On the Matching Density of Tetrahedral and Icosahedral Galois Representations

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Abstract - The matching density of two Galois Representations ρ and η is the Dirichlet density of the set of finite places where ρ and η are unramified and the traces of Frobenius of ρ and η are equal. Using matching density, we can quantify the similarity of two Galois representations. In this paper, we find sharp upper bounds for the matching density of certain pairs of tetrahedral and icosahedral representations.

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1 Introduction

Consider two n-dimensional complex Galois representations, η and ρ over a number field K, which we note are continuous homomorphisms from $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{C})$, where $\overline{K}$ is an algebraic closure of K.

Definition 1.1 *Let $\Sigma_{K,f}$ denote the set of finite places of K. If ρ and η are complex Galois representations over K, which are both unramified at $v \in \Sigma_{K,f}$, let $tr\,\rho(Frob_v)$ be the trace of Frobenius of ρ at v, and similarly for η. We define the matching density $d(\rho, \eta)$ to be the Dirichlet density of the set $V = \{v \in \Sigma_{K,f} : tr\,\rho(Frob_v) = tr\,\eta(Frob_v)\}$.*

If ρ and η are isomorphic representations, then we can see $d(\rho, \eta) = 1$, so we only consider matching densities of representations which are not isomorphic. The above definition has been used to show previous results on Galois representations. In Serre's article [6], he constructs Galois representations of matching density 7/8, and in [8], Walji constructs pairs of representations with matching densities 3/4, 3/5 and 17/32. Furthermore in [9], Walji shows the set of all possible matching densities of n-dimensional Galois representations over $\mathbb{Q}$, and all n, is dense in $[0, 1]$.

For a number field K, we must have that any complex Galois representation over K has finite image. To see this, note $\mathrm{GL}_n(\mathbb{C})$ is equipped with a Lie group topology, so we may take u to be a small open neigbourhood of the identity, I_n, such that u contains no nontrivial subgroup. Consider ρ, a representation of a profinite group G. Then $\rho^{-1}(u)$

is an open neighbourhood of the identity in G, and thus contains an open and normal subgroup H. The set

$$\bigcup_{x \in G} x + H$$

forms an open cover for the compact group G, and thus G/H is finite, and since $H \subseteq \mathrm{Ker}(\rho)$, we have $\mathrm{Im}(\rho)$ finite.

Definition 1.2 *We will call two n-dimensional complex Galois representations ρ_1 and ρ_2 over a finite extension L/K factor equivalent if there exist epimorphisms η_1, η_2, and monomorphisms φ_1 and φ_2, such that*

$$\eta_i : Gal(\overline{K}/K) \to Gal(L/K) \tag{1}$$

$$\varphi_i : Gal(L/K) \to GL_n(\mathbb{C}) \tag{2}$$

$$\varphi_i \circ \eta_i = \rho_i. \tag{3}$$

Note this is a stronger condition than having two n-dimensional representations η and ρ factoring through extensions K_1 and K_2 such that $\mathrm{Gal}(L_1/K_2) \cong \mathrm{Gal}(L_2/K)$. In [9], Walji gives an example of two representations ρ and ρ' factoring through $\mathrm{Gal}(K_1/\mathbb{Q}) \cong \mathrm{Gal}(K_2/\mathbb{Q}) \cong \mathrm{SL}_2(\mathbb{F}_3)$. These are isomorphic as representations of $\mathrm{SL}_2(\mathbb{F}_3)$, however $K_1 \neq K_2$ and as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, η and ρ are not isomorphic. Since we are only considering factor equivalent representations, we use the following theorem to treat our factor equivalent Galois representations as representations of a finite group $G \cong \mathrm{Gal}(L/K)$.

Theorem 1.3 *(Chebotarev Density Theorem). Let L be a finite Galois extension of a number field K, with Galois group $G := Gal(L/K)$. Let $C \subseteq G$ be a conjugacy class, and let S be the set of primes $\mathfrak{p}$ of K, unramified in L with $Frob_{\mathfrak{p}} \in C$, then $d(S) = \frac{\#C}{\#G}$, where $d(S)$ is the Dirichlet Density of S.*

Definition 1.4 *For a complex representation ρ, we take $\chi_\rho(g) : G \to \mathbb{C}$ such that*

$$\chi_\rho(g) = tr\rho(g) \tag{4}$$

to be the character from G to $\mathbb{C}$ arising from the representation ρ.

Remark 1.5 Although characters arise from representations, they may not be group homomorphisms. Indeed, for $n \geq 2$, $\chi_\rho(i) = n$ if ρ is an n-dimensional complex representation, showing χ_ρ does not preserve identity. However, one can see characters are constant on conjugacy classes. We exploit this in the proofs of Theorems 2.3 and 3.11.

Lemma 1.6 *If ρ and η are two Galois representations over K that are factor equivalent by an extension L/K with finite Galois group $Gal(L/K) \cong G$: take their corresponding representations of G to be ρ' and η', then*

$$d(\eta, \rho) = \frac{\#\{g \in G, \chi_{\rho'}(g) = \chi_{\eta'}(g)\}}{\#G},$$

where $d(\eta, \rho)$ is as defined in 1.1.

Therefore, to bound the matching density of two factor equivalent Galois representations, it suffices to bound the matching density of faithful representations of the finite group through which they factor. Now we discuss the motivations behind this paper.

Two-dimensional irreducible complex representations are categorized by their images in $\mathrm{PGL}_2(\mathbb{C})$, which must be isomorphic to one of D_{2n}, A_4, S_4, or A_5 for natural n. This can be found with proof in [1]. In [5], it was shown that the set of matching densities of n-dimensional Galois representations over a number field is bounded above by $1 - \frac{1}{2n^2}$. The Dihedral representations constructed in [6] of matching density 7/8 show this bound is indeed sharp when $n = 2$. So we now ask a new question: if we restrict ourselves to two-dimensional representations that are factor equivalent, and have a projective image other than D_{2n}, can we improve upon the upper-bound for the set of matching densities of these representations? This paper will focus on representations with projective image A_4 or A_5. To do this, it suffices to bound the matching densities of the faithful representations of the finite groups that project to A_4 or A_5. By Schur's lemma, [7], these groups are exactly the cyclic central extensions of A_4 and A_5, respectively. The earlier parts of Section 2 and Section 3 will focus on the construction and classification of these groups. For a shorter description of these groups, and further work in the case of 3-dimensional representations, see [4]. The main result of this article is the following theorem:

Theorem 1.7 *Consider two complex Galois representations over a number field K, ρ and η, both icosahedral or tetrahedral and factor equivalent, then*

$$d(\rho, \eta) \leq 5/8,$$

where this bound is sharp.

The structure of this paper is partitioned into three sections. The first section introduces notation, definitions, and previous results which motivate the paper. The next two sections prove Theorem 1.7, with the Section 2 being the icosahedral case, Theorem 2.3, and Section 3 being the tetrahedral case, Theorem 3.11. The tetrahedral case is significantly longer than the icosahedral case: heuristically, there are fewer possible central extensions of A_5, and consequently, fewer icosahedral representations, because A_5 is simple, whereas A_4 is not. For the second two sections, almost all details are included in the hope that the reader may not have to look elsewhere in the literature.

The following definitions will be used in both Section 2 and Section 3, and so we include them here.

Definition 1.8 *For two groups G, G', we define $G * G'$ in the following manner. If there are subgroups $H \subset Z(G)$ and $H' \subset Z(G')$, with $H \cong H'$, then given an isomorphism $\varphi : H \to H'$, take $\Delta = \{(a, \varphi(a))\} \leq G \times G'$ (which is normal, as it lies in the center of $G \times G'$), and define*

$$G * G' = (G \times G')/\Delta.$$

1.1 A Note on Group Cohomolgy

In this subsection, we provide details on the group cohomolgy that will be used implicitly in Section 2 and explicitly in Section 3. We can generalize the notion of a central extension of groups to that of G-modules. We call E an extension of G by a G-module N if the following short exact sequence exists in G-**Mod**,

$$0 \to N \to E \to G \to 0.$$

From [2], we know that if E is to be a central extension, then the action of G on N must be trivial. Also from [2], we know that for a fixed G action, the set of extensions of G by N is in bijection with $H^2(G, N)$. Consider the split class in $H^2(G, N)$, where the action of G on N is trivial: treating an element of E of this class as a group, we must have that $E \cong G \times N$. In particular, we have the following definition.

Definition 1.9 *A central extension of groups, E of G by N is trivial, if*

$$E \cong G \times N$$

Equivalentely, take E to be a central extension of G-modules G by N, where the action of G on N is trivial. If E lies in the split class of $H^2(G, N)$, then E is the trivial central extension of G by N.

Consider two elements E, E' in the same class in $H^2(G, N)$, then E, E' must satisfy

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & & & \big\downarrow f & & & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

where f is necessarily a G-module isomorphism. And so we see that when treated as groups E and E' must be isomorphic.

Hence in order to count the number of group central extensions of G by N, n, we may upper bound bound this quantity by $|H^2(G, N)| = m$. If we then explicitly construct m non-isomorphic group central extensions of G by N, we can lower bound n by m, giving $n = m$. We will see in the icosahedral case, the work is done for us, however we will have to work through these steps explicitly in the tetrahedral case.

Definition 1.10 *We say two complex representations ρ, ρ' are twist equivalent if*

$$\rho' \cong \rho \otimes \chi,$$

where χ is a character of an abelian group.

Remark 1.11 Consider two groups G and G' such that

$$G \oplus \left(\bigoplus_{j=1}^{N} C_{n_j} \right) \cong G'$$

i.e. G and G' differ by the direct product of finite abelian groups. Then the set of matching densities of faithful representations of G' is bounded above by the set of matching densities of faithful representations of G. To see this, note the representations of G' are twist equivalent to representations of G, and $\mathrm{tr}(\alpha M) = \alpha \mathrm{tr} M$, for a matrix M, and scalar α.

For the rest of this paper, unless otherwise specified, all representations are considered to be factor equivalent, irreducible and two-dimensional complex representations.

2 Icosahedral Representations

Definition 2.1 *We define an icosahedral representation to be a complex two-dimensional representation ρ of a group G, satisfying*

$$\rho(G)/Z(\rho(G)) \cong A_5.$$

Before proceeding to the proof of the upper bound for icosahedral representations, we note the following proposition of Wang, from [10].

Proposition 2.2 *Let F be a number field, ρ an icosahedral Galois representation over F and let G denote the image of $Gal(\overline{F}/F)$ under ρ. Then G is generated by its commutator subgroup G_0, and its center $Z(G) \cong \mu_{2m}$, which is the group of roots of unity of order $2m$. Furthermore, G_0 is isomorphic to $\tilde{A}_5$, the unique non-trival central extension of A_5 by C_2, with center $\{\pm I\}$, and*

$$G \cong (G_0 \times C_{2m})/\{\pm(I, 1)\}.$$

Hence each irreducible representation Λ of G can be expressed (uniquely) as (Λ_0, μ) where $\Lambda_0\big|_G = \Lambda$ is an irreducible representation of G_0 and $\mu = \Lambda\big|_{C_{2m}}$ is a character of C_{2m}, and such that $\Lambda_0(-I) = \mu(-1)I$. Furthermore, each such pair (Λ_0, ρ) gives an irreducible representation of G.

Theorem 2.3 *Consider two complex icosahedral Galois representations ρ, φ, over a number field K that are factor equivalent, then*

$$d(\rho, \varphi) \leq \frac{5}{8}.$$

Proof. *Noting Proposition 2.2, and that $\tilde{A}_5 \cong SL_2(\mathbb{F}_5)$, any icosahedral representation must factor through a group G of the form*

$$G \cong SL_2(\mathbb{F}_5) * C_{2m}.$$

To bound the matching density of factor equivalent icosahedral representations, it suffices to bound the matching density of faithful, irreducible, two-dimensional representations of the above group, or representations of

$$SL_2(\mathbb{F}_5) \times C_{2m},$$

with kernels

$$\{\pm(I_2, 1)\}. \tag{5}$$

We denote the 2 two-dimensional faithful irreducible representations of $SL_2(\mathbb{F}_5)$ *as* X *and* X'*, with respective characters* χ *and* χ'*.*

size	1	1	20	30	12	12	20	12	12
X	2	-2	-1	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
X'	2	-2	-1	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Table 1: Partial character table of $\mathrm{SL}_2(\mathbb{F}_5)$

We can see that both X *and* X' *map* $-I_2$ *to* $-I_2$*. And so, for a character* η *of* C_{2m}*, if we require* $\ker(X \otimes \eta) = \{\pm(I_2, 1)\}$*, we must have* $\eta(-1) = -1$*.*

There are three distinct cases when calculating the matching density of representations of groups of the above form. In the below, we take G *to be* $SL_2(\mathbb{F}_5) * C_{2m}$*.*

Case 1: $d(X \otimes \eta, X' \otimes \eta)$*.*

We can uniquely identify each conjugacy class of G *as the direct sum of a conjugacy classes of* $SL_2(\mathbb{F}_5)$ *and* C_{2m}*. The character table of* $SL_2(\mathbb{F}_5)$ *tells us that for a given* $g \in SL_2(\mathbb{F}_5)$*,*

$$\chi(g) \neq \chi'(g) \iff |\chi(g)| \neq |\chi'(g)|. \tag{6}$$

From (6) we can see that for any η *and* $x \in C_{2m}$*,* $\chi(g)\eta(x) = \chi'(g)\eta(x)$ *if and only if* $\chi(g) = \chi'(g)$*. Hence it suffices to count the elements in* G *where* χ *and* χ' *are equal, and then normalize by the order of* G*. And so by Lemma 1.6 we have*

$$d(X \otimes \eta, X' \otimes \eta) = \frac{1 + 1 + 20 + 30 + 20}{120} = \frac{3}{5}. \tag{7}$$

Now we deal with the second case.

Case 2: $d(X \otimes \eta, X \otimes \eta')$*.*
It is important to note that given two characters η *and* η' *of* C_{2m} *satisfying (5), we have*

$$d(\eta, \eta') \leq \frac{1}{2}. \tag{8}$$

Consider $g \in X$ *that is not in the conjugacy class of order 30 from Table 1. Then for any* $x \in C_{2m}$

$$(X \otimes \eta)(g, x) = (X \otimes \eta')(g, x) \iff \eta(x) = \eta'(x). \tag{9}$$

If g *lies in the conjugacy class of order 30, then for any* η *and* η'*, and* $x \in C_{2m}$*,*

$$(X \otimes \eta)(g, x) = (X \otimes \eta')(g, x). \tag{10}$$

Combining (8), (9) and (10)

$$d((X \otimes \eta), (X \otimes \eta')) \leq \frac{90m + 60m}{240m} = \frac{5}{8}. \tag{11}$$

Case 3: *$d(X \otimes \eta, X' \otimes \eta')$.*

By (5), we can see that given $g \in SL_2(\mathbb{F}_5)$ and any η, η' and $x \in C_{2m}$,

$$\chi(g) \neq \chi'(g) \implies \chi(g)\eta(x) \neq \chi'(g)\eta'(x).$$

Therefore, we have

$$d((X \otimes \eta), (X' \otimes \eta')) \leq \frac{3}{5}.$$

Therefore, given any two representations ρ, φ of G satisfying (5), we have

$$d(\rho, \varphi) \leq \frac{5}{8}.$$

□

Remark 2.4 If one considers η and η', the two distinct representations of C_8 such that $\eta(-1) = -1$ and $\eta'(-1) = (-1)$. Then one can see that $d(\eta, \eta') = \frac{1}{2}$, and so

$$d(X \otimes \eta, X \otimes \eta') = \frac{5}{8}. \tag{12}$$

Hence the bound from Theorem 2.3 is sharp.

3 Tetrahdedral Representations

Definition 3.1 *We define a tetrahedral representation to be a two-dimensional complex representation ρ of a group G such that*

$$\rho(G)/Z(\rho(G)) \cong A_4.$$

Section 3 is split into 3 subsections. In the first, we count the number of central extensions of A_4, in the second we classify these groups, and in the third we bound the matching densities of these groups, proving the Tetrahedral case of Theorem 1.7.

3.1 Counting Central Extensions via Group Cohomology

In this section, we count the possible central extensions of A_4 by C_m using group cohomology. We will follow the procedure described in Section 1. However first we require the following lemma.

Lemma 3.2 *, For distinct primes p, q, if the only central extension of G by C_{p^k} is trivial, then every central extension of G by $C_{p^k q^j}$ is of the form $H \times C_{p^k}$, where H is a central extension of G by C_{q^j}.*

Proof. Consider H, a central extension of G by C_{q^j}, then

$$0 \to C_{p^k q^j} \to H \times C_{p^k} \to G \to 0,$$

gives central extension $H \times C_{p^k}$, of G by $C_{p^k q^j}$

Suppose for sake of contradiction, there exists a central extension K of G by $C_{p^k q^j}$ not of the form $H \times C_{p^k}$ where H is a central extension of G by C_{q^j}. Then this is given by

$$0 \to C_{p^k q^j} \xrightarrow{\varphi} K \to G \to 0.$$

By construction of K, we can see that

$$K/\varphi(i \times C_{q^j}) \ncong G \times C_{p^k}.$$

And therefore

$$0 \to C_{p^k} \to K/\varphi(i \times C_{q^j}) \to G \to 0,$$

is a non-trivial central extension of G by C_{p^k}, which is a contradiction, and proves our lemma. □

Proposition 3.3 *For a fixed m, there exist at most 3 non-trivial central extensions of A_4 by C_m.*

Proof. The extensions we are identifying cannot be trivial since A_4 admits no irreducible two-dimensional complex representations. By Lemma 3.2, we will see it suffices to check for $m = 2^i 3^j$ for $i, j \in \mathbb{N}_{\geq 0}$. We will make use of the method described in Subsection 1.1.

Consider the following short exact sequence,

$$0 \to \mathrm{Ext}^1(H_1(A_4; \mathbb{Z}), C_m) \xrightarrow{f} \mathrm{H}^2(A_4; C_m) \xrightarrow{g} \mathrm{Hom}(H_2(A_4; \mathbb{Z}), C_m) \to 0 \tag{13}$$

whose existence is a result of the Universal Coefficient Theorem for Cohomology, which can be found in more detail and with proof in [3].

We introduce the following two lemmas, whose proofs we omit, but can be found in [3], allowing us to re-write (13).

Lemma 3.4 *For a group T,*

$$\mathrm{Ext}^1_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z}/n_i\mathbb{Z}, T\right) \cong \bigoplus_i T/n_i T.$$

Proof. See [3]. □

Lemma 3.5

$$H_2(A_4; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_1(A_4; \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

Proof. The proof of the first equation follows from Hopf's theorem for $H_2(G;\mathbb{Z})$, and the second from $H_1(G;Z) \cong G^{ab}$. $\square$

Case 1: $(2,m) = (3,m) = 1$.
We use Lemmas 3.4 and 3.5 to arrive at the following short sequence.

$$0 \to 0 \xrightarrow{f} \mathrm{H}^2(A_4;C_m) \xrightarrow{g} 0 \to 0, \tag{14}$$

which tells us that the only central extension of A_4 by C_m with the above divisibility conditions is the trivial extension, and can thus be ignored.

Case 2: $(2,m) = 2$, $(3,m) = 1$
We use the same lemmas, with new divisibility conditions to arrive at the following short exact sequence.

$$0 \to 0 \xrightarrow{f} \mathrm{H}^2(A_4;C_m) \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0,$$

and so there exists a unique non-trivial central extension of A_4 by C_m with the above conditions on m, which we denote U_1^m. We take U_1 to be an arbitrary element of the set $\{U_1^m\}$ over $m \in \mathbb{N}$ with the above divisibility conditions.

Case 3: $(2,m) = 1$, $(3,m) = 3$
We use the same lemmas, with the new divisibility conditions to arrive at the following short exact sequence.

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathrm{H}^2(A_4;C_m) \xrightarrow{g} 0 \to 0,$$

and so there exists a unique non-trivial central extension of A_4 by C_m with the above conditions. We denote it by V_1^m. Similarly as above, we take V_1 to be an arbitrary element of the set of $\{V_1^m\}$ over $m \in \mathbb{N}$ with the above divisibility conditions.

Case 4: $(2,m) = 2$, $(3,m) = 3$
We use the same lemmas, with the new divisibility conditions to arrive at the following short exact sequence.

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathrm{H}^2(A_4;C_m) \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0,$$

and so there exists 3 non-trivial central extensions of A_4 by C_m with the above conditions. We denote these W_1^m, W_2^m and W_2^m. We pick W_1, W_2 and W_3 to be elements of the sets $\{W_1^m\}$, $\{W_3^m\}$ and $\{W_3^m\}$, respectively with the above divisibility conditions.

We can conclude that any tetrahedral representation must factor through one of U_1, V_1, W_1, W_2 or W_3.

However, taking $m = 2^i 3^j$, we can set

$$W_1 = U_1 \times C_{3^j}, W_2 = V_1 \times C_{2^i},$$

Since the representations of the above groups are twist equivalent to those of U_1 and V_1, by Remark 1.11, finding the matching densities of their faithful irreducible representations reduces to finding those of U_1 and V_1. Hence there are 3 main cases to tackle. These being U_1, V_1 and W_3. Now we will classify these groups. $\square$

3.2 Classification of Central Extensions

Proposition 3.6 *For a fixed $k = 3^j$*

$$C_2^2 \rtimes_\varphi C_{3^{j+1}} = V_1$$

acting via $\varphi : C_{3^{j+1}}/C_{3^j} \cong C_3$ is a central extension of A_4 by C_{3^j}.

Proof. We first note that

$$A_4 \cong C_2^2 \rtimes_\varphi C_3.$$

Consider the inclusion map f such that

$$f : C_{3^j} \to i \rtimes_\varphi C_{3^j} \subset C_2^2 \rtimes_\varphi C_{3^{j+1}} = V_1.$$

where $C_{3^j} = \ker(\varphi)$. One sees that $f(C_{3^j}) \subset Z(V_1)$. Consider the projection map

$$g : C_2^2 \rtimes_\varphi C_{3^{j+1}} \to C_2^2 \rtimes_\varphi C_3.$$

It is clear that $\ker(g) = (i, C_{3^j}) = \mathrm{Im}(f)$, giving us

$$0 \to C_{3^j} \xrightarrow{f} C_2^2 \rtimes_\varphi C_{3^{j+1}} \xrightarrow{g} A_4 \to 0,$$

which is the desired central extension. □

Corollary 3.7 *V_1, and consequently W_2 do not admit irreducible two-dimensional representations.*

Proof. By the definition of the semi-direct product, we see that C_2^2 is normal in V_1 and abelian. We note the corollary of Proposition 24 in [7] stating if A is a normal abelian subgroup of G, then the dimension of any irreducible representation of G must divide $[G : A]$. We see $[V_1 : C_2^2] = 3^{j+1}$, which provides the desired result for V_1. Since the representations of W_2 are twist equivalent to those of V_1, by Remark 1.11 we can see W_2 will not admit faithful irreducible two-dimensional complex representations. □

Proposition 3.8 *For $i > 1$, $U_1 = C_{2^i} * SL_2(\mathbb{F}_3)$ with $\Delta = C_2$, is a central extension of A_4 by C_{2^i}.*

Proof. First note that $Z(\mathrm{SL}_2(\mathbb{F}_3)) \cong C_2$ and $\mathrm{SL}_2(\mathbb{F}_3)/Z(\mathrm{SL}_2(\mathbb{F}_3)) \cong A_4$. Take f to be the inclusion map

$$f : C_{2^i} \to (C_{2^i} \times C_2)/C_2 \subset C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3).$$

One can see that $\mathrm{Im}(f) \subset Z(U_1)$. Take g to be the projection map

$$g : C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3) \to (C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3))/((C_{2^i} \times C_2)/C_2).$$

By the isomorphism theorems we have

$$\mathrm{Im}(g) \cong (C_{2^i} \times \mathrm{SL}_2(\mathbb{F}_3))/C_{2^i} \times C_2) \cong \mathrm{SL}_2(\mathbb{F}_3)/C_2 \cong A_4,\ \ker(g) = (C_{2^i} \times C_2)/C_2).$$

Hence we have the central extension

$$0 \to C_{2^i} \xrightarrow{f} C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3) \xrightarrow{g} A_4 \to 0.$$

We also have $W_2 = \mathrm{SL}_2(\mathbb{F}_3) * C_{2^i} \times C_{3^j}$, which we can see is a central extension of A_4 by the above. ☐

Proposition 3.9 *$W_3 = C_{2i} * D_4 \rtimes_\varphi C_{3^{j+1}}$ via $\varphi : C_{3^{j+1}} \to C_3$ is a central extension of A_4 by $C_{3^j 2^i}$.*

Proof. We will construct this group formally to begin. First we note that $Z(D_4) = C_2$ and $Z(C_{2^i} * D_4) \cong C_{2^i}$. Consider the projection map

$$\phi : D_4 \to C_2^2$$

with $\ker(\phi) = Z(D_4)$. Take X, Y, Z to be the cosets of order 2 in $D_4/Z(D_4)$, and identify the cosets with representatives x, y, z. Consider the subgroups of W_3

$$C_{2i}^x, C_{2i}^y, C_{2i}^z,$$

defined by

$$C_{2^i}^x = (C_{2^i} \times \varphi^{-1}(X))/C_2 \cong C_{2^i} \times C_2,$$

and similarly for y, z, where the above $C_{2^i} = Z(W_3)$. Take the automorphism η of order three that satisfies

$$\eta(C_{2^i}^x) = C_{2^i}^y, \eta(C_{2^i}^y) = C_{2^i}^z, \eta(C_{2^i}^z) = C_{2^i}^x.$$

Let the subgroup of $\mathrm{Aut}(W_3)$ generated by η be denoted by A. We define φ to be the map

$$\varphi : C_{3^{j+1}} \to C_3 = A \subset \mathrm{Aut}(W_3).$$

Consider the inclusion map

$$f : C_{2^i 3^j} \to C_{2^i} \rtimes_\varphi C_{3^j} \subseteq W_3,$$

where $C_{3^j} = \ker(\varphi)$. One can see $\mathrm{Im}(f) = Z(G)$. Now consider the projection map

$$g : W_3 \to W_3/ker(f) \cong ((C_{2^i} * D_4)/C_{2^i}) \rtimes_\varphi C_3 \cong C_2^2 \rtimes_\varphi C_3,$$

where C_3 acts faithfully on C_2^2. Therefore, f, g gives us the desired central extension. ☐

Proposition 3.10 *For $k = 2^i 3^j$, $i = 1$, $Q_8 \rtimes_\varphi C_{3^{j+1}}$ acting via $\varphi : C_{3^{j+1}} \to C_3$, is a central extension of A_4 by C_k.*

Proof. Note $Z(Q_8) = C_2$, and $Q_8/C_2 \cong C_2^2$. The proof of this proposition follows from the proof of Proposition 3.9, except that we redefine f to be the inclusion map

$$f : C_2 \times C_{3^j} \to Q_8 \rtimes_\varphi C_{3^{j+1}}$$

with the images $f(C_2 \times C_{3^j}) = Z(Q_8) \times \ker(\varphi)$. Then if we take g to be the quotient map by $Z(Q_8) \times \ker(\varphi)$, we arrive at the desired short exact sequence,

$$0 \to C_{2^i 3^j} \xrightarrow{f} Q_8 \rtimes_\varphi C_{3^{j+1}} \xrightarrow{g} A_4 \to 0.$$

□

We have constructed all of the groups which admit faithful complex irreducible tetrahedral representations. Now we bound the matching densities of the faithful irreducible representations of these groups, which as previously mentioned, is sufficient to bound the set of possible matching densities of factor equivalent tetrahedral Galois representations over number fields.

3.3 Bounding Matching Densities

In this subsection, using the results of the previous two subsections, we prove the following theorem.

Theorem 3.11 *Consider two complex icosahedral Galois representations ρ, η, over a number field K that are factor equivalent, then*

$$d(\rho, \eta) \leq \frac{5}{8}.$$

Proof. By Proposition 3.3, Corollary 3.7, and Propositions 3.8, 3.9, and 3.10, we know that any tetrahedral representation factors through either $U_1 = C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3)$, $W_1 = U_1 \times C_{3^j}$, $W_3 = C_{2^i} * D_4 \rtimes_\varphi C_{3^{j+1}}$ or $W_3^{3^j 2} = Q_8 \rtimes_\varphi C_{3^{j+1}}$. The proof of this theorem is partitioned into four lemmas where we bound the matching density of representations of the finite groups mentioned above.

Lemma 3.12 *Non-isomorphic representations factoring through U_1 and W_1 have matching density of at most* 5/8.

Proof. We denote the 3 two-dimensional faithful irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ as X, X' X'', with respective characters χ, χ' and χ''.

size	1	1	4	4	6	4	4
X	2	-2	-1	-1	0	1	1
X'	2	-2	ζ_6^5	ζ_6	0	ζ_3	ζ_3^2
X''	2	-2	ζ_6	ζ_6^5	0	ζ_3^2	ζ_3

Table 2: Partial character table of $\mathrm{SL}_2(\mathbb{F}_3)$

By construction, we know that any faithful representation of U_1 will be representation of $\mathrm{SL}_2(\mathbb{F}_3)$ sending $-I_2$ to $-I_2$ twisted by a character ϕ of C_{2^i} satisfying

$$\ker(\phi) = \{\pm 1\}. \tag{15}$$

Now we consider the three cases of representations on G satisfying (15).

Case 1: $d(X \otimes \phi, X' \otimes \phi)$.

Note that the image of any element of C_{2^i} satisfying (15) under ϕ will be a $2^{i-1}th$ root of unity. By examining Table 2, we can see that for any $g \in \mathrm{SL}_2(\mathbb{F}_3)$ not in the conjugacy class of order 6, then for two distinct X, X' or X'',

$$\chi'(g)/\chi''(g) = 1, \text{ or } e^{ik\pi/3}$$

where $(k, 3) = 1$. Hence we can see

$$\chi(g) \neq \chi'(g) \implies \chi(g)\phi(x) \neq \chi'(g)\phi(x). \tag{16}$$

Which tells us

$$d(X \otimes \phi, X' \otimes \phi) \leq \frac{1}{4}.$$

Now we deal with the second case.

Case 2: $d(X \otimes \phi, X \otimes \phi')$.

It is important to note that given two characters ϕ and ϕ' of C_{2^i} satisfying (15), we have

$$d(\phi, \phi') \leq \frac{1}{2}. \tag{17}$$

Consider $g \in X$ that is not the conjugacy class 4. Then we have

$$X \otimes \phi(g, x) = X \otimes \phi'(g, x) \iff \phi(x) = \phi'(x). \tag{18}$$

If g lies in a conjugacy class of order 4, then for any ϕ and ϕ', and $x \in C_{2^i}$,

$$X \otimes \phi(g, x) = X \otimes \phi'(g, x). \tag{19}$$

Viewing Cases (17), (18) and (19) together, we see

$$d(X \otimes \phi, X \otimes \phi') \leq \frac{(9)2^i + (6)2^i}{(24)2^i} = \frac{5}{8}. \tag{20}$$

Case 3: $d(X \otimes \phi, X' \otimes \phi')$.

By (15), we can see that given $g \in \mathrm{SL}_2(\mathbb{F}_3)$ and any ϕ, ϕ' and $x \in C_{2^i}$,

$$\chi(g) \neq \chi'(g) \implies \chi(g)\phi(x) \neq \chi'(g)\phi'(x).$$

Therefore, we have

$$d(X \otimes \phi, X' \otimes \phi') \leq \frac{1}{3}$$

and further, given any two representations ρ, φ of U_1 satisfying (15), we have

$$d(\rho, \varphi) \leq \frac{5}{8}.$$

Hence we have found the upper bound for the representations factoring through groups of the form $C_{2^i} * \mathrm{SL}_2(\mathbb{F}_3)$. By Remark 1.11, we have also found the upper bound for matching densities factoring through W_1 to be 5/8. □

Definition 3.13 *We say a subset S of a finite group G has relative density α if $\frac{\#S}{\#G} = \alpha$. In the case where S is a subgroup of G, then the relative density is equal to $[S : G]$.*

Lemma 3.14 *Two non-isomorphic faithful representations of W_3 have matching density of less than $5/8$.*

Proof. We note that the faithful and irreducible two-dimensional representations of $C_4 * D_4 \rtimes_\varphi C_9$ have matching density of at most $5/8$. Every faithful and irreducible two-dimensional representation ρ of $G = C_{2^i} * D_4 \rtimes_\varphi C_{3^{j+1}}$, satisfies the following relation with representations of

$$H = C_4 * D_4 \rtimes_\varphi C_9,$$

$$\rho(G)/\rho(C_{2^{i-2}} \times C_{3^{j-1}}) = \rho'(H)$$

for the unique subgroup of the form $C_{2^{i-2}} \times C_{3^{j-1}}$ lying in $Z(G)$ and a faithful irreducible representation ρ' of H. Now consider two faithful and irreducible two-dimensional representations of G, ρ and η. There are three cases.

Case 1: Given ρ and η as described above, their associated ρ' and η' are not equal, however, $\rho(x) = \eta(x)$ where $\langle x \rangle = C_{2^{i-2}} \times C_{3^{j-1}} \subset Z(G)$.

If the above holds, consider a conjugacy class Y of G that projects to a conjugacy class X of H, then we can see

$$\chi_\rho(Y) = \chi_\eta(Y) \iff \chi_{\rho'}(X) = \chi_{\eta'}(X).$$

By assumption, $d(\rho', \eta') \leq 5/8$ and therefore $d(\rho, \eta) \leq 5/8$.

Case 2: For ρ and η, the associated ρ' and η' are not equal, and $\rho(x) \neq \eta(x)$, for x as defined in the previous case.

If the above holds, consider a conjugacy class Y of G that projects to a conjugacy class X of H, if $\chi_{\rho'}(X) \neq \chi_{\eta'}(X)$, then they must differ by an element not contained in the kernel of the quotient of $G \to G/(C_{2^{i-2}} \times C_{3^{j-1}})$. Hence

$$\chi_{\rho'}(X) \neq \chi_{\eta'}(X) \implies \chi_\rho(Y) \neq \chi_\eta(Y).$$

Hence ρ and η differ on a set of relative density of at least $3/8$. Hence $d(\rho, \eta) \leq 5/8$.

Case 3: ρ and η are not isomorphic, however $\rho' = \eta'$.

If the above holds, we must have that $\rho(x) \neq \eta(x)$ and so

$$0 \neq \chi_{\rho'}(X) = \chi_{\eta'}(X) \implies \chi_\rho(aX) \neq \chi_\eta(aX),$$

where a is a generator of $C_{2^{i-2}} \times C_{3^{j-1}}$. However χ_ρ and χ_η will be equal on all elements that map to 0, which is a subset of relative density less than $1/4$. Hence $d(\rho, \eta) \leq \frac{1}{3} + \frac{2}{3}(\frac{1}{4}) < \frac{5}{8}$. □

Lemma 3.15 *Two non-isomorphic faithful representations of* $W_1^{3^j 2}$ *have matching density less than* $5/8$.

Proof. Take $H = G/C_{3^j}$ and consider two faithful irreducible two-dimensional complex representations of G. Note in the same manner as above, ρ of G corresponds to a representation ρ' of H if

$$\rho(G)/\rho(C_{3^j}) = \rho'(H),$$

and similarly for η. Once again, take x to be a generator of the group of order 3^j contained within $Z(G)$. Note for any 2 irreducible representations of $Q_8 \rtimes_\varphi C_9$, they have matching density of at most $1/3$.

Case 1: Given ρ and η as described above, their associated ρ' and η' are not isomorphic, and $\rho(x) = \eta(x)$ where $\langle x \rangle = C_{3^j} \subset Z(G)$.

If the above holds, consider a conjugacy class Y of G that projects to a conjugacy class X of H, then we can see

$$\chi_\rho(Y) = \chi_\eta(Y) \iff \chi_{\rho'}(X) = \chi_{\eta'}(X).$$

By assumption, $d(\rho', \eta') \leq 1/3$ and therefore $d(\rho, \eta) \leq 1/3$.

Case 2: For ρ and η, the associated ρ' and η' are not isomorphic, and further the image of $\rho(x) \neq \eta(x)$.

If the above holds, consider a conjugacy class Y of G that projects to a conjugacy class X of H, if $\chi_{\rho'}(X) \neq \chi_{\eta'}(X)$, then their images must differ by an element not contained in the kernel of the quotient of $G \to G/C_{3^j}$. Hence

$$\chi_{\rho'}(X) \neq \chi_{\eta'}(X) \implies \chi_\rho(Y) \neq \chi_\eta(Y).$$

Hence ρ and η differ on a set of relative density of at least $2/3$. Hence $d(\rho, \eta) \leq 1/3$.

Case 3: ρ and η are not isomorphic, however $\rho' = \eta'$.

If the above holds, we must have that $\rho(x) \neq \eta(x)$ and so

$$0 \neq \chi_{\rho'}(X) = \chi_{\eta'}(X) \implies \chi_\rho(aX) \neq \chi_\eta(aX),$$

where a is a generator of $C_{2^{i-2}} \times C_{3^{j-1}}$. However χ_ρ and χ_η will be equal on all elements that map to 0, which is a subset of relative mass less than $1/4$. Hence $d(\rho, \eta) \leq \frac{1}{3} + \frac{2}{3}(\frac{1}{4}) < \frac{5}{8}$.
□

Lemmas 3.12, 3.14, 3.15, Proposition 3.3, and Remark 1.11 show us that if ρ, and η are two non-isomorphic representations of a group that projects to A_4, they attain a matching density of at most $5/8$, which proves the desired result. □

Remark 3.16 The bound above is sharp, and there exist tetrahedral representations with matching density equal to $5/8$.

Proof. Consider the table below, note ρ and η are faithful, and have matching density of $5/8$, proving our bound is sharp. □

size	1	1	6	4	4	1	1	6	4	4	4	4	4	4
ρ	2	-2	0	ζ_6^5	ζ_6	$-2i$	$2i$	0	ζ_3^2	ζ_3	$\zeta_4^3\zeta_3$	$\zeta_4^3\zeta_3^2$	$\zeta_4\zeta_3$	$\zeta_4\zeta_3^2$
η	2	-2	0	ζ_6^5	ζ_6	$2i$	$-2i$	0	ζ_3^2	ζ_3	$\zeta_4\zeta_3$	$\zeta_4\zeta_3^2$	$\zeta_4^3\zeta_3$	$\zeta_4^3\zeta_3^2$

Table 3: Partial character table of $C_4 * \mathrm{SL}_2\mathbb{F}_3$.

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