Some Equations Concerning the Sum of Divisors Function

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Abstract - In this article, I will consider four equations involving the sum of divisors function, σ, and I will prove that for the first two there are infinitely many solutions, while for the other two I will provide some particular solutions. I will also study some properties regarding two sequences of numbers.

Keywords : equations; sum of divisors function; lifting the exponent lemma

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1 Introduction

Equations involving the sum of divisors function, σ, date back to Euclid (300 B.C.), who studied the perfect numbers i.e. numbers which satisfy the equation $\sigma(n) = 2 \cdot n$. This equation was lately generalized to $\sigma(n) = k \cdot n$, where k is a positive integer. In this work, I will study equations of the type $\sigma(n) = k \cdot n$, where k is a positive rational number of specific form.

2 Preliminaries

If p is a prime number, we will denote by $v_p(n)$ the largest exponent of p appearing in the prime factor decomposition of $n \in \mathbb{Z}^*$. We will also denote by $\gamma_m(n)$, where m is a positive integer coprime with n, the order of n in the multiplicative group of the units of $\dfrac{\mathbb{Z}}{m\mathbb{Z}}$ and by $(\frac{n}{p})$ the Legendre symbol of n with respect to the prime p.

We will make use of the following theorems ([1]):

Theorem 2.1 (Lifting The Exponent Lemma-First Form) *Let x and y be two integers, n a positive integer and p be an odd prime such that $p \mid x - y$, but $p \nmid x \cdot y$. Then*

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n). \tag{1}$$

Theorem 2.2 (Lifting The Exponent Lemma-Second Form) *Let x and y be two integers, n an odd positive integer and p be an odd prime such that $p \mid x + y$, but $p \nmid x \cdot y$. Then*

$$v_p(x^n + y^n) = v_p(x + y) + v_p(n). \tag{2}$$

Theorem 2.3 (Lifting The Exponent Lemma-Third Form) *Let x and y be two odd integers and n a positive integer such that $4 \mid x - y$.*
Then

$$v_2(x^n - y^n) = v_2(x - y) + v_2(n). \tag{3}$$

Theorem 2.4 (Lifting The Exponent Lemma-Fourth Form) *Let x and y be two odd integers and n an even positive integer. Then*

$$v_2(x^n - y^n) = v_2(x - y) + v_2(x + y) + v_2(n) - 1. \tag{4}$$

Lemma 2.5 *Let p be an odd prime, a an odd positive integer and $b \geq 2$ a positive integer. If $p^a + 1 = 2^b$ then $a = 1$ and $p = 2^b - 1$ is a Mersenne prime.*
Proof. *Suppose by contradiction that $a \geq 3$. Then $a = q \cdot c$, where q is an odd prime and c is an odd positive integer. The equation becomes*

$$p^{q \cdot c} + 1 = 2^b \implies (p + 1) \cdot (p^{q \cdot c - 1} - p^{q \cdot c - 2} + \cdots + 1) = 2^b. \tag{5}$$

Since p, c and q are odd, we get that $p^{q \cdot c - 1} - p^{q \cdot c - 2} + \cdots + 1$ is odd, so $q \cdot c = 1$, which is false. □

3 Four Equations Involving σ

The first equation in x, y and n which I will consider is

$$\sigma(n) = \frac{x}{2^y} \cdot n \tag{6}$$

where x, y and n are positive integers, x odd, $x > 2^y$ and $2^y \mid n$.

Proposition 3.1 *Equation 6 has infinitely many solutions with n of the form $n = 2^{y+\alpha} \cdot p^{\alpha_1}$, where α, α_1 are positive integers and p is an odd prime number.*

Proof. Equation 6 becomes

$$(2^{y+\alpha+1} - 1) \cdot (1 + p + \cdots + p^{\alpha_1}) = 2^\alpha \cdot x \cdot p^{\alpha_1}. \tag{7}$$

Since the right hand side is an even positive integer, we get that α_1 is odd.

Case I: $2^{y+\alpha+1} - 1 = p^{\alpha_1}$ and $1 + p + \cdots + p^{\alpha_1} = x \cdot 2^\alpha$

We show that there are no solutions in this case. Suppose, by contradiction, that there exists a solution. By Lemma 2.5, we get that $\alpha_1 = 1$, which implies that $1 + p = 2^{y+\alpha+1} = x \cdot 2^\alpha$. But x is an odd integer and we get the desired contradiction.

Case II: $2^{y+\alpha+1} - 1 = x \cdot p^{\alpha_1}$ and $1 + p + \cdots + p^{\alpha_1} = 2^\alpha$

We get that $2^\alpha \cdot (p - 1) = p^{\alpha_1+1} - 1$. Using the fourth form of "Lifting The Exponent

Lemma", we obtain $\alpha + v_2(p-1) = v_2(p^{\alpha_1+1} - 1) = v_2(p-1) + v_2(p+1) + v_2(\alpha_1+1) - 1$, so $\alpha = v_2(p+1) + v_2(\alpha_1+1) - 1$.

We write: $p + 1 = 2^r \cdot r'$, where r and r' are positive integers with r' odd and $\alpha_1 + 1 = 2^s \cdot s'$, where s, s' are positive integers with s' odd.

So $\alpha = r+s-1$, from where we obtain: $2^\alpha = 2^{r+s-1} \geq 1+p^{\alpha_1} = 1+(2^r \cdot r' - 1)^{2^s \cdot s' - 1} > (2^{r-1})^{2^s \cdot s'-1} \geq (2^{r-1})^{2^s - 1} \geq 2^{(r-1)\cdot s}$. Consequently, $r+s-1 > (r-1)\cdot s \implies r\cdot(s-1) < 2s-1 \implies s = 1$ or $r = 1$ or $r = 2$.

Case 1: $s = 1$

This implies that $\alpha_1 = 2s' - 1$, $\alpha = r \implies 2^r = 1 + p + \cdots + p^{\alpha_1} = 2^r \cdot r' + \cdots + p^{\alpha_1} \implies r' = \alpha_1 = 1$ and $p = 2^\alpha - 1 \implies p$ is a Mersenne prime and $2^{y+\alpha+1} - 1 = x \cdot (2^\alpha - 1) \implies 2^{y+1} \equiv 1 \pmod{p} \implies \alpha = \gamma_p(2) \mid y + 1$.

Case 2: $r = 1$ and $s > 1$

We show that this case cannot occur. We would get that $p = 2r' - 1$ and $\alpha = s \implies 2^s = 1 + p + \cdots + p^{2^s \cdot s' - 1} \geq 1 + 3^s$, which is false.

Case 3: $r = 2$ and $s > 1$

We show also that this case cannot occur. We would obtain that $p = 4r' - 1$ and $\alpha = s + 1 \implies 2^{s+1} = 1 + p + \cdots + p^{2^s \cdot s' - 1} \geq 1 + 3^{2^s - 1} \geq 1 + 3^{s+1}$, which is again false.

Case III: $1 + p + \cdots + p^{\alpha_1} = 2^\alpha \cdot t$ and $2^{y+\alpha+1} - 1 = p^{\alpha_1} \cdot u$, where t and u are odd positive integers, strictly bigger than 1, such that $t \cdot u = x$

We will construct the infinitely many solutions in the following way: p is arbitrary and $\alpha_1 \geq 3$ is an arbitrarily odd integer. α is defined as $v_2(1 + p + \cdots + p^{\alpha_1}) = v_2(p+1) + v_2(\alpha_1+1) - 1$ and t is defined as $t = \frac{1+p+\cdots+p^{\alpha_1}}{2^\alpha}$. We take $y = \gamma_{p^{\alpha_1}}(2) \cdot k - \alpha - 1$, where k is an arbitrarily large positive integer. Finally, $u = \frac{2^{y+\alpha+1}-1}{p^{\alpha_1}}$. □

The second equation in x, y and n which I will consider is

$$\sigma(n) = \frac{x}{3^y} \cdot n \tag{8}$$

where x, y and n are positive integers, $3 \nmid x$, $x > 3^y$ and $3^y \mid n$.

Proposition 3.2 *Equation 8 has infinitely many solutions with n of the form $n = 3^{y+\alpha} \cdot p^{\alpha_1}$, where α, α_1 are positive integers and $p \geq 5$ is a prime number.*

Proof. The equation considered is equivalent with

$$(1 + 3 + \cdots + 3^{y+\alpha}) \cdot (1 + p + \cdots + p^{\alpha_1}) = 3^\alpha \cdot x \cdot p^{\alpha_1}. \tag{9}$$

We will only study the case $1 + 3 + \cdots + 3^{y+\alpha} = u \cdot p^{\alpha_1}$ and $1 + p + \cdots + p^{\alpha_1} = t \cdot 3^\alpha$, where $t, u \geq 2$ are integers such that $t \cdot u = x$.

As for the first equation considered, we will build an infinite number of solutions. We choose p and α_1 such that $3 \mid 1 + p + \cdots + p^{\alpha_1}$ (i.e. $p \equiv 1 \pmod{3}$ and $\alpha_1 \equiv 2 \pmod{3}$)

or $p \equiv 2 \pmod 3$ and α_1 is odd). α is defined as $v_3(1+p+\cdots+p^{\alpha_1})$, t is defined as $\frac{1+p+\cdots+p^{\alpha_1}}{3^\alpha}$, $y = \gamma_{p^{\alpha_1}}(3) \cdot k - \alpha - 1$, where k is an arbitrarily large positive integer and $u = \frac{1+3+\cdots+3^\alpha}{p_1^\alpha}$. □

Remark 3.3 For the more general equation

$$\sigma(n) = \frac{x}{q^y} \cdot n \tag{10}$$

where $q \geq 5$ is a fixed prime number and x, y are positive integers with $q \nmid x$ and $x > q^y$, we can similarly build an infinite number of solutions with n of the form $q^{\alpha+y} \cdot p^{\alpha_1}$, where $p \equiv 1 \pmod q$ is a prime number.

The third equation in x, y and n which I will consider is

$$\sigma(n) = \frac{2^y}{x} \cdot n \tag{11}$$

where x, y and n are positive integers, x odd and $x < 2^y$. A solution to this equation is $n = x \cdot q_1 \cdot \cdots \cdot q_r$ where $x = p_1 \cdot \cdots \cdot p_r$; $q_1, \ldots, q_r$ are distinct Mersenne primes and $p_j = 2^{n_j} \cdot q_j - 1$ are primes (here, n_j are positive integers). As in [2], I will consider the sequence $A_{n,p} = 2^n \cdot M_p - 1$, where $M_p = 2^p - 1$ is the Mersenne number associated to the prime number p and n is a positive integer and I will study some of its properties (here p is fixed).

Proposition 3.4 *An odd prime number q divides one of the terms in the sequence if and only if $q \mid 2^t - M_p$, for some positive integer t.*

Proof. $q \mid A_{n,p} \iff 2^n \cdot M_p \equiv 1 \pmod q \iff M_p \equiv 2^{-n} \pmod q$. We take $t = (q-1) \cdot s - n$, where s is an arbitrarily large positive integer. □

Remark 3.5 q is among the prime divisors of the numbers $2^t - M_p$.

Proposition 3.6 *If q is a prime number such that $q \equiv 1 \pmod 8$ and $q \equiv -1 \pmod{M_p}$, then q doesn't divide any of the numbers $A_{n,p}$.*

Proof. Suppose by contradiction that there exists such a prime number q. Denote by t the number given by the preceding proposition. Then

$$1 = (\frac{2}{q})^t = (\frac{2^t}{q}) = (\frac{M_p}{q}) = (\frac{q}{M_p}) \cdot (-1)^{\frac{q-1}{2} \cdot \frac{M_p - 1}{2}} = (\frac{q}{M_p}) = (\frac{-1}{M_p}) = -1$$

since $M_p \equiv 3 \pmod 4$. □

Remark 3.7 From Dirichlet's theorem concerning prime numbers in arithmetic progression and from "Chinese Remainder Theorem", we get that there exists an infinite number of primes q which do not divide any of the numbers in the sequence $A_{n,p}$.

Proposition 3.8 *If q is a prime number with $(\frac{2}{q}) = -1$ and $(\frac{M_p}{q}) = 1$, then $q \nmid A_{2n+1,p}$ for any nonnegative integer n.*

Proof. We fix n. Then $q \mid A_{2n+1,p} \iff (2^{n+1} \cdot M_p)^2 \equiv 2M_p \pmod q \implies (\frac{2M_p}{q}) = 1$, which is a contradiction with the hypotheses. □

Proposition 3.9 *Let q be a prime number and suppose that there exists a positive integer l such that $q \mid A_{l,p}$. Let n_0 be the smallest such positive integer. Then $q \mid A_{n,p} \iff n \equiv n_0 \pmod e$, where $e = \gamma_q(2)$.*

Proof. $q \mid A_{n_0,p} \iff M_p \equiv 2^{-n_0} \pmod q$
$q \mid A_{n,p} \iff 2^n \cdot M_p \equiv 1 \pmod q \iff 2^{n-n_0} \equiv 1 \pmod q \iff e \mid n - n_0$. □

Proposition 3.10 *There exists an infinite number of primes that divide the numbers in the sequence $A_{n,p}$.*

Proof. We suppose, by contradiction, that there are a finite number of odd primes, $q_1, \ldots, q_k$ that divide the numbers in the sequence. Then
$2^n \cdot M_p - 1 = \prod_{j=1}^{k} q_j^{e_{j,n}}$, for any positive integer n. So $2^n \cdot M_p \equiv 1 \pmod{\prod_{e_{j,n}>0} q_j}$.
Let $N = \text{l.c.m.}(q_1 - 1, \ldots, q_k - 1) \cdot a$, where a is a positive integer such that $N > p$. Then $2^N \equiv 1 \pmod{\prod q_j}$ and $0 \equiv 2^{N-p} \cdot M_p - 1 \equiv -2^{N-p} \pmod{\prod_{e_{j,N-p}>0} q_j} \implies \prod_{e_{j,N-p}>0} q_j \mid 2^{N-p}$, which cannot be true. □

Remark 3.11 If $A_{n,p}$ is a prime number for some fixed $n \geq 3$, then 2 is not a primitive root modulo $A_{n,p}$, because $A_{n,p} \equiv -1 \pmod 8$.

The last equation in x, y and n which we will consider is

$$\sigma(n) = \frac{3^y}{x} \cdot n \tag{12}$$

where x, y and n are positive integers with $3 \nmid x$ and $x < 3^y$. A particular solution to this equation is $n = 2 \cdot x$, where $x = 2 \cdot 3^{y-1} - 1$ is an odd prime. As for the preceding equation, we will consider the sequence $X_n = 2 \cdot 3^n - 1$ and we will study some of its properties.

Proposition 3.12 *A prime q divides one of the terms in the sequence if and only if $q \mid 3^t - 2$, for some positive integer t.*

Proof. $q \mid X_n \iff 2 \cdot 3^n \equiv 1 \pmod q \iff 2 \equiv 3^{-n} \pmod q$
We take $t = (q-1) \cdot s - n$, where s is a positive integer chosen such that $t > 0$. □

Remark 3.13 q stands among the prime divisors of the numbers $3^t - 2$.

Proposition 3.14 *If q is a prime number such that*
$q \equiv 11 \pmod{24}$ or $q \equiv 13 \pmod{24}$, then q does not divide any of the numbers in the sequence X_n.

Proof. For such a prime q, we have that $(\frac{2}{q}) = -1$ and $(\frac{3}{q}) = 1 \implies 2 \not\equiv 3^n \pmod q$ □

Remark 3.15 From Dirichlet's theorem concerning prime numbers in arithmetic progression and from "Chinese Remainder Theorem", we get that there are infinitely many primes that do not divide any of the numbers in the sequence X_n.

Proposition 3.16 *If q is a prime with $q \equiv 7 \pmod{24}$ or $q \equiv 17 \pmod{24}$, then $q \nmid X_{2n+1}$ for any nonnegative integer n.*

Proof. For such a prime q, we get that $(\frac{6}{q}) = -1$.
But $q \mid X_{2n+1} \iff (6 \cdot 3^n)^2 \equiv 6 \pmod{q}$, which is a contradiction. □

Proposition 3.17 *If q is a prime number such that $q \mid X_{2n}$ for some positive integer n, then $q \equiv \pm 1 \pmod 8$.*

Proof. $q \mid X_{2n} \iff (2 \cdot 3^n)^2 \equiv 2 \pmod{q} \implies (\frac{2}{q}) = 1$
The conclusion now follows. □

Proposition 3.18 *Suppose that there exists a positive integer l such that $q \mid X_l$, where q is a prime number. If n_0 is the smallest such positive integer, then $q \mid X_n \iff n \equiv n_0 \pmod{f}$, where $f = \gamma_q(3)$.*

Proof. $q \mid X_{n_0} \iff 2 \cdot 3^{n_0} \equiv 1 \pmod{q}$
Now, $q \mid X_n \iff 2 \cdot 3^n \equiv 1 \pmod{q} \iff 2 \cdot 3^n \equiv 2 \cdot 3^{n_0} \pmod{q} \iff 3^{n-n_0} \equiv 1 \pmod{q}$. The conclusion follows. □

Proposition 3.19 *There exists an infinite number of primes that divide the terms of the sequence X_n.*

Proof. Suppose by contradiction that there are only finitely many odd primes, $q_1, \ldots, q_k$ that divide the numbers in this sequence. Then
$2 \cdot 3^n - 1 = \prod_{j=1}^{k} q_j^{e_{j,n}} \implies 2 \cdot 3^n \equiv 1 \pmod{\prod_{e_{j,n}>0} q_j}$.
Let $N = \text{l.c.m.}(q_1 - 1, \ldots, q_k - 1)$. Then $3^N \equiv 1 \pmod{\prod_j q_j} \implies 2 \cdot 3^N - 1 \equiv 1 \pmod{\prod_{e_{j,N}>0} q_j}$ contradiction! □

4 Conclusions

The four equations considered are not fully resolved. I believe that there are also other solutions to each of them.

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