequence of Catalan numbers, with two arguments, m and n. Caldwell et al. conjectured corresponding generalized action graphs could be constructed for some cases of the super Catalan numbers [3].

Definition 2.12 [8] *The super Catalan numbers are defined by*

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

When $m = 0$, the first few numbers in the sequence are

$$[1, 2, 6, 20, 70 \ldots].$$

Similar to the generalized action graphs for the Fuss-Catalan numbers, the authors in [3] conjectured a construction of generalized action graphs for the super Catalan numbers. The construction was based on path length as well as the number of paths.

Definition 2.13 [3] *Construct the sequence of directed graphs $\{G_n\}$ inductively. The graph G_0 is a single vertex labeled 0. To construct G_{n+1} from G_n, consider each vertex v in G_n. For each $0 \leq \ell \leq n$, add $p(v,\ell)\frac{2}{2^\ell}$ new vertices labeled n with edges from v, where $p(v,\ell)$ is the number of paths of length ℓ in graph G_n from v to a vertex labeled n.*

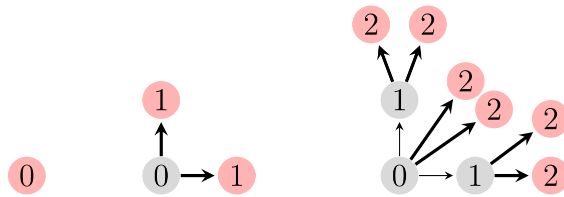


Figure 6: Generalized action graphs G_0 through G_2.

Notation 2.14 [**3**] Since generalized action graphs have many identical subgraphs with the same labels, we collapse them. For an edge from a vertex labeled a to a vertex labeled b, the multiplier $\times m$, indicates the number of such edges from a vertex labeled a to vertices labeled b it corresponds to in the original graph. For a vertex in the condensed form, we can find the number of vertices it represents in the standard form by multiplying the labels along the path from the root to that vertex. For example, the upper right vertex labeled 2 in the condensed graph in Figure 7 represents the $2 \times 2 = 4$ vertices labeled 2 that are adjacent to the two vertices labeled 1 in the original graph.

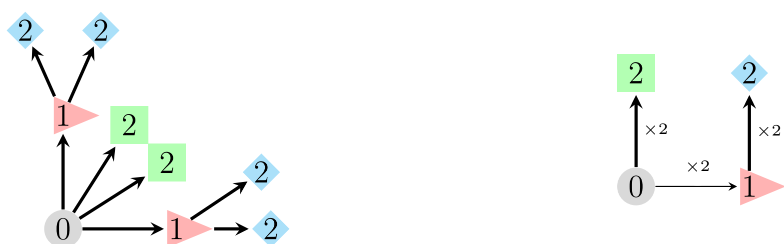


Figure 7: Generalized action graph G_2 in standard and condensed form.

Because these graphs can be cumbersome to draw after the first few graphs, the remaining graphs will be drawn using a condensed notation developed in [3].

The construction of generalized action graphs for the super Catalan numbers is similar to the generalized action graphs for the Fuss-Catalan numbers, where the number of vertices added adjacent to a vertex is dependent on path length. For the action graphs related to the super Catalan numbers, $\frac{2}{2^\ell}$ new vertices are added per path. When $\ell = 1$:

$$\frac{2}{2^1} = \frac{2}{2} = 1.$$

More generally for $\ell = 0, 1, 2, 3 \ldots$, the number of vertices added to the start of each path is scaled by $[2, 1, \frac{1}{2}, \frac{1}{4} \ldots]$. This can be seen in Figure 6, in G_2, where 2 vertices are added to each of the vertices labeled 1, for paths of length 0, and 1 vertex is added to the 0 vertex for each path of length 1.

Initially, one might be concerned that this scaling by $\frac{2}{2^\ell}$ would result in adding fractional vertices. However, we will see in Corollary 2.17 that the number of paths $p(v, \ell)$ will have a large enough power of 2 as a factor to avoid this issue.

Caldwell et al. defined the construction of directed graphs in Definition 2.13 and conjectured that these graphs satisfy the definition of generalized action graphs (Def. 1.1) for the sequence $S(0, n)$, the super Catalan numbers where $m = 0$ [3, Conjecture 5.2.1]. We will show in Theorem 2.16 that generalized action graphs exist for this sequence. While we have not verified that the generalized action graphs guaranteed by Theorem 1.1 are equivalent to the directed graphs in Definition 2.13, we conjecture this to be the case and Corollary 2.17 provides additional evidence in support of the conjecture. The following example applies Theorem 1.1 to the super Catalan numbers in the case of the generalized action graph G_2 as seen in Figure 6.

Example 2.15 From Definition 2.12, when $m = 0$, $S(0, n) = [1, 2, 6, 20 \ldots]$. From Figure 6 and further constructions, the z_n values for these generalized action graphs are $z_n = [2, 2, 4 \ldots]$. In order to apply Theorem 1.1, we must check that

$$S(0, n) = \sum_{i=1}^{n} z_i \cdot S(0, n - i).$$

When $n = 2$:

$$\begin{aligned} S(0,2) &= z_1 \cdot S(0,1) + z_2 \cdot S(0,0) \\ &= 2 \cdot 2 + 2 \cdot 1 \\ &= 6. \end{aligned}$$

Since $S(0, 2) = 6$, our theorem implies the existence of generalized action graphs for the super Catalan numbers when $m = 0$ and $n \leq 2$.

Now we will prove that Theorem 1.1 holds for the sequence of super Catalan numbers, $S(m, n)$, for $m = 0$.

Theorem 2.16 *Sequences* $s_n = S(0, n)$ *and* $z_n = S(1, n - 1) = 2C_{n-1}$ *satisfy the hypotheses of Theorem 1.1.*

We now prove Theorem 2.16 using generating functions. For background knowledge related to generating functions and Catalan numbers, see for instance [7, Chapters 7-9].
Proof. We want to show

$$S(0, n) = \sum_{i=1}^{n} S(1, i - 1) S(0, n - i).$$

Alternatively

$$S(0, n) = \sum_{i=0}^{n-1} S(1, i) S(0, n - 1 - i),$$

or

$$S(0, n + 1) = \sum_{i=0}^{n} S(1, i) S(0, n - i).$$

Note that by Definition 2.12 $S(0,k) = \binom{2k}{k}$ and $S(1,k) = 2C_k$. Generating functions for both of these sequences are known. The generating function for $\binom{2k}{k}$ (see [10, p. 130]) is

$$B(x) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}, \tag{1}$$

and the generating function for the Catalan numbers C_n (see [10, p. 82]) is

$$\sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{2x},$$

so the generating function for $2C_n$ would be

$$C(x) = \sum_{k=0}^{\infty} 2C_k x^k = 2\sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{x}.$$

The sum $\sum_{i=0}^{n} S(1,i)S(0,n-i)$ is a convolution of the sequences $S(1,n) = 2C_n$ and $S(0,n) = \binom{2n}{n}$, which means that it is equal to the coefficient of the x^n term of the product of the two generating functions. Put another way,

$$\begin{aligned}\sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} S(1,i)S(0,k-i) \right) x^k &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} 2C_i \binom{2(k-i)}{k-i} \right) x^k \\ &= C(x)B(x) \\ &= \frac{1-\sqrt{1-4x}}{x} \frac{1}{\sqrt{1-4x}} \\ &= \frac{1}{x} \left(\frac{1}{\sqrt{1-4x}} - 1 \right). \end{aligned} \tag{2}$$

From (1) and the fact that $\binom{0}{0} = 1$, we see that

$$\sum_{k=1}^{\infty} \binom{2k}{k} x^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k - 1 = \frac{1}{\sqrt{1-4x}} - 1,$$

and if we divide by x, distribute, and then shift the index k by 1, we arrive at

$$\frac{1}{x} \left(\frac{1}{\sqrt{1-4x}} - 1 \right) = \frac{1}{x} \left(\sum_{k=1}^{\infty} \binom{2k}{k} x^k \right) = \sum_{k=1}^{\infty} \binom{2k}{k} x^{k-1} = \sum_{k=0}^{\infty} \binom{2(k+1)}{k+1} x^k. \tag{3}$$

In (2) and (3), we have given two different series representations of the same function. So, we conclude

$$\sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} S(1,i)S(0,k-i) \right) x^k = \sum_{k=0}^{\infty} \binom{2(k+1)}{k+1} x^k$$

and, comparing coefficients of x^k on both sides,

$$\left(\sum_{i=0}^{k} S(1,i)S(0,k-i)\right) = \binom{2(k+1)}{k+1} = S(0,k+1).$$

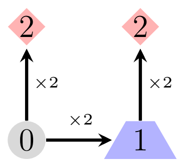
□

The following corollary of Theorem 2.16 illustrates that the construction given in Definition 2.13 is well-defined.

Corollary 2.17 *For the Super Catalan generalized action graphs defined in 2.13, $p(v,\ell)$ will always be a multiple of 2^ℓ.*

Proof. Let $p_n(v,\ell)$ be the number of paths from vertex v of length ℓ in G_n. We will proceed by induction. We start by considering some base cases and observing that in each case, for all ℓ, $p_n(v,\ell)$ is a multiple of 2^ℓ.

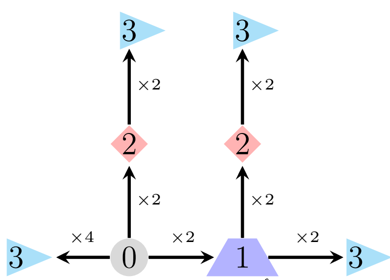
Base case: $n = 2$



$$p_2(0,1) = 2 = 2^1$$
$$p_2(0,2) = 4 = 2^2$$
$$p_2(1,1) = 2 = 2^1$$

So in G_2, $p_2(v,\ell)$ is a multiple of 2^ℓ for all v and ℓ.

Base case: $n = 3$



$p_3(0,1) = 4 = 2^1(2)$ $\quad p_3(1,1) = 2 = 2^1$

$p_3(0,2) = 8 = 2^2(2)$ $\quad p_3(1,2) = 4 = 2^2$

$p_3(0,3) = 8 = 2^3$ $\quad p_3(2,1) = 2 = 2^1$

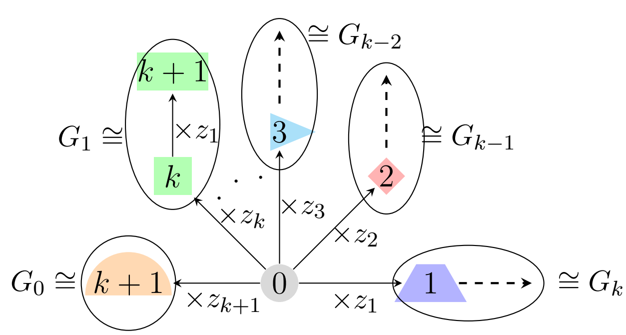
So in G_3, $p_3(v,\ell)$ is a multiple of 2^ℓ for all v and ℓ.

Inductive Hypothesis: Suppose that for the graphs $\{G_i\}_{i=1}^{k}$ and for all v and ℓ:

$$p_i(v,\ell) = 2^\ell \cdot M_{i,v,\ell},$$

where $M_{i,v,\ell}$ is an integer equal to a sum of products.

Inductive Step: Consider graph G_{k+1}.



From the proof of Theorem 1.1 we know that each subtree with root adjacent to the vertex labeled 0 is isomorphic to a previous graph. By induction, we know that the number of paths starting at a vertex v with a length ℓ in each subtree is a multiple of 2^ℓ. Therefore, if v is not the root vertex, then $p_{k+1}(v,\ell)$ is a multiple of 2^ℓ.

Consider $p_{k+1}(0,\ell)$. When $\ell = 1$, $p_{k+1}(0,\ell)$ is the same as the number of vertices labeled $k+1$ adjacent to the root vertex, and thus

$$p_{k+1}(0,1) = z_{k+1}.$$

From Theorem 2.16, we know that $z_{k+1} = 2 \cdot C_k$, and thus $p_{k+1}(0,\ell) = 2^\ell \cdot C_k$ for $\ell = i$.

Now, suppose $\ell > 1$. We can group the paths of length ℓ starting at the root vertex based on which vertex they pass through adjacent to the root. So, consider any vertex v adjacent to the root in G_{k+1} and let j be the label of that vertex. Length ℓ paths starting at the root vertex and passing through v will correspond to paths of length $\ell - 1$ starting at v. The subgraph of G_{k+1} rooted at v is isomorphic to G_{k+1-j}, with labels shifted by j. Applying our inductive hypothesis to G_{k+1-j},

$$p_{k+1-j}(v,\ell-1) = 2^{\ell-1} \cdot M_{k+1-j,v,\ell-1}$$

Now looking at G_{k+1},

$$\begin{aligned} p_{k+1}(0,\ell) &= \sum_{j=1}^{k+1} 2^{\ell-1} \cdot M_{k+1-j,v,\ell-1} \cdot z_j \\ &= \sum_{j=1}^{k+1} 2^{\ell-1} \cdot M_{k+1-j,v,\ell-1} \cdot 2(C_{j-1)} \\ &= \sum_{j=1}^{k+1} 2^{\ell} \cdot M_{k+1-j,v,\ell-1} \cdot (C_{j-1}) \\ &= 2^\ell \sum_{j=1}^{k+1} M_{k+1-j,v,\ell-1} \cdot (C_{j-1}). \end{aligned}$$

We conclude that $p(v,\ell)$ is always a multiple of 2^ℓ.

□

3 Generalized Action Graphs and a Proof of Theorem 1.1

Action graphs were originally defined in the work of Bergner and Hackney on Reedy categories [2]. Subsequently, Alvarez, Bergner and Lopez [1] showed a correspondence between action graphs and Catalan numbers, Cressman et al. [5] showed a correspondence between a generalization of action graphs and the Fuss-Catalan numbers, and finally Caldwell et al. [3] conjectured another generalization for the super Catalan numbers. In [3], they observed properties of the action graphs found in [1] and [5] and formulated a definition of generalized action graphs for any sequence. We use the same definition in this paper.

Definition 1.1 [3] *The sequence G_n of generalized action graphs for a particular sequence s_n of positive integers is a sequence of directed, labeled, rooted trees such that:*

Axiom 1 *Define G_0 as the graph with s_0 vertices labeled 0 and no edges. Construct G_n from G_{n-1} by adding s_n new vertices, which are each labeled n.*

Axiom 2 *For vertex v in G_n, the subtree of G_n with root v is isomorphic to some G_k such that $k \leq n$.*

Axiom 3 *All leaves in the graph G_n have label n.*

Note that *subtree of G with root v* here means the subgraph of G consisting of all descendants of v and edges between them in G.

When constructing generalized action graphs for the super Catalan numbers, the authors in [3] also proved Lemmas 3.1 and 3.2 as necessary properties for generalized action graphs.

Lemma 3.1 [3] *In order for a sequence $\{s_n\}_{n=0}$ to form a valid generalized action graph, s_n must have the property that $s_0 = 1$.*

Lemma 3.2 [3] *In order for a sequence $\{s_n\}_{n=0}$ to form a valid generalized action graph, s_n must have the property $s_2 \geq s_1^2$.*

An important aspect of Theorem 1.1 relies on Axiom 2, regarding graph isomorphism.

3.1 Graph Isomorphism

Graphs are *isomorphic* to each other if they have the same number of vertices and edges, connected in the same way. From Axiom 2, generalized action graphs have isomorphic subtrees, or subgraphs, meaning that previous graphs can be seen in subsequent graphs. Note that when we consider isomorphism in generalized action graphs, the labels on the vertices may be shifted, but the underlying structure of the graphs are the same. We can see this below using an action graph for the super Catalan numbers, G_3, in condensed notation.

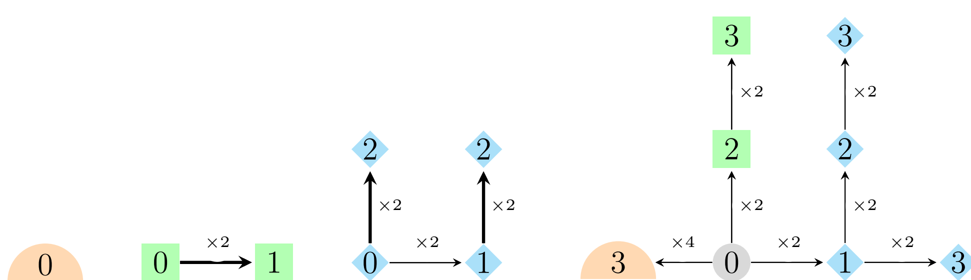


Figure 8: Super Catalan Action graphs G_0 through G_3.

Figure 8 shows that each subtree of G_3 is isomorphic to a previous action graph, using colored vertices. The number of isomorphic subtrees are indicated by the multiplier on the edges connected to the 0 vertex. For example, G_3 has two subtrees isomorphic to G_2, two subtrees isomorphic to G_1, and four subtrees isomorphic to G_0 with roots adjacent to the 0 vertex.

3.2 Building Generalized Action Graphs

Prior to this work, there was no general way of determining if generalized action graphs could be constructed for a given sequence, nor a method for constructing the graphs. However, in [3] the authors showed that some sequences related to the Catalan numbers do not have associated generalized action graphs. Furthermore, [3], [1], and [5] defined action graphs and described their properties for specific sequences. Cochran [4] then conjectured a way to generalize this work for any sequence that satisfies a certain property. In particular, she suggested a way to determine the number of new vertices labeled n adjacent to the 0 vertex in G_n. Said another way, she conjectured the value of the multiplier on the edge from 0 to the vertex labeled n in condensed notation.

Conjecture 3.3 [4] Given an appropriate sequence, s_n, of positive integers, in order to build action graph G_n, the number of new vertices that must be added to the 0 vertex is

$$z_n = s_n - \sum_{i=1}^{n-1} z_i \cdot s_{n-i}.$$

The following theorem is the main result of our paper. Rather than answering Conjecture 3.3, we prove the converse. Our result confirms the relevance of the formula for z_n and constructs graphs that satisfy Definition 1.1. We suspect Conjecture 3.3 is indeed true, which, together with Theorem 1.1, would result in a classification of sequences for which generalized action graphs exist.

Theorem 1.1 *If two sequences of positive integers $\{s_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ satisfy (i) $s_0 = 1$ and (ii) for all $n \geq 1$,*

$$s_n = \sum_{i=1}^{n} z_i s_{n-i},$$

then there are generalized action graphs G_n for all $n \geq 0$ such that z_i is the number of vertices labeled i adjacent to the root, and s_i is the total number of vertices labeled i for all $0 \leq i \leq n$. Moreover, for a vertex v labeled i adjacent to the root in G_n, the subtree with root v is isomorphic G_{n-i} with vertex labels shifted up by i.

Proof. Let s_n be a positive sequence for $n \geq 0$ such that $s_0 = 1$ and there exist positive integers z_n such that

$$s_n = \sum_{i=1}^{n} z_i \cdot s_{n-i}. \tag{4}$$

Note that by hypothesis *(ii)*, $z_1 = s_1$ since $s_0 = 1$. By Definition 1.1 and since $s_0 = 1$, G_0 is a single vertex labeled 0. The construction of the remaining graphs will proceed by induction. We include more base cases than necessary in order to clearly illustrate the construction.
Base Case: Suppose $n = 1$. By assumption, $s_1 = z_1$ since $s_0 = 1$. Adding $s_1 = z_1$ new vertices labeled 1 to the previous graph, we arrive G_1 shown in Figure 9.

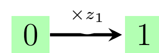


Figure 9: Construct G_1 by adding z_1 vertices adjacent to the root vertex.

To see that G_1 satisfies Definition 1.1, we can see that z_1 vertices labeled 1 are added to the vertex labeled 0. Since $z_1 = s_1$, we have added a total number of s_1 vertices labeled 1 to the previous graph and thus satisfy Axiom 1.

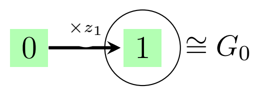


Figure 10: Isomorphic subtrees in G_1.

Figure 10 shows that any subtree with root vertex labeled 1 is isomorphic to the generalized action graph G_0, satisfying Axiom 2.

Note that all leaves in this graph are labeled 1, satisfying Axiom 3. Thus G_1 satisfies the definition of generalized action graph given in Definition 1.1.
Base Case: Suppose $n = 2$. By plugging $n = 2$ into Formula 4, we see z_2 must satisfy

$$\begin{aligned} s_2 &= \sum_{i=1}^{2} z_i \cdot s_{2-i} \\ &= z_1 \cdot s_1 + z_2 \cdot s_0. \end{aligned}$$

By assumption, we know $s_1 = z_1$ and $s_0 = 1$, so $s_2 = z_1 \cdot z_1 + z_2$, or alternatively z_2.

To construct G_2, we will add z_1 vertices labeled 2 adjacent to each vertex labeled 1 in order to make the subtrees rooted at vertices labeled 1 isomorphic to graph G_1, with labels shifted up by 1. We will also add $z_2 = s_2 - z_1 \cdot z_1$ vertices labeled 2 adjacent to the vertex labeled 0.

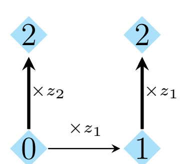


Figure 11: Construct G_2 by adding s_2 vertices to G_1.

Figure 11 shows z_2 vertices labeled 2 are added to the vertex labeled 0, and $z_1 \cdot z_1$ vertices labeled 2 are added to the vertices labeled 1. From this, we know a total of s_2 vertices labeled 2 have been added to the previous generalized action graph, showing that G_2 satisfies Axiom 1. Since z_1 is assumed to be positive, all leaves are labeled 2, and thus G_2 also satisfies Axiom 3.

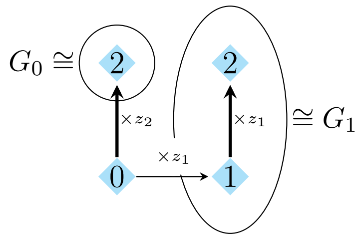


Figure 12: Isomorphic subtrees in G_2.

Figure 12 illustrates that subtrees with root vertex 1 are isomorphic to the generalized action graph G_1 with labels shifted by one, and leaves are isomorphic to G_0 with labels shifted by two. So we have confirmed Axiom 3 and conclude that G_2 satisfies the definition of generalized action graph.
Base Case: Suppose $n = 3$. By plugging $n = 3$ into Formula 4, we have

$$\begin{aligned} s_3 &= \sum_{i=1}^{3} z_i \cdot s_{3-i} \\ &= z_1 \cdot s_2 + z_2 \cdot s_1 + z_3 \cdot s_0. \end{aligned}$$

By assumption $s_1 = z_1$ and $s_0 = 1$, and from base case $n = 2$ we know $s_2 = z_1 \cdot z_1 + z_2$, so

$$\begin{aligned} s_3 &= z_1(z_2 + z_1 \cdot z_1) + z_2 \cdot z_1 + z_3 \\ &= z_1 \cdot z_2 + z_1 \cdot z_1 \cdot z_1 + z_2 \cdot z_1 + z_3. \end{aligned}$$

We use the values z_1, z_2, and z_3 to construct G_3 from G_2 as seen in Figure 13.

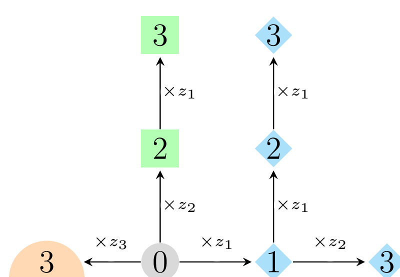


Figure 13: Construct G_3 by adding s_3 vertices G_2.

Figure 13 shows z_3 vertices labeled 3 adjacent to the vertex labeled 0, as well as z_1 vertices labeled 3 adjacent to each vertex labeled 2, and z_2 vertices labeled 3 adjacent to each vertex labeled 1. From this we know the total number of vertices added is s_3, so G_3 satisfies Axiom 1 of the definition of generalized action graph. Since all leaves in G_2 were labeled 2 and z_1 is positive, we see all leaves in G_3 are labeled 3, satisfying Axiom 3.

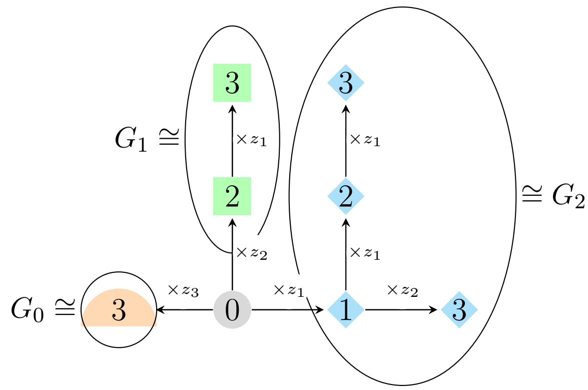


Figure 14: Isomorphic subtrees in G_3.

Figure 14 illustrates that subtrees with root vertex labeled 1 are isomorphic to G_2 with labels shifted by one, subtrees with root vertex labeled 2 are isomorphic to G_1 with labels shifted by two, and subtrees with root vertex labeled 3 are isomorphic to G_0 with labels shifted by three. So, we have checked Axiom 2, and conclude G_3 satisfies 1.1.

Inductive Hypothesis: Consider the case where $n = k \geq 0$. In particular, assume that for some $k \geq 0$, there exists a sequence of generalized action graphs $G_0, \ldots, G_k$ that are constructed by adding s_i vertices labeled i to the action graph G_{i-1}, that there are z_i vertices labeled i adjacent to the vertex labeled 0, and each subtree in G_k with the root vertex labeled t adjacent to the 0 vertex is isomorphic to G_{k-t}, where $t \leq k$, with all leaves labeled k.

Since k is arbitrary, there is no way to draw these action graphs in their completeness. However, we give a sketch in Figure 15. By assumption, for each vertex v labeled i adjacent the vertex labeled 0, the subtree with root v is isomorphic to G_{k-i}, and we will use a dashed edge to represent the remainder of the subtree.

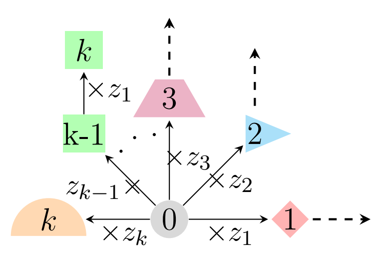


Figure 15: Graph G_k.

Figure 15 shows that there are z_k vertices labeled k adjacent to the vertex labeled 0. By the induction hypothesis, we also know that there are a total of s_k vertices labeled k in the graph G_k.

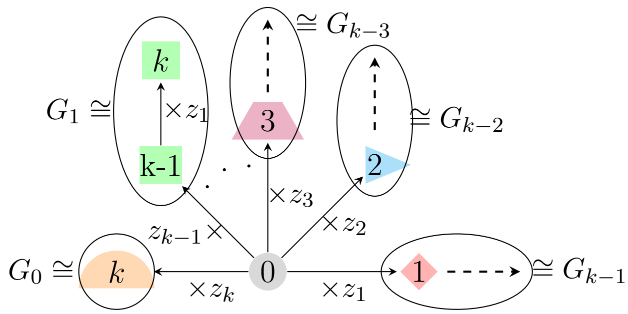


Figure 16: Isomorphic subtrees in G_k.

Figure 16 illustrates each subtree with root vertex labeled t adjacent to the vertex labeled 0 is isomorphic to G_{k-t} where $t \leq k$.

Inductive Step: We will construct G_{k+1} using G_k. We construct G_{k+1} in the following way, as illustrated by Figure 17.

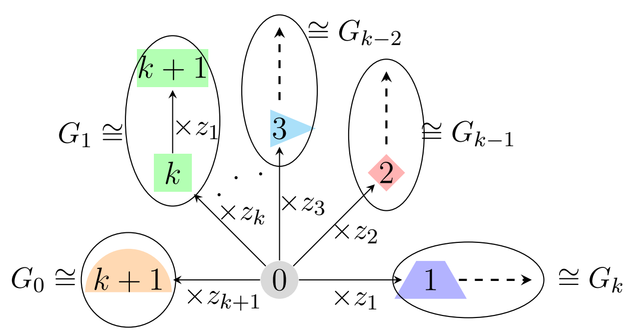


Figure 17: Isomorphic subtrees in G_{k+1}.

We add z_{k+1} vertices, which are each isomorphic to G_0 with labels shifted by $k+1$, adjacent to the vertex labeled 0.

In general, we consider the vertices labeled i adjacent to the vertex labeled 0 in G_k. There are z_i such vertices and they are each roots of subtrees isomorphic to G_{k-i} with labels shifted by i. By the induction hypothesis, we already know how to add new vertices labeled $k-i+1$ to G_{k-i} in order to construct G_{k-i+1}. We use this same process to add vertices labeled $k+1$ to the subtrees isomorphic to G_{k-i} in G_k. This will result in z_i subtrees isomorphic to G_{k-i+1} with with roots labeled i adjacent to the vertex labeled 0 in G_{k+1}. Once we have done this for all $1 \leq i \leq k$, the construction is complete.

For example: In G_k, each vertex labeled k adjacent to 0 is the root of a subtree isomorphic to G_0. So, to each vertex labeled k adjacent to 0, we add z_1 adjacent new vertices. This results in z_k subtrees isomorphic to G_1 with roots labeled k adjacent to the 0 vertex in G_{k+1}. Note that this step adds $z_k \cdot s_1$ new vertices.

As a further example: In G_k, each vertex labeled $k-1$ adjacent to 0 is the root of a subtree isomorphic to G_1. To each of these subtrees isomorphic to G_1, we add new vertices labeled k in the same way that we added vertices labeled 2 to construct G_2 from G_1. This will result in z_{k-1} subtrees isomorphic to G_2 with roots labeled $k-1$ adjacent to the 0 vertex in G_{k+1}.

By construction and the induction hypothesis, G_{k+1} satisfies Axiom 2. Since G_{k+1} satisfies Axiom 2, each leaf in G_k now has z_1 new leaves labeled $k+1$ adjacent to it in G_{k+1}. Thus, G_{k+1} satisfies Axiom 3.

By the induction hypothesis and Axiom 1, this construction adds a total of $z_i \cdot s_{k+1-i}$ new vertices for each $1 \leq i \leq k$. Summing over i and including the z_{k+1} vertices labeled $k+1$ adjacent to the 0 vertex, the total number of vertices labeled $k+1$ in G_{k+1} is

$$\sum_{i=1}^{k+1} z_i \cdot s_{k+1-i},$$

which, by assumption, is equal to s_{k+1}. Thus G_{k+1} satisfies the definition of generalized action graph, and our proof by induction is complete.

□

4 Further Questions

This project opens several directions for future analysis. Theorem 1.1 provides sufficient conditions to construct action graphs. Theorem 1.1 can also help us determine other sequences that could possibly yield action graphs. By finding other sequences that do yield action graphs, we could further analyze what makes sequences like the Catalan, Fuss-Catalan, and super Catalan numbers unique. Another open question is whether the hypotheses of Theorem 1.1 are in fact necessary conditions for generalized action graphs to exist.

Acknowledgments

We appreciate the reviewer's detailed and helpful recommendations.

References

[1] G. Alvarez, J.E. Bergner, R. Lopez, Action Graphs and Catalan Numbers, *J. Integer Seq.*, **18** (2015), Article 15.7.2, 7 pp, available online at the URL: `https://arxiv.org/abs/1503.00044v1`.

[2] J.E. Bergner, P. Hackney, Reedy categories which encode the notion of category, *Fund. Math.*, **228** (2015), 193–222. Also available online at the URL: `http://eudml.org/doc/282637`.

[3] D. Caldwell, A. Cochran, N. Glisson, B. Jennings, K. McDicken, L.J. Proctor, S. Klanderman, A. Tebbe, Catalan Number Sequences and Generalized Action Graphs, *Mathematics Exchange.*

Ball State University, **18** (2024), available online at the URL: `https://openjournals.bsu.edu/mathexchange/article/view/5831`.

[4] A. Cochran, Action Graphs for Catalan Sequences, *MathFest Poster Session* (2023).

[5] D. Cressman, J. Lin, A. Nguyen, L. Wiljanen, Generalized action graphs, (in preparation).

[6] H.W. Gould, J. Quaintance, Combinatorial Identities: Table II: Advanced Techniques for Summing Finite Series, available online at the URL: `https://math.wvu.edu/~hgould/Vol.5.PDF`.

[7] J. Morris,*Combinatorics: an upper-level introductory course in enumeration, graph theory, and design theory*, available online at the URL: `https://opentext.uleth.ca/Combinatorics/frontmatter-1.html`.

[8] R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[9] W. Młotkowski, Fuss-Catalan numbers in noncommutative probability, *Doc. Math.*, **15** (2010), 939–955. Also available online at the URL: `https://ems.press/journals/dm/articles/8965250`.

[10] J. Riordan, *Combinatorial Identities*, Wiley, 1968.

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Received: July 10, 2025 **Accepted**: March 23, 2026
Communicated by Ha Nguyen