Geometry of a Family of Cubic Polynomials

C. Frayer and J. Wallace

Abstract - Let P_a be the family of complex-valued polynomials of the form $p(z) = (z - a)(z - r_1)(z - r_2)$ with $a \in [0, 1]$ and $|r_1| = |r_2| = 1$. The Gauss-Lucas Theorem implies that the critical points of a polynomial in P_a lie in the unit disk. This paper characterizes the location and structure of these critical points. We show that the unit disk contains a 'desert' region, the open disk $\{z \in \mathbb{C} : |z - \frac{2}{3}a| = \frac{1}{3}\}$, in which critical points of polynomials in P_a do not occur. Furthermore, almost every c inside the unit disk and outside of the desert region is the critical point of a unique polynomial in P_a.

Keywords : geometry of polynomials; critical points; Gauss-Lucas Theorem

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1 Introduction

The Gauss-Lucas Theorem implies that the critical points of a complex-valued polynomial lie in the convex hull of its roots [4]. Several recent papers [1], [2], [3], [5] have explored the geometry of complex-valued polynomials with three roots. Critical points of polynomials of the form $p(z) = (z - 1)(z - r_1)(z - r_2)$ with $|r_1| = |r_2| = 1$ are investigated in [1]. For such a polynomial, the Gauss-Lucas Theorem guarantees that its critical points lie in the unit disk. In this case, there is more to say. It is shown in [1] that no critical point occurs in the 'desert' region $\left\{z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3}\right\}$, and a critical point of such a polynomial almost always determines the polynomial uniquely. Furthermore, an underlying structure relates the critical points of such a polynomial. If one critical point lies on the circle $C = \left\{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\right\}$, then the other critical point lies on C. Otherwise the critical points lie on opposite sides of C.

For $a \in [0, 1]$, a natural generalization of [1] is to investigate the family of polynomials

$$P_a = \{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z - a)(z - r_1)(z - r_2), |r_1| = |r_2| = 1, a \in [0, 1]\}.$$

Critical points of polynomials in P_0 are characterized in [3]. Similar to [1], the unit disk contains a desert, $\left\{z \in \mathbb{C} : |z| < \frac{1}{3}\right\}$, in which critical points do not occur, and a critical point almost always determines a polynomial uniquely. This paper completes the generalization by investigating critical points of polynomials in P_a for $a \in (0, 1)$. We used GeoGebra to visualize the critical points of polynomials in P_a. In Figure 1, we set r_1 and r_2 in motion around the unit circle and traced the trajectories of the critical points in grey. Similar to [1] and [3], the unit disk contains a desert region in which critical points do not occur, and almost every c inside the unit disk and outside the desert region is the critical point of a unique polynomial in P_a (see our Theorem 4.1).

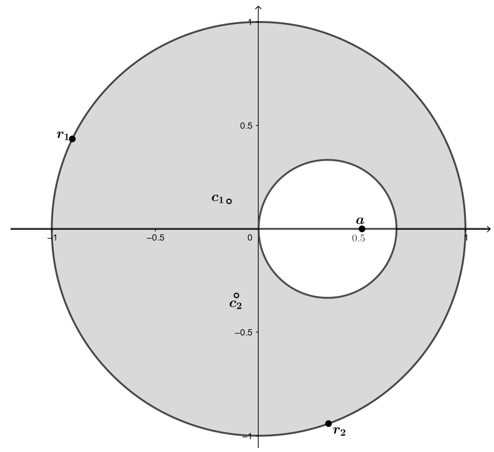

Figure 1: Setting r_1 and r_2 in motion around the unit circle and tracing the trajectories of the critical points in grey allows us to visualize the desert region.

2 Critical Points

We begin by introducing some notation. For $a \in [0,1]$, define the circle $A_b = \{z \in \mathbb{C} : |z-(1-b)a| = b, b \geq 0\}$. When necessary, we express the a dependence of A_b as A_b^a. Observe that A_1 is the unit circle and $A_0 = \{a\}$. Furthermore, for $a \in [0,1)$, a given z in the unit disk lies on a unique A_b with $b \geq 0$. When $a = 1$, all the circles A_b^1 contain the point $z = 1$. In this case, each $z \neq 1$ in the unit disk lies on a unique A_b^1 with $b > 0$.

A polynomial of the form $p(z) = (z-a)(z-r_1)(z-r_2)$ has two critical points. To characterize these critical points, we investigate how they are related to their associated roots. If c is a critical point of $p(z)$, then

$$0 = p'(c) = 3c^2 - 2(a + r_1 + r_2)c + r_1 r_2 + ar_1 + ar_2$$

and it follows that

$$0 = r_2[r_1 - (2c-a)] - [(2c-a)r_1 - (3c^2 - 2ac)]. \tag{1}$$

If $r_1 \neq 2c - a$, (1) becomes

$$r_2 = \frac{(2c-a)r_1 - (3c^2 - 2ac)}{r_1 - (2c-a)} \tag{2}$$

and when $r_1 = 2c - a$, (1) implies $c = a$.

Definition 2.1 *Given $c \neq a$, we define*

$$f_c(z) = \frac{(2c-a)z - (3c^2 - 2ac)}{z - (2c-a)} \tag{3}$$

and $S_c = f_c(A_1)$.

Observe that f_c is a linear fractional transformation with $f_c(r_1) = r_2$. Furthermore, for $c \neq a$, direct calculation shows $(f_c)^{-1} = f_c$. Therefore, $f_c(r_2) = r_1$ and we have established the following result.

Theorem 2.2 *A polynomial $p(z) = (z-a)(z-r_1)(z-r_2)$ has a critical point at c if and only if $f_c(r_1) = r_2$.*

To gain intuition, we recall Gauss's physical interpretation relating roots and critical points of a polynomial [4]. The critical points of a polynomial are the equilibrium points of a force field. The field is generated by particles placed at the roots of the polynomial, the particles having masses equal to the multiplicity of the roots and attracting with a force inversely proportional to the distance from the particle. With this in mind, when a critical point of a polynomial in P_a occurs at a repeated root on the unit circle, the second critical point is forced to be as close as possible to the root at $z = a$, and hence lie on the boundary of the desert region.

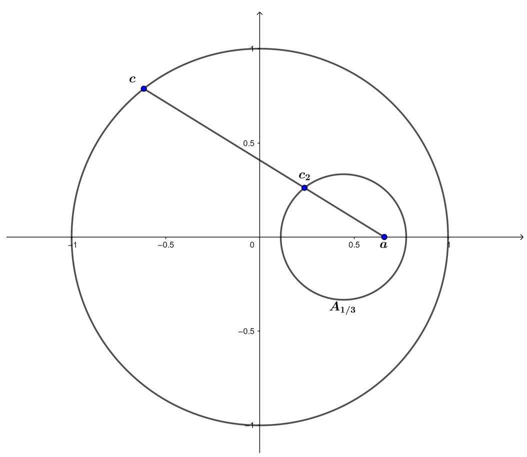

Figure 2: The polynomial $p(z) = (z-a)(z-c)^2 \in P_a$ has critical points $c \in A_1$ and $c_2 \in A_{1/3}$. Observe that c_2 is in the convex hull of $\{c, a\}$.

Example 2.3 Suppose $p \in P_a$ has a critical point $c \in A_1$. The Gauss-Lucas Theorem implies that c is a repeated root of p. Therefore, $p(z) = (z-a)(z-c)^2$ is the only polynomial in P_a with a critical point at $c \in A_1$. Furthermore,

$$p'(z) = (z-c)(3z-(2a+c))$$

so the other critical point of $p(z)$ satisfies $3c_2 - (2a+c) = 0$. Therefore, as $3c_2 - 2a = c$ implies $\left|c_2 - \frac{2}{3}\right| = \frac{1}{3}$, it follows that $c_2 \in A_{1/3}$. See Figure 2.

This example provides us with a conjecture. We hypothesize that no polynomial in P_a will have a critical point strictly inside $A_{1/3}$ (See Theorem 4.1.) Proving this hypothesis requires a better understanding of S_c.

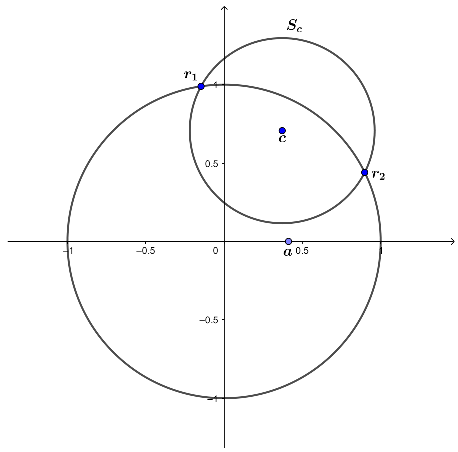

Figure 3: If c is a critical point of $p(z) = (z-a)(z-r_1)(z-r_2) \in P_a$, then $\{r_1, r_2\} \subseteq S_c \cap A_1$.

To further explore S_c, recall that $\{r_1, r_2\} \subseteq A_1$, $f_c(r_1) = r_2$, $f_c(r_2) = r_1$, and $S_c = f_c(A_1)$. These facts imply $\{r_1, r_2\} \subseteq S_c \cap A_1$. That is, if c is a critical point of $p(z) = (z-a)(z-r_1)(z-r_2) \in P_a$, then $\{r_1, r_2\} \subseteq S_c \cap A_1$. See Figure 3. Lemma 2.4 investigates the cardinality of $S_c \cap A_1$ and is the direct extension of a result in [3, Lemma 4].

Lemma 2.4 *Suppose $c \neq a$. The circles S_c and A_1 coincide, are disjoint, or intersect in one or two distinct points.*

1. *If $S_c \cap A_1 = \varnothing$, then no polynomial in P_a has a critical point at c.*
2. *If $S_c = A_1$, then for each $r \in A_1$ there is a unique polynomial $(z-a)(z-r)(z-f_c(r)) \in P_a$ with a root at r and a critical point at c.*
3. *Otherwise, $S_c \cap A_1 = \{r, f_c(r)\}$ consists of two points, or one point if $f_c(r) = r$. In this case, $(z-a)(z-r)(z-f_c(r))$ is the unique polynomial in P_a with a critical point at c.*

3 Properties of S_c

Suppose $c \neq a$. As

$$f_c(z) = \frac{(2c-a)z - (3c^2 - 2ac)}{z - (2c-a)}$$

is a linear fractional transformation and A_1 is a circle, S_c will be a circle or a line. Furthermore, S_c will be a line whenever there exists a $z \in A_1$ with $z - (2c-a) = 0$. As $|z| = 1$, this occurs when $\frac{1}{2} = |c - \frac{a}{2}|$. Therefore, S_c is a line if and only if $c \in A_{1/2}$.

To determine when $S_c = A_1$, we need an additional fact related to linear fractional transformations.

Theorem 3.1 *[2] A linear fractional transformation T sends the unit circle to the unit circle if and only if*

$$T(z) = \frac{\overline{\alpha} z - \overline{\beta}}{\beta z - \alpha}$$

for some $\alpha, \beta \in \mathbb{C}$ with $\left|\frac{\alpha}{\beta}\right| \neq 1$.

Example 3.2 When $a = 0$, it is shown in [3] that $S_c = A_1$ if and only if $|c| = \sqrt{1/3}$ and when $a = 1$, it is shown in [1] that $S_c = A_1$ if and only if $c = -\frac{1}{3}$.

Suppose $a \in (0, 1)$. Applying Theorem 3.1 to (3) implies $S_c = A_1$ whenever

$$\overline{2c - a} = 2c - a \ \text{ and } \ 3c^2 - 2ac = \overline{1}.$$

The left equality implies c is real, and for a real-valued c the right equation gives $c = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3} \in \mathbb{R}$. That is, if $c = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3}$, then $S_c = A_1$.

Conversely, if $S_c = A_1$, then Theorem 3.1 implies

$$f_c(z) = \frac{(2c - a)z - (3c^2 - 2ac)}{z - (2c - a)} = \frac{\overline{\alpha} z - \overline{\beta}}{\beta z - \alpha}.$$

In this case, there exists a nonzero complex number v such that

$$v((2c - a)z - (3c^2 - 2ac)) = \overline{\alpha} z - \overline{\beta} \ \text{ and } \ v(z - (2c - a)) = \beta z - \alpha.$$

Therefore,

$$v(2c - a) = \overline{\alpha} = \overline{v}\,\overline{(2c - a)} \ \text{ and } \ v(3c^2 - 2ac) = \overline{\beta} = \overline{v}$$

so

$$3c^2 - 2ac = \frac{2c - a}{\overline{2c - a}}. \tag{4}$$

Setting $c = x + iy$ in (4) and equating real and imaginary parts gives

$$2x - a = (2x - a)(3x^2 - 3y^2 - 2ax) + 4y^2(3x - a) \tag{5}$$

and

$$2y = 2y\left[(2x - a)(3x - a) - (3x^2 - 3y^2 - 2ax)\right]. \tag{6}$$

If $y \neq 0$, (6) implies

$$y^2 = \frac{1 - 3x^2 + 3ax - a^2}{3}. \tag{7}$$

Substituting (7) into (5) eventually gives $0 = a(a^2 + 2 - 3ax)$ so $x = \dfrac{a^2 + 2}{3a}$. Substituting into (7) implies

$$y^2 = -\frac{a^4 - 5a^2 + 4}{9a^2},$$

a contradiction. It follows that $y = 0$. In this case we have that $c = x$ and from (5)we get

$$0 = (2x - a)(3x^2 - 2ax - 1).$$

If $2x - a = 0$, then $c = \frac{a}{2}$. In this case we have $f_c(z) = \frac{-3c^2+2ac}{2} = \frac{a^2}{4z}$ so that $1 = |f_c(1)| = \frac{a^2}{4}$, a contradiction. Therefore $3x^2 - 2ax - 1 = 0$ and

$$c = x = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3}.$$

To summarize, $S_c = A_1$ if and only if $c = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3}$.

For $c \notin A_{1/2}$, S_c is a circle. By the definition of S_c, $z \in S_c$ if and only if there exists some $w \in A_1$ with $f_c(w) = z$. Equivalently, $f_c(z) = w$ implies $|f_c(z)| = 1$, so that

$$\left| \frac{(2c-a)z - (3c^2 - 2ac)}{z - (2c-a)} \right| = 1.$$

Thus, $z \in S_c$ if and only if

$$|z - (2c - a)| = |2c - a| \left| z - \frac{3c^2 - 2ac}{2c - a} \right|. \tag{8}$$

For $k \neq 1$, an introductory complex analysis result [6] states that the solution set of $|z - u| = k|z - v|$ is a circle with center C and radius R satisfying

$$C = \frac{k^2 v - u}{k^2 - 1} \text{ and } R = |v - u| \left| \frac{k}{k^2 - 1} \right|.$$

When $k = |2c - a| = 1$ in (8), $c \in A_{1/2}$ and S_c is a line. This leads to the following lemma.

Lemma 3.3 *Suppose $c \notin A_{1/2}$. Then S_c is a circle with center γ and radius r given by*

$$\gamma = \frac{(2\bar{c} - a)(3c^2 - 2ac) - (2c - a)}{|2c - a|^2 - 1} \text{ and } r = \left| \frac{3c^2 - 2ac}{2c - a} - (2c - a) \right| \left| \frac{|2c - a|}{|2c - a|^2 - 1} \right|.$$

For future use, we illustrate Lemma 3.3 with an example.

Example 3.4 For $c = \frac{2}{3}a$, the center and radius of S_c are

$$\gamma = \frac{3a}{9 - a^2} \quad \text{and} \quad r = \frac{a^2}{9 - a^2}.$$

As $|\gamma + r| = \left| \frac{3a + a^2}{9 - a^2} \right| = \left| \frac{a}{3 - a} \right| < 1$, S_c is contained inside A_1. Therefore, by Lemma 2.4, no polynomial in P_a has a critical point at $c = \frac{2}{3}a$.

According to Lemma 2.4, if S_c is tangent to A_1 at r, then c is the critical point of the unique polynomial

$$p(z) = (z-a)(z-r)^2 \in P_a.$$

Furthermore, as seen in Example 2.3, the critical points of p are $r \in A_1$ and $\frac{r+2a}{3} \in A_{1/3}$. That is, if S_c is tangent to A_1, then $c \in A_1 \cup A_{1/3}$. Example 3.5 investigates the converse of this statement.

Example 3.5 Suppose $c \in A_1$. Then $c = e^{i\theta}$ and by Lemma 3.3 the center of S_c is

$$\begin{aligned}\gamma &= \frac{(2e^{-i\theta}-a)(3e^{2i\theta}-2ae^{i\theta})-(2e^{i\theta}-a)}{(2e^{-i\theta}-a)(2e^{i\theta}-a)-1} \\ &= \frac{-3ae^{2i\theta}+(4+2a^2)e^{i\theta}-3a}{3+a^2-2ae^{-i\theta}-2ae^{i\theta}} \\ &= \frac{-3ae^{i\theta}-3ae^{-i\theta}+4+2a^2}{3+a^2-4a\cos(\theta)}e^{i\theta} \\ &= \frac{4+2a^2-6a\cos(\theta)}{3+a^2-4a\cos(\theta)}c.\end{aligned}$$

Additionally, for $a \in (0,1)$, one can verify that $\frac{4+2a^2-6a\cos(\theta)}{3+a^2-4a\cos(\theta)} > 1$. Since $c \in A_1$ and $f_c(c) = c \in S_c$, c is a point of intersection of S_c and A_1 that lies on the line connecting their centers. Therefore, S_c is externally tangent to A_1 at c.

Similar calculations show that if $c \in A_{1/3}$, then S_c is internally tangent to A_1 at $3c-2a$. See Figure 4.

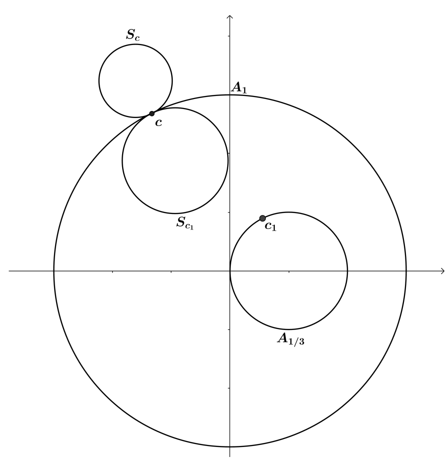

Figure 4: Circles S_{c_1} and S_c for $c_1 \in A_{1/3}$ and $c = 3c_1 - 2a \in A_1$.

We have established the following result.

Lemma 3.6 *Suppose $c \in \mathbb{C}$.*

1. *S_c is internally tangent to A_1 if and only if $c \in A_{1/3}$.*

2. *S_c is externally tangent to A_1 if and only if $c \in A_1$*

For $c \in A_{1/3} \cup A_1$, S_c is tangent to A_1 and it follows that $|S_c \cap A_1| = 1$. Lemmas 3.7 and 3.8 determine $|S_c \cap A_1|$ when $c \notin A_{1/3} \cup A_1$.

Lemma 3.7 *If $c \in A_b$ with $b \in (\frac{1}{3}, 1)$ and $c \neq \frac{a \pm \sqrt{a^2+3}}{3}$, then $|S_c \cap A_1| = 2$.*

Proof. Let $c \in A_b$ with $b \in (\frac{1}{3}, 1)$ and $c \neq \frac{a \pm \sqrt{a^2+3}}{3}$. Without loss of generality, suppose to the contrary that $|S_c \cap A_1| = 0$ and S_c is contained inside A_1. As we drag c to A_1 along a line segment avoiding $A_{1/3}$, S_c is continuously transformed into a circle externally tangent to A_1. By the Intermediate Value Theorem, there exists a c_0 on the line segment with S_{c_0} internally tangent to A_1. However, as c does not cross $A_{1/3}$, this contradicts Lemma 3.6 and it follows that $|S_c \cap A_1| = 2$. □

A similar 'dragging' argument shows that $S_c \cap A_1 = \emptyset$ whenever c is contained inside $A_{1/3}$.

Lemma 3.8 *If c is contained inside $A_{1/3}$, then $S_c \cap A_1 = \emptyset$.*

Proof. Suppose to the contrary that c is contained inside $A_{1/3}$ with $|S_c \cap A_1| = 2$. As we drag c to $\frac{2}{3}a$, the center of $A_{1/3}$, along a line segment, it follows from Example 3.4 that S_c is continuously transformed into a circle contained inside A_1. By the Intermediate Value Theorem, there must exist a c_0 on the line segment with S_{c_0} internally tangent to A_1. However, as c does not cross $A_{1/3}$, this contradicts Lemma 3.6 and it follows that $S_c \cap A_1 = \emptyset$. □

4 Main Result

We are now ready to characterize the critical points of polynomials in P_a.

Theorem 4.1 *Let $c \in \mathbb{C}$.*

1. *If $c \in A_b$ with $b \in [0, \frac{1}{3}) \cup (1, \infty)$, then no polynomial in P_a has a critical point at c.*
2. *If $c \in A_b$ with $b \in [\frac{1}{3}, 1]$ and $c \neq \frac{a \pm \sqrt{a^2+3}}{3}$, then c is the critical point of exactly one polynomial in P_a.*
3. *If $c = \frac{a \pm \sqrt{a^2+3}}{3}$, then infinitely many polynomials in P_a have a critical point at c.*

Proof. Let $c \in \mathbb{C}$.

1. If $c \in A_b$ with $b \in [0, \frac{1}{3})$, then Lemmas 3.8 and 2.4 imply that no polynomial in P_a has a critical point at c. Furthermore, if $c \in A_b$ with $b > 1$, the Gauss-Lucas Theorem implies that no polynomial in P_a has a critical point outside of the unit disk.

2. If $c \in A_b$ with $b \in [\frac{1}{3}, 1]$ and $c \neq \frac{a \pm \sqrt{a^2+3}}{3}$, then Example 3.2 and Lemmas 3.6 and 3.8 imply $|S_c \cap A_1| \in \{1, 2\}$. Therefore, by Lemma 2.4, c is the critical point of exactly one polynomial in P_a.

3. If $c = \frac{a \pm \sqrt{a^2+3}}{3}$, then Example 3.2 implies $S_c = A_1$ and by Lemma 2.4,

$$p(z) = (z-a)(z-r)(z-f_c(r)) \in P_a$$

has a critical point at c for each $r \in A_1$.

□

This completes the characterization of critical points of polynomials in P_a. The unit disk contains a single desert region, $\{z \in A_b \ : \ b \in [0, \frac{1}{3})\}$, in which critical points do not occur, and almost every c inside the unit disk and outside the desert region is the critical point of a unique polynomial in P_a.

5 Structure of Critical Points

Having determined where the critical points of a polynomial in P_a are located, we now investigate how they are related to each other. This relationship involves inversion over a circle. Recall that if C is a circle centered at O with radius r and X is a point distinct from O, then the inversion of X over C is the point Y on the ray $\overrightarrow{OX}$ such that $|OX| \cdot |OY| = r^2$.

For motivation, we revisit the results of [1]. Polynomials of the form

$$p(z) = (z-1)(z-r_1)(z-r_2)$$

with $|r_1| = |r_2| = 1$ have two critical points. If one critical point, c_1, lies on the circle $A^1_{1/2}$, then the other critical point, c_2, also lies on $A^1_{1/2}$ with $c_2 = \overline{c_1}$. Otherwise, the critical points lie on opposite sides of $A^1_{1/2}$. In this case, A^1_1, the unit circle, is the inversion of $A^1_{1/3}$, the boundary of the desert region, across $A^1_{1/2}$.

For $a = 0$, [3] establishes similar results with $A^0_{\sqrt{1/3}}$ being the circle of inversion. As this structure is present for the extreme values of $a = 0$ and $a = 1$, one might expect similar results for $a \in (0, 1)$.

Definition 5.1 *For $a \in [0, 1]$, we define C_I to be the circle*

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{1}{3} - \frac{a^2}{12}.$$

Observe that when $a = 1$, we have $A^1_{1/2} = C_I$, and when $a = 0$, we have $A^0_{\sqrt{1/3}} = C_I$. However, when $a \notin \{0, 1\}$, C_I is not an A_b circle. We claim that A_1 is the inversion of $A_{1/3}$ across C_I.

Lemma 5.2 *The circles A_1 and $A_{1/3}$ are inversions with respect to C_I.*

Proof. Since A_1, $A_{1/3}$, and C_I are symmetric across the real axis, it suffices to show that the two pairs of real numbers $\frac{2}{3}a + \frac{1}{3} \in A_{1/3}$ and $1 \in A_1$, and $\frac{2}{3}a - \frac{1}{3} \in A_{1/3}$ and $-1 \in A_1$ are inversions across C_I. Setting $X = \frac{2}{3}a - \frac{1}{3}$, $Y = -1$ and denoting the center and radius of C_I as $O = \left(\frac{a}{2}, 0\right)$ and $r = \sqrt{\frac{1}{3} - \frac{a^2}{12}}$ gives

$$\begin{aligned}|OX| \cdot |OY| &= \left|\frac{a}{2} - \left(\frac{2}{3}a - \frac{1}{3}\right)\right| \cdot \left|\frac{a}{2} + 1\right| \\ &= \left(\frac{1}{3} - \frac{a}{6}\right)\left(1 + \frac{a}{2}\right) \\ &= \frac{1}{3} - \frac{a^2}{12} \\ &= r^2.\end{aligned}$$

It follows that $\frac{2}{3}a - \frac{1}{3} \in A_{1/3}$ and $-1 \in A_1$ are inversions across C_I. Similar calculations verify the claim for the other set of points. Therefore, the circles A_1 and $A_{1/3}$ are inversions with respect to C_I. □

We are now ready to describe the structure relating critical points of polynomials in P_a.

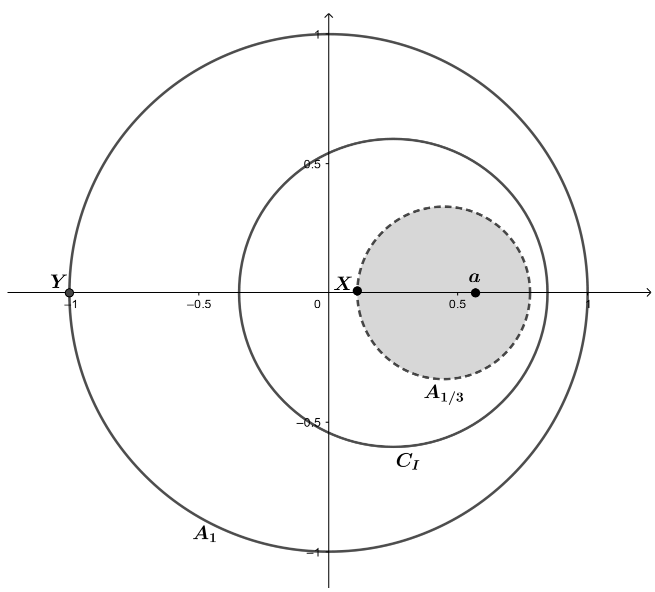

Figure 5: The circles A_1 and $A_{1/3}$ are inversions with respect to C_I.

Theorem 5.3 *Let c_1 and c_2 be the critical points of $p \in P_a$. If $c_1 \in C_I$, then $c_2 = \overline{c_1} \in C_I$. Otherwise, c_1 and c_2 lie on opposite sides of C_I.*

Proof. Let $p \in P_a$ have critical points c_1 and c_2.

Suppose $c_1 = x + iy \in C_I$. For $\cos(\theta) = \frac{3x-a}{2}$, $r = e^{i\theta}$, and

$$p_r(z) = (z-a)(z-r)(z-\overline{r}) \in P_a,$$

we will verify that c_1 is a critical point of p_r. Direct calculations give

$$\begin{aligned} p_r(z) &= 3z^2 - 2(a + r + \overline{r})z + r\overline{r} + ar + a\overline{r} \\ &= 3z^2 - 2(a + 2\cos(\theta))z + 1 + 2a\cos(\theta) \\ &= 3z^2 - 6xz + 1 + 3ax - a^2. \end{aligned}$$

As $c_1 \in C_I$, $c_1\overline{c_1} = ax + \frac{1}{3} - \frac{1}{3}a^2$ and it follows that

$$p_r(z) = 3z^2 - 3(c_1 + \overline{c_1})z + 3c_1\overline{c_1} = 3(z-c_1)(z-\overline{c_1}).$$

Therefore, p_r has critical points at c_1 and $\overline{c_1} \in C_I$. By uniqueness, $p_r(z) = p(z)$, and if $c_1 \in C_I$, then $c_2 = \overline{c_1} \in C_I$.

Suppose $c_1 \notin C_I$. We use a dragging argument to show that c_1 and c_2 lie on opposite sides of C_I. Without loss of generality, suppose to the contrary that c_1 and c_2 lie strictly inside C_I. As we drag c_1 along a line segment, contained strictly inside C_I, to a point on $A_{1/3}$, Example 2.3 and the uniqueness from Theorem 4.1 implies that c_2 travels along a continuous path to A_1. By the Intermediate Value Theorem, c_2 must cross C_I at some point c_{2,t_0}. By the first part of this theorem, $\overline{c_{2,t_0}} = c_{1,t_0} \in C_I$ which contradicts the fact that c_1 is strictly inside C_I. Therefore, c_1 and c_2 lie on opposite sides of C_I.

□

As an interesting observation, we note that for $r = e^{i\theta}$, $p_r(z) = (z-a)(z-r)(z-\overline{r})$ will not always have critical points on C_I. As r moves around the unit circle, there are threshold values of θ where the critical points move from C_I to the real axis. For

$$\theta_{\pm} = \arccos\left(\frac{a \pm \sqrt{12 - 3a^2}}{4}\right),$$

the set of points $\{a,\ e^{i\theta_+},\ e^{-i\theta_+}\}$ and $\{a,\ e^{i\theta_-},\ e^{-i\theta_-}\}$ form equilateral triangles. With this observation in mind, direct calculations give: If $\theta \in [\theta_-, \theta_+]$, then $p_r(z)$ has critical points on C_I; If $\theta \in (0, \theta_-) \cup (\theta_+, \pi)$, then $p_r(z)$ has real-valued critical points not on C_I.

This completes the characterization of critical points of polynomials in P_a. Our results can be extended to polynomials of the form

$$p(z) = (z-a)^k (z-r_1)^m (z-r_2)^n$$

with $|r_1| = |r_2| = 1$ and $\{k, m, n\} \subseteq \mathbb{N}$ with $m = n$. Similar to P_a, the unit disk contains a desert region bounded by $A_{\frac{k}{2m+k}}$ and a critical point almost always determines a polynomial uniquely. However, many open questions remain. For example, what happens when $m \neq n$?

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