A Compendium of Motzkin-Like Path Generating Functions

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Abstract - We catalog fifteen types of Motzkin-like paths with various restrictions and use the symbolic method to determine their generating functions, several of which are new. We also introduce a new type of Motzkin path corresponding to a walk on a circle, rather than on a line.

Keywords : Motzkin path; Motzkin numbers; symbolic method; generating functions

Mathematics Subject Classification (2020) **:** 05A15

1 Introduction

Motzkin paths are lattice paths in the plane with three permitted steps: up, down, and horizontal. These combinatorial objects are generalizations of Dyck paths, which consist of only up and down steps and are enumerated by the ubiquitous Catalan numbers, A000108 in [10]. For a sampling of recent articles about Motzkin paths, see [4, 5, 6, 11]. In [3, 9] the authors study Motzkin paths with various restrictions, such as a specified start height, bounding height, and end height. Unlike these articles, our derivations of explicit generating functions of restricted Motzkin paths rely solely on the symbolic approaches described in [7]. Our main contributions are obtaining new explicit expressions in Equations (25), (27), and (29) for the symbolic constructions and generating functions of bounded Motzkin meanders, bounded Motzkin paths with given start and end heights, and Motzkin paths with a specified maximum height, respectively. We also introduce a new Motzkin path variation, the *modular Motzkin path*, and obtain its corresponding generating function in Equation (31).

In section 2 we review the symbolic method for the derivation of generating functions and apply them to standard examples of Motzkin-like paths. In the subsequent three sections we provide symbolic constructions for additional classes of Motzkin path variations and derive the corresponding generating functions, in each case starting with a subsection on classes of paths that start at the origin followed by a subsection on more generalized classes starting at arbitrary heights. Section 3 does so with Motzkin paths bounded by a single line, section 4 with unbounded Motzkin paths, and section 5 with Motzkin paths bounded by two lines, which has an additional subsection on classes of Motzkin-like paths with exact maximum heights. Section 6 introduces modular Motzkin paths and derives their constructions and generating functions. We conclude in section 7 with a table of our results.

2 Background

2.1 The Symbolic Method

The symbolic method presented in Part A of [7] forms the foundation of much of our work. For the convenience of readers unfamiliar with the method, we provide an overview here. Those already acquainted may proceed directly to Section 2.2. At the top level, the symbolic method allows combinatorial objects to be defined by dividing them into different cases or decomposing them into smaller components that have known specifications. With the symbolic definition in hand, there is an almost effortless translation to a corresponding generating function that provides the key to analyzing desirable attributes of the class, such as closed-form formulas or asymptotics. As the authors of [7] put it, "If you can specify it, you can analyze it."

A combinatorial class is a set of objects on which a size function is defined such that the size of an element is a nonnegative integer, and the number of elements of any given size is finite. If $\mathcal{A}$ is a combinatorial class, then $\alpha \in \mathcal{A}$ means that α is an object in class $\mathcal{A}$, with $|\alpha|$ denoting its size, and we can define the ordinary generating function (OGF) of $\mathcal{A}$ as $A(x) = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|}$. Letting a_n be the number of elements in $\mathcal{A}$ with size n gives an equivalent representation of the generating function, $A(x) = \sum_{n=0}^{\infty} a_n x^n$. There are two fundamental combinatorial classes known as the neutral class $\mathcal{E}$, which contains a single object of size 0 and has generating function $E(x) = 1$ and the atomic class $\mathcal{X}$, which contains a single object of size 1 and has generating function $X(x) = x$. A wide range of combinatorial objects can be built by combining these two fundamental classes with the operations defined in [7]. Here, we list the relevant operations for Motzkin path construction, along with their notations and implications for the corresponding generating functions.

If class $\mathcal{A}$ is the disjoint union of classes $\mathcal{B}$ and $\mathcal{C}$, then the construction is expressed as

$$\mathcal{A} = \mathcal{B} + \mathcal{C} \implies A(x) = B(x) + C(x). \tag{1}$$

So, the generating function of a class that is a disjoint union is simply the sum of the generating functions of its constituent disjoint classes. Although a subtraction operation is not formally defined for combinatorial classes, it is clear from Equation (1) that $\mathcal{A} = \mathcal{B} + \mathcal{C}$ must also imply that $B(x) = A(x) - C(x)$, which is a useful implication when the generating function of a constituent class, rather than the union class, happens to be the unknown.

If every element in $\mathcal{A}$ consists of any element in $\mathcal{B}$ paired with any element in $\mathcal{C}$, then the construction is expressed as

$$\mathcal{A} = \mathcal{B} \times \mathcal{C} \implies A(x) = B(x)C(x). \tag{2}$$

If every element in class $\mathcal{A}$ consists of a k-tuple of elements from class $\mathcal{B}$, then the construction is expressed as

$$\mathcal{A} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} = \mathcal{B}^k \implies A(x) = \big(B(x)\big)^k. \tag{3}$$

If every element in class $\mathcal{A}$ consists of a finite sequence (possibly empty) of objects from class $\mathcal{B}$, then the construction is expressed as

$$\mathcal{A} = \mathcal{E} + \mathcal{B} + \mathcal{B}^2 + \mathcal{B}^3 + \cdots = \textsc{Seq}(\mathcal{B}) \implies A(x) = \frac{1}{1 - B(x)}. \tag{4}$$

If every element in class $\mathcal{A}$ consists of a finite sequence of k or less objects from class $\mathcal{B}$, then the construction is expressed as

$$\mathcal{A} = \mathcal{E} + \mathcal{B} + \mathcal{B}^2 + \cdots + \mathcal{B}^k = \textsc{Seq}_{\leq k}(\mathcal{B}) \implies A(x) = \frac{1 - \big(B(x)\big)^{k+1}}{1 - B(x)}. \tag{5}$$

If there exists a size-preserving bijective function between two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, meaning $a_n = b_n$ for all n and each element of size n in $\mathcal{A}$ is mapped to a unique element of size n in $\mathcal{B}$, then we say $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and write

$$\mathcal{A} \cong \mathcal{B} \implies A(x) = B(x). \tag{6}$$

In the context of lattice paths, isomorphic relationships help identify the generating functions of classes of paths that are simple translations or reflections of classes of paths with known generating functions.

2.2 Symbolic Examples

The Motzkin path is a fitting combinatorial object to showcase common aspects of symbolic specifications, including the disjoint union, the Cartesian product, and recursion, in which a combinatorial class is specified in terms of itself. A Motzkin path is a lattice path that starts at the origin; remains on or above the x-axis; consists of up steps with displacement $(1,1)$, down steps with displacement $(1,-1)$, and horizontal steps with displacement $(1,0)$; and ends back on the x-axis. The nth Motzkin number m_n is the number of distinct Motzkin paths that take exactly n steps, where n is referred to as the length of the path.

The class of Motzkin paths $\mathcal{M}$ can be divided into three disjoint cases, as illustrated in Figure 1. First, there is the empty path of length 0, represented by a single point at the origin and belonging to the neutral class $\mathcal{E}$. Next, we condition on the first step of the Motzkin path, which must be either an up step or a horizontal step, each of which is represented by the atomic class $\mathcal{X}$. If the first step is a horizontal step, then the rest of the path may proceed as another Motzkin path of class $\mathcal{M}$, this time translated to the right by one unit. If the first step is an up step, then the rest of the path must look like another Motzkin path translated by $(1,1)$ followed by the first down step back to the x-axis

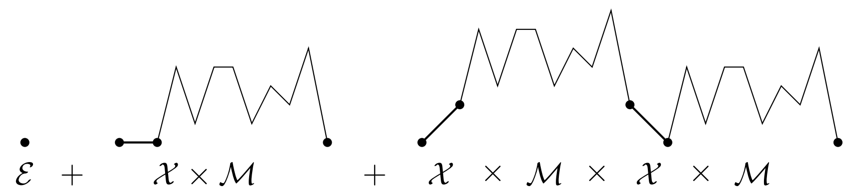

Figure 1: Symbolic construction of the class of standard Motzkin paths $\mathcal{M}$

followed by a final translated Motzkin path. In our constructions, we adopt the common practice of using the same symbol for a class of lattice paths and its translations and reflections. In cases where isomorphism is less obvious, we use different symbols to distinguish between classes, and we use congruency, rather than equality, in the construction. Therefore, putting the three cases together results in the specification

$$\mathcal{M} = \mathcal{E} + \mathcal{X} \times \mathcal{M} + \mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{M} \implies M(x) = 1 + xM(x) + x^2\big(M(x)\big)^2, \quad (7)$$

where solving for $M(x)$ results in the two roots

$$M_\pm(x) = \frac{1 - x \pm \sqrt{1 - 2x - 3x^2}}{2x^2}. \quad (8)$$

Since there is only one empty Motzkin path and $M(0)$ must equal m_0, $M(x)$ must be the negative root $M_-(x)$, which generates A001006, the Motzkin numbers, in [10]. For the sake of brevity, we suppress the dependence on x in later equations. When necessary, we also maintain the distinction between the positive and negative roots; otherwise, we simply refer to $M_-(x)$ as M.

With a nod towards continued fractions, the quadratic relation for M in Equation (7) can be rewritten as $M = 1/(1 - x - x^2M)$. Repeated application of this relation gives the generating function for the Motzkin numbers as a continued fraction

$$M = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - x - \ddots}}}. \quad (9)$$

The objects of the above class $\mathcal{M}$ are typically referred to simply as Motzkin paths. However, for the sake of specificity, we refer to these objects as *standard Motzkin paths* for the remainder of the paper. We then define a *Motzkin-like path* to be any lattice path in the plane consisting of the same three steps as a standard Motzkin path. A *Motzkin meander* is a Motzkin-like path that starts at the origin, not necessarily ending on the x-axis. The *minimum* and *maximum heights* of a Motzkin-like path are h_1 and h_2, respectively, if the lowest points and highest points of the path are on the horizontal lines $y = h_1$ and $y = h_2$, which may be negative. A Motzkin-like path is *bounded below* by k_1

if the path has a minimum height greater than or equal to k_1, and similarly, is *bounded above* by k_2 if it has a maximum height less than or equal to k_2. A class of Motzkin-like paths is considered *singly bounded* if either every path in the class is bounded below by the same value or every path in the class is bounded above by the same value. For instance, $\mathcal{M}$ is a singly bounded class because all paths in $\mathcal{M}$ are bounded below by 0. A class of Motzkin-like paths is *unbounded* if there are no bounding values common to every path in the class. Conversely, a class of Motzkin-like paths is *bounded* if all paths in the class share a common lower bound and upper bound.

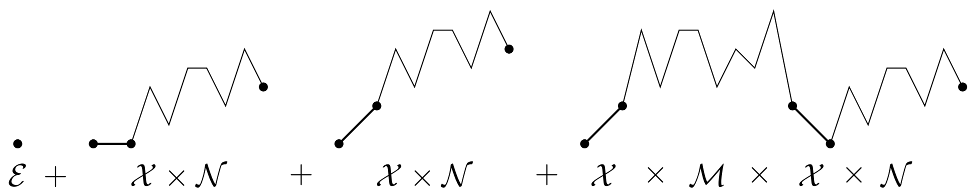

Figure 2: Symbolic construction of the class of Motzkin meanders, bounded below by the x-axis, $\mathcal{N}$

A notable example of a Motzkin-like class is the class $\mathcal{N}$ of Motzkin meanders that are bounded below by the x-axis. The paths in this class are also known as *Motzkin prefixes* or *left factors* of Motzkin paths. A path belonging to $\mathcal{N}$ must be either the empty path; a horizontal step followed by a meander from $\mathcal{N}$; an up step followed by a meander (that never returns to the x-axis); or an up step followed by a Motzkin path, the first down step returning to the x-axis, and a final meander, as illustrated in Figure 2. That is,

$$\begin{aligned} \mathcal{N} &= \mathcal{E} + \mathcal{X} \times \mathcal{N} + \mathcal{X} \times \mathcal{N} + \mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{N} \\ \implies N(x) &= \frac{1}{1-2x-x^2M} = \frac{2}{1-3x+\sqrt{1-2x-3x^2}}, \end{aligned} \tag{10}$$

where $N(x)$ generates A005773 in [10] and counts other combinatorial objects involving directed animals, directed n-ominoes, and pattern avoiding permutations.

Based on Equation (10), N must satisfy the quadratic relation $x(1-3x)N^2 + (1-3x)N - 1 = 0$ or, equivalently, $N = 1\big/(1-3x)(1+xN)$. Repeated application of this relation gives the generating function for meanders as a continued fraction

$$N(x) = \cfrac{1}{1-3x+\cfrac{x}{1+\cfrac{x}{1-3x+\cfrac{x}{1+\ddots}}}}. \tag{11}$$

Grand Motzkin paths, *super Motzkin paths*, or *Motzkin bridges* all refer to objects in the unbounded class of Motzkin-like paths that start and end at height 0, denoted $\mathcal{W}$.

Figure 3 demonstrates that such a path is either the empty path; a horizontal step followed by a Grand Motzkin path; an up step followed by a Motzkin path, the first down step to the x-axis, and a final Grand Motzkin path; or a down step followed by an "upside-down" Motzkin path, the first up step to the x-axis, and a final Grand Motzkin path, which symbolically yields

$$\begin{aligned} \mathcal{W} &= \mathcal{E} + \mathcal{X} \times \mathcal{W} + \mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{W} + \mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{W} \\ \implies W(x) &= \frac{1}{1 - x - 2x^2 M} = \frac{1}{\sqrt{1 - 2x - 3x^2}}. \end{aligned} \tag{12}$$

$W(x)$ generates the central trinomial coefficients, the coefficient of x^n in the expansion of $(1 + x + x^2)^n$, which appear as A002426 in [10] and have a variety of other combinatorial interpretations involving ordered trees, increasing sequences, and ordered ballots, to name a few.

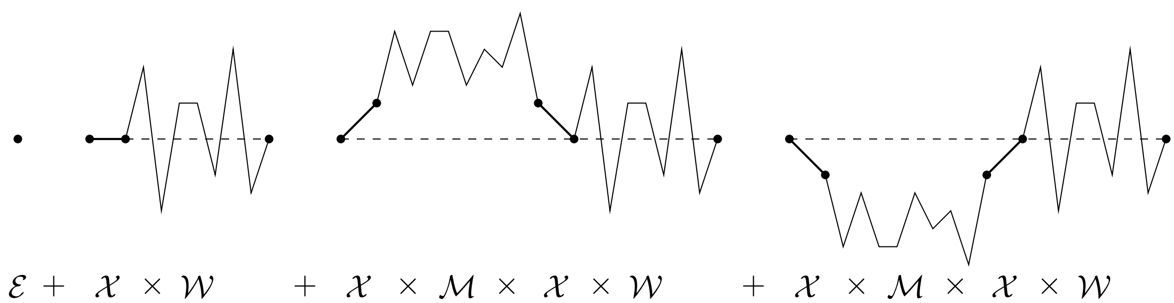

Figure 3: Symbolic construction of the unbounded class of Motzkin-like paths that start and end at height 0, $\mathcal{W}$

3 Singly Bounded Classes of Motzkin-Like Paths

The previous section showcased the symbolic method derivation of standard Motzkin paths. These paths start at the origin, remain on or above the x-axis, and end back on the x-axis. To generalize beyond paths that must start and end at height 0, we use $\mathcal{M}^{(a,b)}$ to denote the class of Motzkin-like paths starting at height a, remaining on or above the x-axis, and ending at height b, with $M^{(a,b)}(x)$ denoting the corresponding generating function. Such generating functions are discussed, for example, in Sections I.5.3 and V.4.1 of [7], but here, we provide novel derivations in Equations (14) and (15).

3.1 Singly Bounded Classes of Motzkin Meanders

We start with a symbolic derivation of $\mathcal{M}^{(0,a)}$, the class of Motzkin-like paths that begin at the origin, remain on or above the x-axis, and end at height a, or equivalently, the class of Motzkin meanders, bounded below by 0, that end at height a. Such a path must have at least a up steps in order to ascend from height 0 to height a. To isolate these guaranteed steps from the rest of the path, we identify each up step that is the final time the path ascends from one height to the next. In other words, we mark each of the final up steps from heights 0 to 1, 1 to 2, ..., and $a-1$ to a. This is known as *last passage decomposition* and is also demonstrated, for instance, in Figure 2 of [1] and Figure V.8 of [7]. Before and after each of these marked up steps, the path may ascend to any height, so long as it returns to the height of the up step, meaning it takes the form of a standard Motzkin path. Therefore, $\mathcal{M}^{(0,a)}(x)$ can be decomposed into a standard Motzkin path followed by a repetitions of an up step followed by a standard Motzkin path as shown in Figure 4. Symbolically, we have

$$\mathcal{M}^{(0,a)} = \mathcal{M} \times (\mathcal{X} \times \mathcal{M})^a \implies M^{(0,a)}(x) = x^a M^{a+1}, \tag{13}$$

where $M^{(0,a)}(x)$ generates many entries in [10], including A002026, A005322, A005323, A005324, and A005325, for $a = 1, 2, \ldots, 5$, respectively. It also generates the ath diagonal in A026300, the so-called Motzkin triangle, and ath column in A064189. These entries have well documented correspondence to Motzkin-like paths.

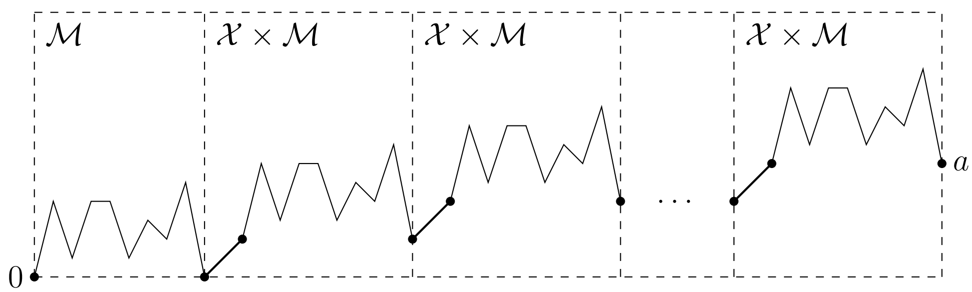

Figure 4: Symbolic construction of the class of Motzkin meanders, bounded below by 0, ending at height a, $\mathcal{M}^{(0,a)}$

The class $\mathcal{N}$ of Motzkin meanders bounded below by the x-axis are illustrated in Figure 2 and specified recursively in Equation (10). However, the above result in Equation (13) yields an alternate explicit specification by simply summing over all possible end heights of the meander, resulting in

$$\mathcal{N} = \sum_{a=0}^{\infty} \mathcal{M}^{(0,a)} \implies N(x) = \sum_{a=0}^{\infty} x^a M^{a+1} = \frac{M}{1-xM} = \frac{1}{1-2x-x^2M}, \tag{14}$$

where the equivalency $1 = M - xM - x^2M^2$ evident in Equation (7) allows a substitution that produces the same expression for $N(x)$ as in Equation (10).

3.2 Singly Bounded General Class

We now consider the general class $\mathcal{M}^{(a,b)}$, initially making the assumption that $0 \leq a \leq b$. A path in this class must eventually start an ascent to height b, but before it does so, it may descend to any height so that it remains on or above the x-axis. Say the initial descent drops down d levels, meaning the path reaches a minimum height of $a-d$ and consists of at least d down steps, where $0 \le d \le a$. It follows that the ascending portion of the path must have at least $b-a+d$ up steps. In the "descent phase", we mark each of the d down steps that is the *first* time the path descends from one height to the next, and in the "ascent phase", we mark each of the $b-a+d$ up steps that is the *last* time the path ascends from one height to the next. As before, each segment of the path that precedes or proceeds a marked step is in the form of a standard Motzkin path. Figure 5 shows what the decomposition looks like for a given value of d. To get the complete construction, we sum over all a possible values of d, resulting in

$$\begin{aligned} \mathcal{M}^{(a,b)} &= \sum_{d=0}^{a} \left((\mathcal{M} \times \mathcal{X})^{d} \times \mathcal{M} \times (\mathcal{X} \times \mathcal{M})^{b-a+d} \right) \\ &\cong \mathcal{M} \times (\mathcal{X} \times \mathcal{M})^{b-a} \times \textsc{Seq}_{\le a}\left((\mathcal{X} \times \mathcal{M})^{2} \right) \\ \implies M^{(a,b)}(x) &= x^{b-a} M^{b-a+1} \left(\frac{1-(xM)^{2a+2}}{1-(xM)^{2}} \right) \end{aligned} \tag{15}$$

for $0 \le a \le b$. Once again, specific values of a and b generate sequences in [10]. $M^{(1,b)}(x)$ generates A026107, A026134, A026109, and A026110 for $b = 1, 2, 3,$ and 4, and $M^{(2,b)}(x)$ generates A026325, A026163, A026327, A026328, and A026329 for $b = 2, 3, 4, 5,$ and 6.

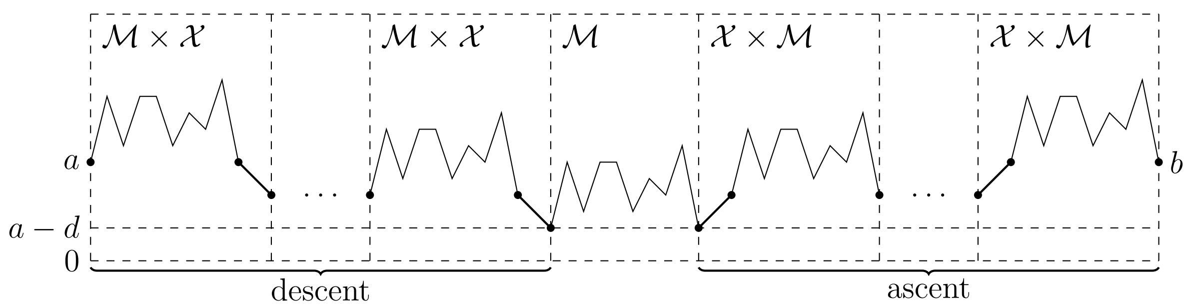

Figure 5: Symbolic construction of Motzkin-like paths that start at height a, descend to minimum height $a-d$, and end at height b, a summand in $\mathcal{M}^{(a,b)}$

If instead, we swap the parameters a and b such that the class of paths start at height b and end at height a, each path in this class is a horizontal reflection of exactly one unique path in the original class, meaning $\mathcal{M}^{(b,a)} \cong \mathcal{M}^{(a,b)} \implies M^{(b,a)}(x) = M^{(a,b)}(x)$. Further generalizations of singly bounded Motzkin-like classes may have an arbitrary bounding line other than the x-axis, which may bound the paths in the class above rather than below. Once again, these new classes would be reflections or translations of and therefore isomorphic to $\mathcal{M}^{(a,b)}$.

4 Unbounded Classes of Motzkin-Like Paths

We now consider the classes of paths without any common bounding lines. While Section 7 of [9] provides an outline for studying such paths using recurrence relations, no explicit formulas are given. In contrast, in Equations (17) and (18) we use the symbolic method to provide novel constructions that yield explicit generating functions.

4.1 Unbounded Classes of Motzkin Meanders

As before, we let $\mathcal{W}$ denote the combinatorial class of such unbounded Motzkin-like paths that start at the origin and end on the x-axis, or the unbounded class of Motzkin meanders that end at height 0. Figure 3 and Equation (12) demonstrate the recursive specification that results from conditioning on the first step of the path. Here, we show an alternate construction of $\mathcal{W}$ that employs a technique also used for bounded Motzkin-like paths in the subsequent section, which results in an explicit specification in terms of $\mathcal{M}$. We condition on the number of times the path takes a down step from height 0 to height -1. If the path never takes such a down step, it is a standard Motzkin path. If the path does take the down step, then it must be followed by an "upside-down" standard Motzkin path, an up step back to the x-axis, and another standard Motzkin path before it takes its next down step to negative heights. The structure resulting from this single down step is symbolized by $\mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{M}$, as illustrated in Figure 6. Since any path in class $\mathcal{W}$ consists of 0 or more of these down steps, $\mathcal{W}$ can be defined in terms of the sequence structure as follows

$$\mathcal{W} = \mathcal{M} \times \textsc{Seq}(\mathcal{X} \times \mathcal{M} \times \mathcal{X} \times \mathcal{M}) \implies W(x) = \frac{M}{1 - x^2M^2} = \frac{1}{1 - x - 2x^2M}. \quad (16)$$

As before, equivalencies from Equation (7) aid in algebraic manipulation to the same expression as in Equation (12).

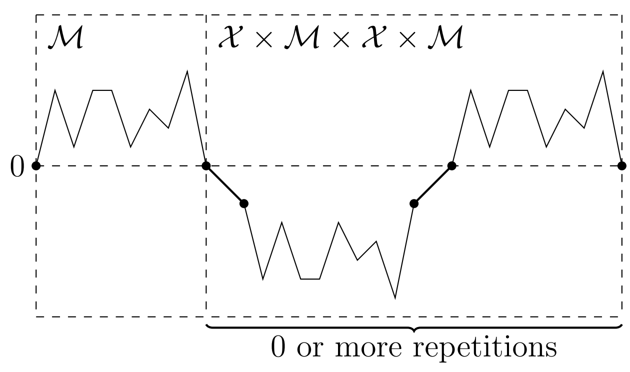

Figure 6: Symbolic construction of the unbounded class of Motzkin-like paths that start and end at height 0, $\mathcal{W}$

For $\mathcal{W}^{(0,a)}$, the unbounded class of Motzkin meanders that end at height a, we first assume $0 \leq a$ and once again observe that the path must have at least a up steps as marked in Figure 7. Before the first marked up step, the path has the form of a Grand Motzkin path from $\mathcal{W}$. After each marked up step, the path has the form of a standard Motzkin path. Symbolically we have

$$\mathcal{W}^{(0,a)} = \mathcal{W} \times (\mathcal{X} \times \mathcal{M})^a \implies W^{(0,a)}(x) = W(xM)^a = \frac{M(xM)^a}{1-(xM)^2}, \tag{17}$$

where $W^{(0,1)}(x), \ldots, W^{(0,5)}(x)$ appear in [10] as A005717, A014531, A014532, A014533, and A098740, respectively.

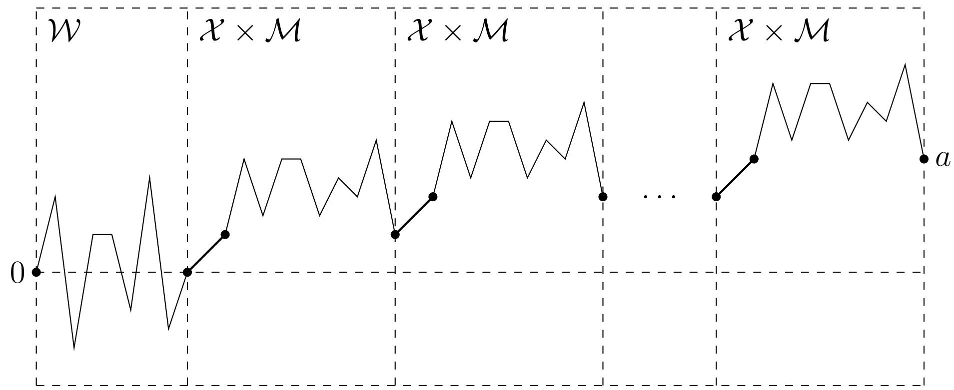

Figure 7: Symbolic construction of the unbounded class of Motzkin meanders that end at height a, $\mathcal{W}^{(0,a)}$

The unbounded class $\mathcal{A}$ of Motzkin meanders can be constructed recursively by conditioning on the first step. An object in $\mathcal{A}$ can be the empty path or one of the three steps followed by another meander. That is, $\mathcal{A} = 1 + 3(\mathcal{X} \times \mathcal{A}) \implies A(x) = 1/(1-3x)$, where $A(x)$ generates the powers of three. Once again, there is an alternate specification in terms of $\mathcal{W}^{(0,a)}$ by summing over all possible end heights for the meanders in $\mathcal{A}$. Keeping in mind that $\mathcal{W}^{(0,a)} \cong \mathcal{W}^{(0,-a)}$, we get

$$\begin{aligned} \mathcal{A} = \sum_{a=-\infty}^{\infty} \mathcal{W}^{(0,a)} &\cong \mathcal{W} + 2\sum_{a=1}^{\infty} \mathcal{W}^{(0,a)} \\ \implies A(x) = W\left(1 + 2\sum_{a=1}^{\infty}(xM)^a\right) &= \frac{(1+xM)}{(1-x-2x^2M)(1-xM)}, \end{aligned} \tag{18}$$

where, amazingly, substitutions from Equation (7) simplify the above expression to $1/(1-3x)$.

4.2 Unbounded General Class

The generalized unbounded class of Motzkin-like paths that start at height a and end at height b, denoted $\mathcal{W}^{(a,b)}$, where $a \leq b$, is a simple translation of the unbounded class of paths that start at the origin. Thus, $\mathcal{W}^{(a,b)} \cong \mathcal{W}^{(b,a)} \cong \mathcal{W}^{(0,b-a)} \implies W^{(a,b)}(x) = W^{(0,b-a)}(x)$.

5 Bounded Classes of Motzkin-Like Paths

We use $\mathcal{M}_{(h_1,h_2)}^{(a,b)}$ to denote the class of Motzkin-like paths starting at height a, ending at height b, bounded below by h_1, and bounded above by h_2. In cases where certain parameters are equal to 0, we define the abbreviations $\mathcal{M}_{(0,h)}^{(a,b)} = \mathcal{M}_h^{(a,b)}$, $\mathcal{M}_{(h_1,h_2)}^{(0,0)} = \mathcal{M}_{(h_1,h_2)}$, and $\mathcal{M}_{(0,h)}^{(0,0)} = \mathcal{M}_h$. These classes are similar to the restricted Motzkin classes defined by Oste and Van der Jeugt in [9], which start with a given number of up steps, are bounded above by a given height, and end with a given number of down steps. These authors define recurrence relations for their classes, which result in continued fraction representations of the corresponding generating functions. Here, we derive explicit specifications and generating functions for our above classes, essentially ignoring the initial a up steps that would take a Motzkin path from height 0 to height a and the final b down steps that would take the path from height b to height 0.

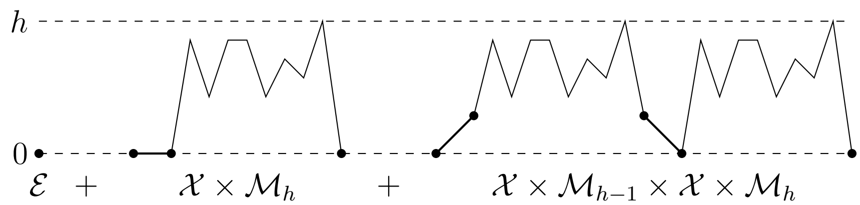

Figure 8: Symbolic construction of the class of standard Motzkin paths bounded below by 0 and above by h, $\mathcal{M}_h$

5.1 Bounded Classes of Motzkin Meanders

A path belonging to $\mathcal{M}_h$ must be a standard Motzkin path with a height of at most h. That is, it can either be the empty path, a horizontal step followed by a standard Motzkin path of height at most h, or an up step followed by a standard Motzkin path of height at most $h-1$ and a down step followed by another standard Motzkin path of height at most h, as illustrated in Figure 8, which results in the specification

$$\begin{aligned} \mathcal{M}_h &= \mathcal{E} + \mathcal{X} \times \mathcal{M}_h + \mathcal{X} \times \mathcal{M}_{h-1} \times \mathcal{X} \times \mathcal{M}_h \\ \implies M_h(x) &= 1 + xM_h(x) + x^2 M_{h-1}(x) M_h(x) \\ \implies M_h(x) &= \frac{1}{1 - x - x^2 M_{h-1}(x)} \end{aligned} \tag{19}$$

for $h \geq 1$. Since there is only one path of height 0 for any length and it consists entirely of horizontal steps, $M_0(x) = 1/(1-x)$. Repeated application of Equation (19) gives the first h levels of the continued fraction

$$M_h(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - x - \cfrac{\ddots}{1 - x - \cfrac{x^2}{1 - x}}}}}. \tag{20}$$

This continued fraction representation of $M_h(x)$, originally due to Flajolet in [6], is well known and also appears as Theorem 10.9.1 in [3], Example V.21 in [7], and Equation (25) in [9]. So, $M_h(x)$ coincides with the first h levels of the continued fraction in Equation (9). In [8] the authors derive an equivalent explicit generating function

$$M_h(x) = \frac{1}{x^2} \frac{M_-^{h+1} - M_+^{h+1}}{M_-^{h+2} - M_+^{h+2}}, \tag{21}$$

revealing the remarkable fact that the *bounded* classes of standard Motzkin paths can be generated from an algebraic combination of powers of both the generating function $M_-(x)$ of the *singly bounded* class of standard Motzkin paths and its conjugate $M_+(x)$ from Equation (8). Employing the same technique as in Identity 235 in [2], which is attributed to Catalan, they use the binomial theorem to expand the powers of M_- and M_+ and then cancel like terms to get the equivalent rational representation

$$M_h(x) = 2\frac{V_h(x)}{V_{h+1}(x)}, \tag{22}$$

where

$$V_h(x) = \sum_{j=0}^{\lfloor h/2 \rfloor} \binom{h+1}{2j+1} (1-x)^{h-2j} \left(1 - 2x - 3x^2\right)^j.$$

The polynomials $V_h(x)$ are closely related to the Chebyshev polynomials of the second kind $U_h(x)$, namely $V_h(x) = (2x)^h U_h(\frac{1-x}{x})$. Furthermore, in [10] $M_1(x), \dots, M_6(x)$ appear as A011782, A171842, A005207, A094286, A094287, A094288, respectively.

With any of these explicit expressions for $M_h(x)$, we can derive explicit generating functions for further generalizations of bounded Motzkin-like paths, whether in terms of the Motzkin roots or ratios of polynomials with integer coefficients. For example, in Figure 9 and Equation (23), $\mathcal{M}_h^{(0,a)}$, the class of Motzkin meanders that end at height a and are bounded by 0 and h, is specified in the same way as the singly bounded $\mathcal{M}^{(0,a)}$ in Figure 4 and Equation (13), except this time as a product of bounded Motzkin classes rather than the standard Motzkin class. A path in $\mathcal{M}_h^{(0,a)}$ must be a standard Motzkin path bounded above by h followed by a repetitions of an up step followed by another bounded standard Motzkin path, where each repetition results in a decreased bounding height. Symbolically, this yields

$$\begin{aligned}\mathcal{M}_h^{(0,a)} &= \mathcal{M}_h \times \prod_{i=1}^{a} (\mathcal{X} \times \mathcal{M}_{h-i}) \cong \mathcal{X}^a \times \prod_{i=0}^{a} \mathcal{M}_{h-i} \\ \implies M_h^{(0,a)}(x) &= \frac{1}{x^{a+2}} \frac{M_-^{h-a+1} - M_+^{h-a+1}}{M_-^{h+2} - M_+^{h+2}}\end{aligned} \tag{23}$$

which has the rational representation

$$M_h^{(0,a)}(x) = 2^{a+1} x^a \frac{V_{h-a}(x)}{V_{h+1}(x)}. \tag{24}$$

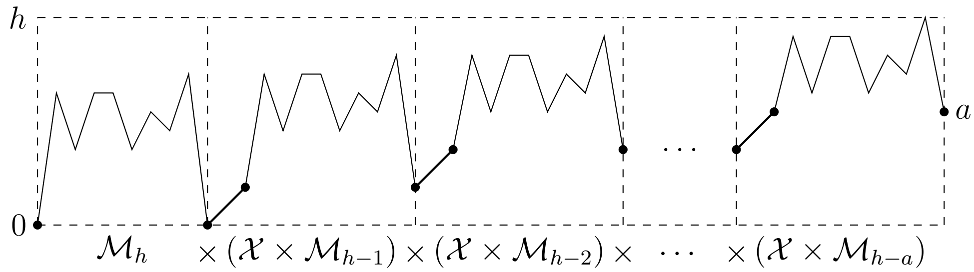

Figure 9: Symbolic construction of the class of Motzkin meanders, bounded below by 0 and above by h, ending at height a, $\mathcal{M}_h^{(0,a)}$

Summing over all h possible end heights for a bounded meander results in the following specification for $\mathcal{N}_h$, the class of Motzkin meanders bounded by 0 and h,

$$\begin{aligned}\mathcal{N}_h &= \sum_{a=0}^{h} \mathcal{M}_h^{(0,a)} \\ \implies N_h(x) &= \sum_{a=0}^{h} M_h^{(0,a)}(x) = \frac{1}{x^2(M_-^{h+2} - M_+^{h+2})} \left(\sum_{a=0}^{h} \frac{M_-^{h-a+1}}{x^a} - \sum_{a=0}^{h} \frac{M_+^{h-a+1}}{x^a} \right) \\ &= \frac{M_-(1-(xM_-)^{h+1})(1-xM_+) - M_+(1-(xM_+)^{h+1})(1-xM_-)}{x^{h+2}(M_-^{h+2} - M_+^{h+2})(1-xM_-)(1-xM_+)},\end{aligned} \tag{25}$$

where $N_h(x)$ is a novel explicit expression for the generating function of bounded meanders in terms of the Motzkin roots. For odd h, it is also true that $N_h(x)$ coincides with the first h levels of the continued fraction representation of $N(x)$ in Equation (11), that is

$$N_h(x) = \cfrac{1}{1 - 3x - \cfrac{x}{1 + \cfrac{x}{1 - 3x - \cfrac{\ddots}{1 + \cfrac{x}{1 - 3x + x}}}}}. \tag{26}$$

Additionally, $N_0(x), \ldots, N_5(x)$ appear in [10] as A000012, A000079, A000129, A122367, A057960, and A085810 respectively. Currently, there is no entry in [10] for the two variable $N_h(x)$.

5.2 Bounded General Classes

We can decompose $\mathcal{M}_h^{(a,b)}$, assuming $0 \le a \le b \le h$, as follows. At a minimum, any path in $\mathcal{M}_h^{(a,b)}$ must be a path that ascends from height a to height b, but it does not necessarily traverse above the line $y = b$. So we can represent the initial portion of the path as one belonging to $\mathcal{M}_b^{(a,b)}$, the class of paths that start at a, end at b, and are bounded by 0 and b. Reflecting $\mathcal{M}_b^{(a,b)}$ across the line $y = b/2$ makes it evident that $\mathcal{M}_b^{(a,b)} \cong \mathcal{M}_b^{(b-a,0)}$, and an additional vertical reflection makes clear that $\mathcal{M}_b^{(a,b)} \cong \mathcal{M}_b^{(0,b-a)}$, which allows use of Equation (23). After this initial portion of the path, we condition on the number of up steps the path takes from height b to height $b+1$, similar to the conditioning on the number of down steps performed in Equation (16). When the path takes such an up step, it must be followed by a path bounded by heights $b+1$ and h, that is, a path from $\mathcal{M}_{h-b-1}$, before taking the first down step that returns to height b. After this, the path may traverse on or below height b before returning to height b, taking the form of an upside-down path from $\mathcal{M}_b$, and then potentially taking another up step, as illustrated in Figure 10. Since the path can take 0 or more of these up steps, it gives rise to the sequence structure as specified below

$$\begin{aligned} \mathcal{M}_h^{(a,b)} &\cong \mathcal{M}_b^{(0,b-a)} \times \textsc{Seq}\left(\mathcal{X} \times \mathcal{M}_{h-b-1} \times \mathcal{X} \times \mathcal{M}_b\right) \\ \implies M_h^{(a,b)}(x) &= \frac{M_b^{(0,b-a)}}{1 - x^2 M_b M_{h-b-1}}. \end{aligned} \tag{27}$$

The above equation implies $M_h^{(a,b)}(x)$ has a novel explicit representation in terms of the Motzkin roots using Equations (21) and (23), which is omitted for brevity. It also must be rational as evident from Equations (22) and (24). In Equation 10.74 of [3] Krattenthaler gives another representation of $M_h^{(a,b)}(x)$ as a ratio of Chebyshev polynomials of the second kind, which have the advantage of having explicitly known zeros, allowing him to obtain a closed-form formula in Equation 10.75 for the coefficients of the generating function.

To further generalize bounded classes of Motzkin-like paths, we can have lower bounds other than $y = 0$. However, every path that starts at height a, ends at height b, is bounded below by h_1 and is bounded above by h_2 is isomorphic to another path that is bounded below by 0 by a simple translation by h_1 units. Specifically,

$$\mathcal{M}_{(h_1,h_2)}^{(a,b)} \cong \mathcal{M}_{h_2-h_1}^{(a-h_1,b-h_1)} \implies M_{(h_1,h_2)}^{(a,b)}(x) = M_{h_2-h_1}^{(a-h_1,b-h_1)}(x) \tag{28}$$

where $h_1 \le a \le b \le h_2$. In [10] $M_{(-1,1)}(x) = M_2^{(1,1)}(x)$ appears as A001333 and $M_{(-2,2)}(x) = M_4^{(2,2)}(x)$ appears as A052948.

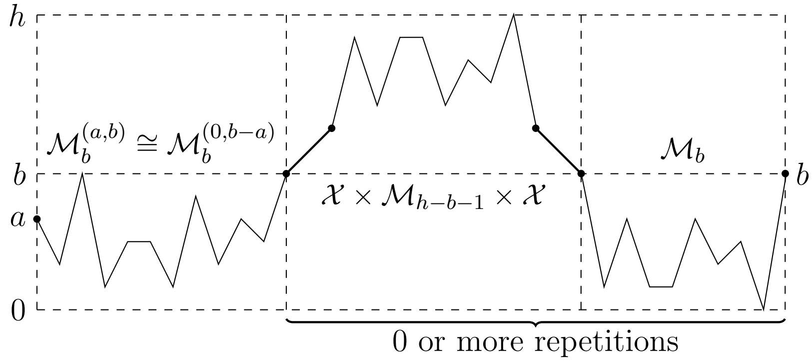

Figure 10: Symbolic construction of the class of Motzkin-like paths that start at height a, end at height b, and are bounded by the x-axis and height h, $\mathcal{M}_h^{(a,b)}$

5.3 Bounded Classes of Motzkin-Like Paths with Exact Maximum Heights

The specifications for bounded classes provide an effortless specification for classes of paths that have exact maximum heights, albeit, the corresponding generating functions become a bit unwieldy. Letting $\mathcal{M}_{(=h)}$ denote the class of standard Motzkin paths that have a maximum height of exactly h, we observe that the union of this class with the class of standard Motzkin paths bounded by $h-1$ must specify the class of standard Motzkin paths that are bounded by h. That is,

$$\begin{aligned} \mathcal{M}_h &= \mathcal{M}_{h-1} + \mathcal{M}_{(=h)} \\ \implies M_{(=h)}(x) &= M_h(x) - M_{h-1}(x) \\ &= \frac{1}{x^2}\frac{\left(M_-^{h+1} - M_+^{h+1}\right)^2 - \left(M_-^{h+2} - M_+^{h+2}\right)\left(M_-^{h} - M_+^{h}\right)}{\left(M_-^{h+2} - M_+^{h+2}\right)\left(M_-^{h+1} - M_+^{h+1}\right)} \end{aligned} \tag{29}$$

where $M_{(=h)}(x)$ is a novel explicit generating function for the hth column of A097862 in [10], which currently documents only a recursively defined generating function. As before, $M_{(=h)}$ has a rational representation using Equation (22). Similarly, the class $\mathcal{N}_{(=h)}$ of Motzkin meanders that remain on or above the x-axis and have a maximum height of exactly h are specified as

$$\mathcal{N}_h = \mathcal{N}_{(=h)} + \mathcal{N}_{h-1} \implies N_{(=h)}(x) = N_h(x) - N_{h-1}(x) \tag{30}$$

where the full expression for $N_{(=h)}(x)$ is omitted for the sake of brevity. $N_{(=h)}(x)$ generates the hth column of A283595 in [10], an entry with a currently undocumented generating function.

6 Classes of Modular Motzkin Paths

To motivate the introduction of a new class of Motzkin-like paths, called *modular Motzkin paths*, we first review a well-known correspondence between Motzkin-like paths and walks on the number line that results from equating the up, down, and horizontal steps of a path to the right, left, and idle steps of a walk. With this correspondence, the graph of the Motzkin-like path plots the position of the walker over time. Since a standard Motzkin path starts at the origin, remains on or above the x-axis, and ends on the x-axis, the corresponding walk must begin at the origin, remain strictly on the nonnegative portion of the number line, and finish at the origin. For other singly bounded classes of Motzkin paths, the corresponding walk must have a fixed position on the number line that the walker may not traverse past. Similarly, the classes of unbounded Motzkin-like paths correspond to walks on the entire number line of integers, and the classes of bounded Motzkin-like paths correspond to walks on a finite line segment.

Instead, consider a walk that takes place on a circle consisting of n positions, where the clockwise, counterclockwise, and idle steps of the circular walk correspond to the up, down, and horizontal steps of a Motzkin-like path. Say the walk starts and ends at position 0. Then a corresponding Motzkin-like path could be a standard Motzkin path that starts and ends at height 0, but it could also be a Motzkin-like path that starts at height 0 and ends at height n, indicating the walker has completed a full revolution of the circle. More generally, this class $\mathcal{C}_n$ of walks on a circle of size n corresponds to the class of Motzkin-like paths that start at height 0 and end at height 0 (mod n). To get the symbolic construction, we sum over all unbounded Motzkin meanders, illustrated in Figure 7, that have an end height that is an integer multiple of n. Keeping in mind that $\mathcal{W}^{(0,a)} \cong \mathcal{W}^{(0,-a)}$, we get

$$\begin{aligned} \mathcal{C}_n &= \sum_{i=-\infty}^{\infty} \mathcal{W}^{(0,in)} \cong \mathcal{W} + 2\sum_{i=1}^{\infty} \mathcal{W}^{(0,in)} \\ \implies C_n(x) &= W\left(1 + \frac{2(xM)^n}{1-(xM)^n}\right) = \frac{M\left(1+(xM)^n\right)}{\left(1-(xM)^2\right)\left(1-(xM)^n\right)}, \end{aligned} \tag{31}$$

where $C_2(x),\ C_3(x),\ C_4(x)$, and $C_6(x)$ appear in [10] as A046717, A133494, A122983, and A087432, respectively. However, [10] does not currently include a tabular entry for the two variable $C_n(x)$. A further generalization of modular Motzkin paths is $\mathcal{C}_n^{(0,a)}$, the class of Motzkin-like paths that start at height 0 and end at height a (mod n). To get the construction, we sum over all Motzkin meanders that are a more than an integer multiple of n. That is,

$$\begin{aligned} \mathcal{C}_n^{(0,a)} &= \sum_{i=-\infty}^{\infty} \mathcal{W}^{(0,in+a)} \cong \sum_{i=0}^{\infty} \mathcal{W}^{(0,in+a)} + \sum_{i=1}^{\infty} \mathcal{W}^{(0,in-a)} \\ \implies C_n^{(0,a)}(x) &= W\left(\frac{(xM)^a + (xM)^{n-a}}{1-(xM)^n}\right) = \frac{M\left((xM)^a + (xM)^{n-a}\right)}{\left(1-(xM)^2\right)\left(1-(xM)^n\right)}, \end{aligned} \tag{32}$$

where $C_2^{(0,1)}(x)$, $C_3^{(0,1)}(x)$, $C_4^{(0,1)}(x)$, $C_4^{(0,2)}(x)$, $C_5^{(0,1)}(x)$, $C_5^{(0,2)}(x)$, $C_6^{(0,1)}(x)$, and $C_6^{(0,2)}(x)$ appear in [10] as A152011, A140429, A015518, A081251, A098703, A094688, A094039, and A004054, respectively. The general class $\mathcal{C}_n^{(a,b)}$ of Motzkin-like paths that start at height a and end at height b (mod n) is isomorphic to $\mathcal{C}_n^{(0,b-a)}$ by translation, assuming

$a \le b$. Classes of modular Motzkin paths may aid in enumerating walks on other types of graphs that contain cycles, such as the pan, tadpole, kayak paddle, Dutch windmill, double cone, and wheel graphs described in [12].

7 Table of Results

Class	Start	End	Low. bnd.	Up. bnd.	Generating function	Eqn
$\mathcal{M}$	0	0	0	∞	$M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$	(8)
$\mathcal{M}^{(0,a)}$	0	a	0	∞	$M^{(0,a)}(x) = x^a M^{a+1}$	(13)
$\mathcal{N}$	0	$[0,\infty)$	0	∞	$N(x) = \frac{1}{1-2x-x^2M} = \frac{2}{1-3x-\sqrt{1-2x-3x^2}}$	(14)
$\mathcal{M}^{(a,b)}$	a	b	0	∞	$M^{(a,b)}(x) = x^{b-a}M^{b-a+1}\left(\frac{1-(xM)^{2a+2}}{1-(xM)^2}\right)$	(15)
$\mathcal{W}$	0	0	$-\infty$	∞	$W(x) = \frac{1}{1-x-2x^2M} = \frac{1}{\sqrt{1-2x-3x^2}}$	(16)
$\mathcal{W}^{(0,a)}$	0	a	$-\infty$	∞	$W^{(0,a)}(x) = \frac{M(xM)^a}{1-(xM)^2}$	(17)
$\mathcal{A}$	0	$(-\infty,\infty)$	$-\infty$	∞	$A(x) = \frac{(1+xM)}{(1-x-2x^2M)(1-xM)} = \frac{1}{1-3x}$	(18)
$\mathcal{M}_h$	0	0	0	h	$M_h(x) = \frac{1}{x^2}\frac{M_-^{h+1}-M_+^{h+1}}{M_-^{h+2}-M_+^{h+2}}$	(21)
$\mathcal{M}_h^{(0,a)}$	0	a	0	h	$M_h^{(0,a)}(x) = \frac{1}{x^{a+2}}\frac{M_-^{h-a+1}-M_+^{h-a+1}}{M_-^{h+2}-M_+^{h+2}}$	(23)
$\mathcal{N}_h$	0	$[0,h]$	0	h	$N_h(x) = \sum_{a=0}^{h} M_h^{(0,a)}(x)$	(25)
$\mathcal{M}_h^{(a,b)}$	a	b	0	h	$M_h^{(a,b)}(x) = \frac{M_b^{(0,b-a)}}{1-x^2M_bM_{h-b-1}}$	(27)
$\mathcal{M}_{(=h)}$	0	0	0	$=h$	$M_{(=h)}(x) = M_h(x) - M_{h-1}(x)$	(29)
$\mathcal{N}_{(=h)}$	0	$[0,h]$	0	$=h$	$N_{(=h)}(x) = N_h(x) - N_{h-1}(x)$	(30)
$\mathcal{C}_n$	0	$0 \pmod n$	$-\infty$	∞	$C_n(x) = \frac{M(1+(xM)^n)}{(1-(xM)^2)(1-(xM)^n)}$	(31)
$\mathcal{C}_n^{(0,a)}$	0	$a \pmod n$	$-\infty$	∞	$C_n^{(0,a)}(x) = \frac{M\left((xM)^a+(xM)^{n-a}\right)}{(1-(xM)^2)(1-(xM)^n)}$	(32)

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References

[1] J. Baril, H. Prodinger, Enumeration of partial Łukasiewicz paths, *Enumer. Combin. Appl.*, **3** (2022), #S2R2, available online at the URL: `https://arxiv.org/abs/2205.01383`

[2] A. Benjamin, J. Quinn, W. Watkins, *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.

[3] M. Bona, *Handbook of Enumerative Combinatorics*, CRC Press, 2015, available online at the URL: `https://doi.org/10.1201/b18255`

[4] A. Bu, Automated counting of restricted Motzkin paths, *Enumer. Combin. Appl.*, **1:2** (2021), #SR12, available online at the URL: `https://arxiv.org/abs/2010.02389`

[5] R. Donaghey, L.W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A*, **23** (1977), 291–301, available online at the URL: `https://www.sciencedirect.com/science/article/pii/0097316577900206`

[6] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Math*, **32** (1980), 125–161, available online at the URL: `https://www.sciencedirect.com/science/article/pii/0012365X80900503`

[7] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009, available online at the URL: `https://ac.cs.princeton.edu/home/AC.pdf`

[8] R. May, J. Vogelpohl, A probabilistic approach to the enumeration of bounded Motzkin paths, *SIAM Undergrad. Res. Online*, **18** (2025), 95–105, available online at the URL: `https://doi.org/10.1137/24S1694586`

[9] R. Oste, J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, *Elect. J. Combin.*, **22** (2015), P2.8, available online at the URL: `https://doi.org/10.37236/4781`

[10] OEIS Foundation Inc. (2025), The On-Line Encyclopedia of Integer Sequences, available online at the URL: `https://oeis.org`

[11] R.A. Sulanke, Bijective recurrence for Motzkin paths, *Adv. in Appl. Math.*, **27** (2001), 627–640, available online at the URL: `https://www.sciencedirect.com/science/article/pii/S0196885801907535`

[12] Wolfram MathWorld, Simple Graphs, available online at the URL: `https://mathworld.wolfram.com/topics/SimpleGraphs.html`

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