Fixed-Term Decompositions Using Even-Indexed Fibonacci Numbers

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Abstract - As a variant of Zeckendorf's theorem, Chung and Graham proved that every positive integer can be uniquely decomposed into a sum of even-indexed Fibonacci numbers, whose coefficients are either $0, 1$, or 2 so that between two coefficients 2, there must be a coefficient 0. This paper characterizes all positive integers that do not have F_{2k} $(k \geq 1)$ in their decompositions. This continues the work of Kimberling, Carlitz et al., Dekking, and Griffiths, to name a few, who studied such a characterization for Zeckendorf decomposition.

Keywords : Zeckendorf decomposition; even-indexed Fibonacci numbers; fixed terms

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1 Introduction

We define the Fibonacci sequence $(F_n)_{n=1}^{\infty}$ as $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Zeckendorf's theorem [28] states that every positive integer can be uniquely written as a sum of nonadjacent Fibonacci numbers from $(F_n)_{n=2}^{\infty}$. The sum is called the *Zeckendorf decomposition* of a positive integer. Note that we start from F_2 since otherwise, $F_1 = F_2 = 1$ ruins uniqueness. Zeckendorf-type decompositions have been extensively studied in the literature: to name a few, see [5, 6, 8, 10, 11, 14, 15, 20, 24, 25] for various generalizations to other sequences, [3, 12, 13, 16, 17, 21, 27] for digits in the decomposition, and [1, 2, 23, 26] for Zeckendorf games.

A beautiful Zeckendorf-type decomposition that uses even-indexed Fibonacci numbers only is due to Chung and Graham [9]:

Theorem 1.1 *[9, Lemma 1] Every positive integer n can be uniquely represented as a sum $n = \sum_{i \geq 1} c_i F_{2i}$, where c_i's are in $\{0, 1, 2\}$ so that if $c_i = c_j = 2$ with $i < j$, then for some k, $i < k < j$, we have $c_k = 0$.*

We call the decomposition in Theorem 1.1 the *Chung-Graham decomposition* of an integer. This paper answers the following question.

Question 1.2 Given an even-indexed Fibonacci number F_{2N}, $N \geq 1$, what are the positive integers whose Chung-Graham decomposition contains neither F_{2N} nor $2F_{2N}$?

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The analog of Question 1.2 is well-known for Zeckendorf decomposition. Kimberling [22] studied numbers without 1 in their decomposition. Another pioneering paper is due to Carlitz et al. [4] who described the set $Z(N)$ of all positive integers having the summand F_N in their Zeckendorf decomposition, depending on the parity of N. Later, Griffiths [19] gave

$$Z(N) = \left\{ F_N \left\lfloor \frac{n+\phi^2}{\phi} \right\rfloor + nF_{N+1} + j \, : \, 0 \leq j \leq F_{N-1} - 1, n \geq 0 \right\}.$$

Dekking [13] characterized all integers that share the same *initial* Zeckendorf decomposition, using the so-called compound Wythoff sequences and generalized Beatty sequences.

It is worth mentioning that Griffiths' analysis [19] can also be used to determine all positive integers having $\{F_N : N \in A\}$ for some certain sets A in their Zeckendorf decomposition. The idea is to analyze consecutive rows of the table of all numbers having F_N as the minimum summand in their Zeckendorf decomposition and employed properties of the golden string, which we shall discuss in Section 2. Recently, Chu [7] generalized the golden string to study a generalized Zeckendorf decomposition.

In the present paper, we answer Question 1.2 using the same method as in [7, 19] while dealing with a considerably more involved table due to the appearance of the coefficient 2 in the Chung-Graham decomposition. In the process, we need to utilize more properties of the golden string (see Propositions 2.2 and 2.4, for example). We state our main result.

Theorem 1.3 *For $N \geq 1$, the set of all positive integers that do not have F_{2N} nor $2F_{2N}$ in their Chung-Graham decomposition is given by*

$$\begin{aligned} B_{2N} &:= [1, F_{2N} - 1] \cup \\ & \bigcup_{k=N+1}^{\infty} \left\{ j + F_{2k}, j + (n+2)F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k-1} \, : \, 0 \leq j \leq F_{2N} - 1, n \geq 0 \right\}, \end{aligned}$$

where $\phi = (\sqrt{5}+1)/2$.

For example, we list integers at most 30 that belong to the following sets

$$\begin{aligned} B_2 &= \{3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, \ldots\}, \\ B_4 &= \{1, 2, 8, 9, 10, 16, 17, 18, 21, 22, 23, 29, 30\}, \\ B_6 &= \{1, 2, 3, 4, 5, 6, 7, 21, 22, 23, 24, 25, 26, 27, 28\}. \end{aligned}$$

To facilitate our writing, we introduce some notation that distinguish the two coefficients 1 and 2. Given $n \in \mathbb{N}$, let $\mathcal{CG}(n)$ denote the set of all Fibonacci numbers in the Chung-Graham decomposition of n. Let $\mathcal{CG}_1(n)$ be the set of all numbers in $\mathcal{CG}(n)$ that have coefficient 1 in the Chung-Graham decomposition of n, and let $\mathcal{CG}_2(n) := \mathcal{CG}(n) \backslash \mathcal{CG}_1(n)$ be the set of all numbers in $\mathcal{CG}(n)$ that have coefficient 2. For example,

$$\begin{aligned} \mathcal{CG}(2F_2 + F_4 + 2F_8 + F_{14}) &= \{F_2, F_4, F_8, F_{14}\}, \\ \mathcal{CG}_1(2F_2 + F_4 + 2F_8 + F_{14}) &= \{F_4, F_{14}\}, \text{ and} \\ \mathcal{CG}_2(2F_2 + F_4 + 2F_8 + F_{14}) &= \{F_2, F_8\}. \end{aligned}$$

For example, $\max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{14}$ means that the largest Fibonacci number that appears in the Chung-Graham decomposition of n is F_{14} whose coefficient is 1.

The paper is structured as follows: in Section 2, we define the golden string $\mathcal{S}$ and collect several properties that will be used in due course; Section 3 investigates the ordered list of integers having F_{2k} as the smallest Fibonacci number in their Chung-Graham decomposition; finally, Section 4 gathers some auxiliary results before proving Theorem 1.3.

2 The Golden String

For two finite strings of symbols X and Y, we write $X : Y$ to mean the concatenation of X and Y. The golden string, denoted by $\mathcal{S}$, is an infinite string consisting of the letters A and B, built recursively as follows: $S_1 = B$, $S_2 = BA$, and $S_k = S_{k-1} : S_{k-2}$ for $k \geq 3$. For example,

$$\begin{aligned} S_3 &= S_2 : S_1 = BAB, \\ S_4 &= S_3 : S_2 = BABBA, \\ S_5 &= S_4 : S_3 = BABBABAB. \end{aligned}$$

The first few letters of $\mathcal{S}$ are

$$BABBABABBABBABABBABAB\ldots.$$

We record several properties of $\mathcal{S}$ and $(S_n)_{n=1}^{\infty}$.

Lemma 2.1 *The following are true.*

a) *For $n \in \mathbb{N}$, the length of S_n, denoted by $|S_n|$, is equal to F_{n+1}.*

b) *The substring S_n gives the first F_{n+1} letters of $\mathcal{S}$.*

c) *For each $n \geq 2$, the substring consisting of the first F_n letters of $\mathcal{S}$ is the same as the substring consisting of all the letters of $\mathcal{S}$ between the $F_{n+1} + 1^{th}$ and the $F_{n+2}{}^{th}$ positions, inclusively.*

d) *If $n = F_{c_1} + F_{c_2} + \cdots + F_{c_\ell}$ is the Zeckendorf decomposition of n, then $S_{c_k} : S_{c_{k-1}} : \cdots : S_{c_1}$ gives the first n letters of the golden string.*

e) *If $N_B(n)$ denotes the number of B's in the first n letters of $\mathcal{S}$, then*

$$N_B(n) = \left\lfloor \frac{n+1}{\phi} \right\rfloor, \tag{1}$$

where $\phi = (1 + \sqrt{5})/2$, the golden ratio.

f) *For $m \in \mathbb{N}$,*

(f1) *the* $(F_{2m+1}-1)^{th}$ *letter of* $\mathcal{S}$ *is* B.

(f2) *the* ${F_{2m+1}}^{th}$ *letter of* $\mathcal{S}$ *is* A.

(f3) *the* $(2F_{2m+1}-1)^{th}$ *letter of* $\mathcal{S}$ *is* B.

(f4) *if* $m \geq 2$, *the* $2{F_{2m+1}}^{th}$ *letter of* $\mathcal{S}$ *is* A.

Proof. Property a) and b) follow from the construction of the golden string $\mathcal{S}$.

Property c) is immediate from the following:

- by Property b), the first F_n letters of $\mathcal{S}$ are given by S_{n-1};
- also by Property b), the first F_{n+2} letters of $\mathcal{S}$ are given by

$$S_{n+1} \;:=\; S_n : S_{n-1}; \text{ and}$$

- by Property a), $|S_n| = F_{n+1}$.

Property d) is [18, Lemma 3.2].

Property e) is [18, Lemma 3.3].

We prove (f1) and (f2) by induction. For the base case $m = 1$, $F_{2m+1} = 2$, and the first and second letters of $\mathcal{S}$ are B and A, respectively. Assume that for some $k \geq 1$, the $(F_{2k+1}-1)^{\text{th}}$ letter of $\mathcal{S}$ is B, and the ${F_{2k+1}}^{\text{th}}$ letter of $\mathcal{S}$ is A. By construction, an initial segment of $\mathcal{S}$ is

$$\underbrace{S_{2k}}_{\text{length } F_{2k+1}} \;:\; \underbrace{S_{2k-1}}_{\text{length } F_{2k}} \;:\; \underbrace{S_{2k}}_{\text{length } F_{2k+1}} .$$

Therefore, the $(F_{2k+3}-1)^{\text{th}}$ and ${F_{2k+3}}^{\text{th}}$ letters of $\mathcal{S}$ are the same as the $(F_{2k+1}-1)^{\text{th}}$ and ${F_{2k+1}}^{\text{th}}$ letters of $\mathcal{S}$, which are B and A, respectively.

We can verify that (f3) holds for $m = 1$. For $m \geq 2$, (f3) and (f4) are true due to (f1), (f2), and Proposition 2.2 below. □

Proposition 2.2 *For* $n \geq 4$, *the substring of* $\mathcal{S}$ *consisting of the first* F_n *letters of* $\mathcal{S}$ *is the same as the substring of* $\mathcal{S}$ *consisting of the next* F_n *letters.*

Proof. Fix $n \geq 4$. Since for $k < \ell$, S_k gives the initial letters of S_ℓ, we can write $S_{n-1} = S_{n-3} : L$ for some finite string L. We have

$$\begin{aligned} S_{n+1} \;=\; S_n : S_{n-1} \;&=\; (S_{n-1} : S_{n-2}) : (S_{n-3} : L) \\ &=\; S_{n-1} : (S_{n-2} : S_{n-3}) : L \\ &=\; S_{n-1} : S_{n-1} : L. \end{aligned}$$

Since S_{n+1} gives the initial letters of $\mathcal{S}$ and $|S_{n-1}| = F_n$, we are done. □

We shall use the following notation. For a string W, we write $W - 2$ to mean the string formed by deleting the last two letters of W.

Lemma 2.3 *For $n \geq 1$, we have*

$$S_n : S_{n+1} - 2 \ = \ S_{n+1} : S_n - 2.$$

Proof. We prove by induction. The equality is true for $n = 1$. Inductive hypothesis: suppose that it is true for $n = \ell \geq 1$. We show that

$$S_{\ell+1} : S_{\ell+2} - 2 \ = \ S_{\ell+2} : S_{\ell+1} - 2.$$

We have

$$\begin{aligned} S_{\ell+2} : S_{\ell+1} - 2 \ &= \ (S_{\ell+1} : S_\ell) : S_{\ell+1} - 2 \ = \ S_{\ell+1} : (S_\ell : S_{\ell+1} - 2) \\ &= \ S_{\ell+1} : (S_{\ell+1} : S_\ell - 2) \ = \ S_{\ell+1} : S_{\ell+2} - 2. \end{aligned}$$

□

Proposition 2.4 *For $n \geq 5$, the substring of $\mathcal{S}$ consisting of the first $(F_{n-1} - 2)$ letters is the same as the substring of $\mathcal{S}$ consisting of all the letters between the $(2F_n + 1)^{th}$ letter and the $(F_{n+2} - 2)^{th}$ letter, inclusively.*

Proof. Pick $n \geq 5$. We have

$$\begin{aligned} S_{n+1} \ = \ S_n : S_{n-1} \ &= \ S_{n-1} : S_{n-2} : S_{n-2} : S_{n-3} \\ &= \ S_{n-1} : S_{n-2} : S_{n-3} : S_{n-4} : S_{n-3} \\ &= \ S_{n-1} : S_{n-1} : S_{n-4} : S_{n-3}. \end{aligned}$$

Observe that $|S_{n-1}| = F_n$ and $|S_{n-4} : S_{n-3}| = F_{n-3} + F_{n-2} = F_{n-1}$. Hence, the substring of $\mathcal{S}$ consisting of all the letters between the $(2F_n + 1)^{\text{th}}$ letter and the $(F_{n+2} - 2)^{\text{th}}$ letter, inclusively is

$$W \ := \ S_{n-4} : S_{n-3} - 2.$$

By Lemma 2.3,

$$W \ = \ S_{n-3} : S_{n-4} - 2 \ = \ S_{n-2} - 2,$$

which gives the first $F_{n-1} - 2$ letters of $\mathcal{S}$. □

3 The Ordered List of Positive Integers n With $F_{2k} = \min \mathcal{CG}(n)$

For $k \geq 1$, let $A_{2k} = \{n : F_{2k} = \min \mathcal{CG}(n)\}$. We form a table whose rows are numbers in A_{2k} arranged in increasing order $q(1) < q(2) < q(3) < \cdots$. Let us look at the first few rows of the table.

$q(1)$	F_{2k}		
$q(2)$	$2F_{2k}$		
$q(3)$	F_{2k}	F_{2k+2}	
$q(4)$	$2F_{2k}$	F_{2k+2}	
$q(5)$	F_{2k}	$2F_{2k+2}$	
$q(6)$	F_{2k}		F_{2k+4}
$q(7)$	$2F_{2k}$		F_{2k+4}
$q(8)$	F_{2k}	F_{2k+2}	F_{2k+4}
$q(9)$	$2F_{2k}$	F_{2k+2}	F_{2k+4}
$q(10)$	F_{2k}	$2F_{2k+2}$	F_{2k+4}
$q(11)$	F_{2k}		$2F_{2k+4}$
$q(12)$	$2F_{2k}$		$2F_{2k+4}$
$q(13)$	F_{2k}	F_{2k+2}	$2F_{2k+4}$
$\vdots$			

Table 1. The numbers in A_{2k} in increasing order

This section shows a way to construct new rows of Table 1 recursively. First, we prove that a number with a larger maximum Fibonacci number in its Chung-Graham decomposition belongs to a lower row. Consequently, the numbers n with the same $\max \mathcal{CG}(n)$ form consecutive rows in the table.

Lemma 3.1 *For $m \geq 0$, the largest positive integer $n \in A_{2k}$ with $\max \mathcal{CG}(n) = F_{2k+2m}$, denoted by $N(m)$, is*

$$F_{2k} + F_{2k+2} + \cdots + F_{2k+2m-2} + 2F_{2k+2m}.$$

Proof. Write $N(m)$ as $\sum_{i=0}^{m} c_i F_{2k+2i}$, where the c_i's are in $\{0, 1, 2\}$ and satisfy the Chung-Graham condition. Suppose, for a contradiction, that $c_j = 0$ for some $1 \leq j \leq m-1$. If changing c_j to 1 does not violate the Chung-Graham condition, then $N(m) + F_{2k+2j}$ is greater than $N(m)$, while $\max \mathcal{CG}(N(m) + F_{2k+2j}) = F_{2k+2m}$. This contradicts the maximality of $N(m)$. Hence, changing c_j to 1 violates the Chung-Graham condition; that is, there are $1 \leq j' < j < j'' \leq m$ such that $c_{j'} = c_{j''} = 2$. Here we choose the largest j' and the smallest j'' that satisfy these conditions. We change both $c_{j'}$ and c_j to 1. Then the new coefficients still satisfy the Chung-Graham condition, but since

$$F_{2k+2j'} + F_{2k+2j} \;>\; 2F_{2k+2j'} + 0F_{2k+2j},$$

the new number is greater than $N(m)$. This again contradicts the maximality of $N(m)$. Therefore, $c_i \geq 1$ for all i, which clearly implies that

$$N(m) \;=\; F_{2k} + F_{2k+2} + \cdots + F_{2k+2m-2} + 2F_{2k+2m},$$

as desired. □

Corollary 3.2 *Given $n, m \in A_{2k}$, if $\max \mathcal{CG}(n) < \max \mathcal{CG}(m)$, then $n < m$.*

Proof. Suppose that $\max \mathcal{CG}(n) = F_{2k+2j_1}$ and $\max \mathcal{CG}(m) = F_{2k+2j_2}$ with $j_1 < j_2$. Then $m \geq F_{2k} + F_{2k+2j_2}$. By Lemma 3.1,

$$\begin{aligned} n &\leq F_{2k} + F_{2k+2} + \cdots + F_{2k+2j_1-2} + 2F_{2k+2j_1} \\ &= (F_{2k-1} + F_{2k} + F_{2k+2} + \cdots + F_{2k+2j_1-2}) + 2F_{2k+2j_1} - F_{2k-1} \\ &= F_{2k+2j_1-1} + 2F_{2k+2j_1} - F_{2k-1} \;=\; F_{2k+2j_1+2} - F_{2k-1} \;<\; F_{2k+2j_2} \;<\; m. \end{aligned}$$

□

Thanks to Corollary 3.2, we know that the numbers having $F_{2k+2\ell}$ ($\ell \geq 1$) with either coefficient 1 or 2 as the maximum Fibonacci number in its decomposition form consecutive rows in Table 1. The next result tells us their location.

Lemma 3.3 *For $\ell \geq 1$, the numbers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2\ell}\}$ lie between the $(F_{2\ell+1}+1)^{th}$ and the ${F_{2\ell+3}}^{th}$ rows, inclusively.*

Proof. We prove by induction. Base case: it is easy to verify that the lemma holds for $\ell = 1$. Inductive hypothesis: suppose the lemma is true for $\ell \leq m$ for some $m \geq 1$. We prove that the lemma holds for $\ell = m+1$; that is, the numbers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2m+2}\}$ lie between the $(F_{2m+3}+1)^{\text{th}}$ to the ${F_{2m+5}}^{\text{th}}$ rows, inclusively. By the inductive hypothesis,

$$\begin{aligned} &|\{n \in A_{2k} : \max \mathcal{CG}(n) \leq F_{2k+2m}\}| \\ &= \sum_{\ell=0}^{m} |\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2\ell}\}| \\ &= 2 + \sum_{\ell=1}^{m} |\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2\ell}\}| \\ &= F_3 + \sum_{\ell=1}^{m} (F_{2\ell+3} - F_{2\ell+1}) \;=\; F_3 + \sum_{\ell=1}^{m} F_{2\ell+2} \;=\; F_{2m+3}. \end{aligned}$$

Therefore, the numbers with the largest summand $F_{2k+2m+2}$ start at the $(F_{2m+3}+1)^{\text{th}}$ row.

It suffices to prove that

$$|\{n \in A_{2k} \,:\, \max \mathcal{CG}(n) = F_{2k+2m+2}\}| \;=\; F_{2m+4}.$$

Note that

$$\begin{aligned} &\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2m+2}\} \\ &= \{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2m+2}\} \\ &\quad \cup \{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2m+2}\}. \end{aligned}$$

Let

$$\begin{aligned} A_{2k,m,1} &:= \{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2m+2}\} \text{ and} \\ A_{2k,m,2} &:= \{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2m+2}\}. \end{aligned}$$

Since all numbers in $A_{2k,m,1}$ are created by adding $F_{2k+2m+2}$ to the numbers in $\{n \in A_{2k} : \max \mathcal{CG}(n) \le F_{2k+2m}\}$, we know that $|A_{2k,m,1}| = F_{2m+3}$.

It remains to show $|A_{2k,m,2}| = F_{2m+2}$. All numbers in $A_{2k,m,2}$ are formed by adding $2F_{2k+2m+2}$ to a subset of $\{n \in A_{2k} : \max \mathcal{CG}(n) \le F_{2k+2m}\}$. Pick $s \in \{n \in A_{2k} : \max \mathcal{CG}(n) \le F_{2k+2m}\}$ such that $s + 2F_{2k+2m+2} \in A_{2k,m,2}$. Equivalently, if the Chung-Graham decomposition of s is $\sum_{i=0}^{m} c_i F_{2k+2i}$, then $(c_0, c_1, \dots, c_m)$ must have one of the following forms, based on the largest i (if any) with $c_i = 0$:

$$\begin{array}{rl} f_1 & (c_0, c_1, \dots, c_{m-3}, c_{m-2}, c_{m-1}, 0) \\ f_2 & (c_0, c_1, \dots, c_{m-3}, c_{m-2}, 0, 1) \\ f_3 & (c_0, c_1, \dots, c_{m-3}, 0, 1, 1) \\ \vdots & \\ f_m & (c_0, 0, \dots, 1, 1, 1, 1) \\ f_{m+1} & (1, 1, \dots, 1, 1, 1, 1). \end{array}$$

For $1 \le i \le m-1$, the number of s having the form f_i is equal to $|\{n \in A_{2k} : \max \mathcal{CG}(n) \le F_{2k+2m-2i}\}|$, which, by the inductive hypothesis, is

$$\sum_{j=0}^{m-i} |\{n \in A_{2k} : \max \mathcal{CG}(n) = F_{2k+2j}\}| \;=\; 2 + \sum_{j=1}^{m-i} F_{2j+2} \;=\; F_{2m-2i+3}.$$

The number of s having the form f_m and f_{m+1} is 2 and 1, respectively. Therefore,

$$|A_{2k,m,2}| \;=\; \sum_{i=1}^{m-1} F_{2m-2i+3} + 2 + 1 \;=\; F_4 + F_5 + \cdots + F_{2m+1} \;=\; F_{2m+2}.$$

□

The proof of Lemma 3.3 also reveals the counts of numbers having $F_{2k+2\ell}$ and $2F_{2k+2\ell}$ as the maximum summand in their decomposition.

Corollary 3.4 *For $\ell \ge 1$, we have*

$$\begin{aligned} |\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\}| &= F_{2\ell+1}, \text{ and} \\ |\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\}| &= F_{2\ell}. \end{aligned}$$

We now show that the numbers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\}$ are larger than the numbers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\}$.

Lemma 3.5 *For $\ell \ge 0$,*

$$\begin{aligned} & \min\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\} \\ > \; & \max\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\}. \end{aligned}$$

Proof. When $\ell = 0$, we are comparing F_{2k} and $2F_{2k}$, and the lemma is obviously true. Suppose that $\ell \geq 1$. Since $\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\}$ is formed by adding $F_{2k+2\ell}$ to each number in $\{n \in A_{2k} : \max \mathcal{CG}(n) \leq F_{2k+2\ell-2}\}$, Lemma 3.1 gives

$$\begin{aligned}\max\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\} \\ = F_{2k} + F_{2k+2} + \cdots + F_{2k+2\ell-4} + 2F_{2k+2\ell-2} + F_{2k+2\ell}.\end{aligned}$$

On the other hand,

$$\min\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\} = F_{2k} + 2F_{2k+2\ell}.$$

We need only to verify that

$$F_{2k} + 2F_{2k+2\ell} > F_{2k} + F_{2k+2} + \cdots + F_{2k+2\ell-4} + 2F_{2k+2\ell-2} + F_{2k+2\ell}.$$

Equivalently,

$$F_{2k+1} + F_{2k+2\ell} > F_{2k+1} + F_{2k+2} + \cdots + F_{2k+2\ell-4} + 2F_{2k+2\ell-2},$$

which is true because the right side of the inequality is equal to $F_{2k+2\ell}$. □

We have used in the proof of Lemmas 3.3 and 3.5 the fact that for $\ell \geq 1$,

$$\begin{aligned}&\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\} \\ &= \{n \in A_{2k} : \max \mathcal{CG}(n) \leq F_{2k+2\ell-2}\} + F_{2k+2\ell}.\end{aligned}$$

In other words, integers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\}$ are formed by adding $F_{2k+2\ell}$ to all previous rows in Table 1. We now describe how to form integers in $\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\}$.

Lemma 3.6 *For $\ell \geq 1$, we have*

$$\begin{aligned}&\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\} \\ &= \left\{n \in A_{2k} : n \leq \sum_{i=0}^{\ell-1} F_{2k+2i}\right\} + 2F_{2k+2\ell}.\end{aligned}$$

Proof. By Lemma 3.1, the largest integer $n \in A_{2k}$ with $\max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}$ is

$$\sum_{i=0}^{\ell-1} F_{2k+2i} + 2F_{2k+2\ell};$$

hence,

$$\begin{aligned}&\{n \in A_{2k} : \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\} \\ &\subset \left\{n \in A_{2k} : n \leq \sum_{i=0}^{\ell-1} F_{2k+2i}\right\} + 2F_{2k+2\ell}.\end{aligned}$$

We prove the reverse inclusion. Pick $n \in A_{2k}$ such that

$$n \;\leq\; \sum_{i=0}^{\ell-1} F_{2k+2i}. \tag{2}$$

Write the Chung-Graham decomposition of n as

$$n \;=\; c_0 F_{2k} + c_1 F_{2k+2} + \cdots + c_{\ell-1} F_{2k+2\ell-2},$$

for $c_i \in \{0,1,2\}$. Suppose, for a contradiction, that

$$c_0 F_{2k} + c_1 F_{2k+2} + \cdots + c_{\ell-1} F_{2k+2\ell-2} + 2F_{2k+2\ell}$$

is not a Chung-Graham decomposition. Then there is $0 \leq j \leq \ell - 1$ such that $c_j = 2$ and $c_i = 1$ for all $i \in [j+1, \ell-1]$. It follows that

$$n \;\geq\; 2F_{2k+2j} + \sum_{i=j+1}^{\ell-1} F_{2k+2i} \tag{3}$$

From (2) and (3),

$$\sum_{i=0}^{\ell-1} F_{2k+2i} \;\geq\; 2F_{2k+2j} + \sum_{i=j+1}^{\ell-1} F_{2k+2i}.$$

Equivalently,

$$\sum_{i=0}^{j} F_{2k+2i} \;\geq\; 2F_{2k+2j},$$

which is a contradiction. Hence,

$$c_0 F_{2k} + c_1 F_{2k+2} + \cdots + c_{\ell-1} F_{2k+2\ell-2} + 2F_{2k+2\ell}$$

is a Chung-Graham decomposition. Therefore,

$$n + 2F_{2k+2\ell} \in \{n \in A_{2k} \,:\, \max \mathcal{CG}(n) = \max \mathcal{CG}_2(n) = F_{2k+2\ell}\}.$$

□

4 Proof of the Main Theorem

Using what we know about Table 1 from Section 3, we now prove an identity that equates the difference between two earlier consecutive rows with the difference between two later consecutive rows in the table.

Proposition 4.1 *Fix $\ell \geq 1$. For $1 + F_{2\ell+1} \leq j \leq F_{2\ell+3} - 1$, we have*

$$q(j+1) - q(j) \;=\; q(j - F_{2\ell+1} + 1) - q(j - F_{2\ell+1}). \tag{4}$$

Proof. Choose $\ell \geq 1$. By Lemma 3.3, the numbers in $\{n : \max \mathcal{CG}(n) \leq F_{2k+2\ell-2}\}$ lie from the 1st row to the $F_{2\ell+1}$th row, inclusively. Since

$$\begin{aligned}\{n : \max \mathcal{CG}(n) \leq F_{2k+2\ell-2}\} &+ F_{2k+2\ell} \\ &= \{n : \max \mathcal{CG}(n) = \max \mathcal{CG}_1(n) = F_{2k+2\ell}\},\end{aligned}$$

we know that

$$q(i+1) - q(i) \;=\; q(i + F_{2\ell+1} + 1) - q(i + F_{2\ell+1}) \tag{5}$$

whenever $1 \leq i \leq F_{2\ell+1} - 1$. Applying the change of variable $j = i + F_{2\ell+1}$, we obtain

$$q(j+1) - q(j) \;=\; q(j - F_{2\ell+1} + 1) - q(j - F_{2\ell+1}), 1 + F_{2\ell+1} \leq j \leq 2F_{2\ell+1} - 1. \tag{6}$$

Furthermore, by Corollary 3.4 and Lemmas 3.5 and 3.6,

$$q(i+1) - q(i) \;=\; q(i + 2F_{2\ell+1} + 1) - q(i + 2F_{2\ell+1}), 1 \leq i \leq F_{2\ell} - 1. \tag{7}$$

From (5) and (7),

$$q(i + F_{2\ell+1} + 1) - q(i + F_{2\ell+1}) \;=\; q(i + 2F_{2\ell+1} + 1) - q(i + 2F_{2\ell+1}), 1 \leq i \leq F_{2\ell} - 1.$$

Using the change of variable $j = i + 2F_{2\ell+1}$, we have

$$q(j+1) - q(j) = q(j - F_{2\ell+1} + 1) - q(j - F_{2\ell+1}), 1 + 2F_{2\ell+1} \leq j \leq F_{2\ell+3} - 1. \tag{8}$$

Thanks to (6) and (8), it remains to verify (4) when $j = 2F_{2\ell+1}$; that is,

$$q(2F_{2\ell+1} + 1) - q(2F_{2\ell+1}) \;=\; q(F_{2\ell+1} + 1) - q(F_{2\ell+1}). \tag{9}$$

By Lemmas 3.1, 3.3, 3.5, and Corollary 3.4,

$$\begin{aligned} q(F_{2\ell+1}) &= \sum_{i=0}^{\ell-2} F_{2k+2i} + 2F_{2k+2\ell-2}, \\ q(F_{2\ell+1} + 1) &= F_{2k} + F_{2k+2\ell}, \\ q(2F_{2\ell+1}) &= F_{2k} + F_{2k+2} + \cdots + F_{2k+2\ell-4} + 2F_{2k+2\ell-2} + F_{2k+2\ell}, \\ q(2F_{2\ell+1} + 1) &= F_{2k} + 2F_{2k+2\ell}. \end{aligned}$$

These confirm (9), and we are done. □

We are now ready to prove the following key lemma to describe all integers in A_{2k}.

Lemma 4.2 *For $j \geq 2$,*

$$q(j+1) - q(j) \;=\; \begin{cases} F_{2k} \text{ if the } (j-1)^{th} \text{ letter of } \mathcal{S} \text{ is } A, \\ F_{2k+1} \text{ if the } (j-1)^{th} \text{ letter of } \mathcal{S} \text{ is } B. \end{cases} \tag{10}$$

Proof. It suffices to prove that (10) is true for all $j \le F_{2m+1}-1$ for any arbitrary $m \in \mathbb{N}$. We do so by induction.

Base case: for $m = 3$, we can see from Table 1 that (10) is true for all $j \le F_7 - 1$.

Inductive hypothesis: suppose that (10) is true for $j \le F_{2m+1} - 1$ for some $m \ge 3$. We need to show that it is true for all $j \le F_{2m+3} - 1$. We proceed by a case analysis.

Case 1: $j = F_{2m+1}$. By Lemmas 3.1 and 3.3, we have

$$\begin{aligned} q(F_{2m+1}+1) - q(F_{2m+1}) &= F_{2k} + F_{2k+2m} - (F_{2k} + \cdots + F_{2k+2m-4} + 2F_{2k+2m-2}) \\ &= F_{2k+1}. \end{aligned}$$

By Property f) in Section 2, (10) is true when $j = F_{2m+1}$.

Case 2: $j = F_{2m+1} + 1$. By Lemma 3.3,

$$q(F_{2m+1}+2) - q(F_{2m+1}+1) = (2F_{2k} + F_{2k+2m}) - (F_{2k} + F_{2k+2m}) = F_{2k}.$$

By Property f) in Section 2, (10) is true when $j = F_{2m+1} + 1$.

Case 3: $F_{2m+1} + 2 \le j \le 2F_{2m+1} - 1$. It follows from Proposition 4.1 that

$$q(j+1) - q(j) = q(j+1-F_{2m+1}) - q(j - F_{2m+1}).$$

Thanks to Proposition 2.2 and the fact that

$$j - F_{2m+1} \le 2F_{2m+1} - 1 - F_{2m+1} = F_{2m+1} - 1,$$

the inductive hypothesis guarantees that (10) is true for $F_{2m+1} + 2 \le j \le 2F_{2m+1} - 1$.

Case 4: $j = 2F_{2m+1}$. It follows from Lemmas 3.1, 3.3, 3.5, and Corollary 3.4 that

$$\begin{aligned} & q(2F_{2m+1}+1) - q(2F_{2m+1}) \\ &= (F_{2k} + 2F_{2k+2m}) - (F_{2k} + \cdots + 2F_{2k+2m-2} + F_{2k+2m}) = F_{2k+1}. \end{aligned}$$

Property f) in Section 2 confirms that (10) is true for $j = 2F_{2m+1}$.

Case 5: $j = 2F_{2m+1} + 1$. By Lemmas 3.3, 3.5, and Corollary 3.4,

$$q(2F_{2m+1}+2) - q(2F_{2m+1}+1) = (2F_{2k} + 2F_{2k+2m}) - (F_{2k} + 2F_{2k+2m}) = F_{2k}.$$

Property f) in Section 2 confirms that (10) is true for $j = 2F_{2m+1} + 1$.

Case 6: $2F_{2m+1} + 2 \le j \le F_{2m+3} - 1$. According to Proposition 4.1,

$$q(j+1) - q(j) = q(j+1-2F_{2m+1}) - q(j - 2F_{2m+1}).$$

Since

$$j - 2F_{2m+1} \le F_{2m+3} - 1 - 2F_{2m+1} = F_{2m} - 1,$$

the inductive hypothesis can be applied. Together with Proposition 2.4, we know that (10) holds for $2F_{2m+1} + 2 \le j \le F_{2m+3} - 1$. □

Proposition 4.3 *For $k \geq 1$, we have*

$$A_{2k} \;=\; \left\{F_{2k}, (n+2)F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k-1} \;:\; n \geq 0\right\}.$$

Proof. By Lemma 4.2, we have

$$A_{2k} \;=\; \{F_{2k}, 2F_{2k} + a(n)F_{2k} + b(n)F_{2k+1} \;:\; n \geq 0\},$$

where $a(n)$ and $b(n)$ are the number of A's and B's, respectively, among the first n letters of $\mathcal{S}$. Due to (1),

$$\begin{aligned} A_{2k} \;&=\; \left\{F_{2k}, 2F_{2k} + \left(n - \left\lfloor \frac{n+1}{\phi} \right\rfloor\right) F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k+1} \;:\; n \geq 0\right\} \\ &=\; \left\{F_{2k}, (n+2)F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k-1} \;:\; n \geq 0\right\}. \end{aligned}$$

□

Proof. [Proof of Theorem 1.3] For $1 \leq N < k$, the set of integers whose Chung-Graham decomposition has F_{2k} but none of $F_{2N}, \ldots, F_{2k-2}$ is

$$\left\{j + F_{2k}, j + (n+2)F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k-1} \;:\; 0 \leq j \leq F_{2N} - 1, n \geq 0\right\}.$$

Indeed, all Chung-Graham decompositions of the form $\sum_{i=1}^{N-1} c_i F_{2i}$, when added to an integer in A_{2k}, gives an integer whose Chung-Graham decomposition has F_{2k} but none of $F_{2N}, \ldots, F_{2k-2}$. Meanwhile,

$$\left\{\sum_{i=1}^{N-1} c_i F_{2i} \;:\; c_i\text{'s satisfy the Chung-Graham decomposition}\right\} \;=\; [0, F_{2N} - 1].$$

Therefore,

$$\begin{aligned} B_{2N} \;:=\; &[1, F_{2N} - 1] \cup \\ &\bigcup_{k=N+1}^{\infty} \left\{j + F_{2k}, j + (n+2)F_{2k} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{2k-1} \;:\; 0 \leq j \leq F_{2N} - 1, n \geq 0\right\}. \end{aligned}$$

□

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