Geometry of a Family of Cubic Polynomials

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Abstract - Let \( P_a \) be the family of complex-valued polynomials of the form \( p(z) = (z - a)(z - r_1)(z - r_2) \) with \( a \in [0, 1] \) and \( |r_1| = |r_2| = 1 \). The Gauss-Lucas Theorem implies that the critical points of a polynomial in \( P_a \) lie in the unit disk. This paper characterizes the location and structure of these critical points. We show that the unit disk contains a ‘desert’ region, the open disk \( \{ z \in \mathbb{C} : |z - \frac{2}{3}a| = \frac{1}{3} \} \), in which critical points of polynomials in \( P_a \) do not occur. Furthermore, almost every \( c \) inside the unit disk and outside of the desert region is the critical point of a unique polynomial in \( P_a \).

Keywords: geometry of polynomials; critical points; Gauss-Lucas Theorem

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1 Introduction

The Gauss-Lucas Theorem implies that the critical points of a complex-valued polynomial lie in the convex hull of its roots [4]. Several recent papers [1], [2], [3], [5] have explored the geometry of complex-valued polynomials with three roots. Critical points of polynomials of the form \( p(z) = (z - 1)(z - r_1)(z - r_2) \) with \( |r_1| = |r_2| = 1 \) are investigated in [1]. For such a polynomial, the Gauss-Lucas Theorem guarantees that its critical points lie in the unit disk. In this case, there is more to say. It is shown in [1] that no critical point occurs in the ‘desert’ region \( \{ z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3} \} \), and a critical point of such a polynomial almost always determines the polynomial uniquely. Furthermore, an underlying structure relates the critical points of such a polynomial. If one critical point lies on the circle \( C = \{ z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2} \} \), then the other critical point lies on \( C \). Otherwise the critical points lie on opposite sides of \( C \).

For \( a \in [0, 1] \), a natural generalization of [1] is to investigate the family of polynomials \( P_a = \{ p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z - a)(z - r_1)(z - r_2), |r_1| = |r_2| = 1, a \in [0, 1] \} \).

Critical points of polynomials in \( P_0 \) are characterized in [3]. Similar to [1], the unit disk contains a desert, \( \{ z \in \mathbb{C} : |z| < \frac{1}{3} \} \), in which critical points do not occur, and a critical point almost always determines a polynomial uniquely. This paper completes the generalization by investigating critical points of polynomials in \( P_a \) for \( a \in (0, 1) \). We used GeoGebra to visualize the critical points of polynomials in \( P_a \). In Figure [1] we set \( r_1 \) and \( r_2 \) in motion around the unit circle and traced the trajectories of the critical points in grey. Similar to [1] and [3], the unit disk contains a desert region in which critical points do not occur, and almost every \( c \) inside the unit disk and outside the desert region is the critical point of a unique polynomial in \( P_a \) (see our Theorem [1]).
Figure 1: Setting $r_1$ and $r_2$ in motion around the unit circle and tracing the trajectories of the critical points in grey allows us to visualize the desert region.

2 Critical Points

We begin by introducing some notation. For $a \in [0, 1]$, define the circle $A_b = \{z \in \mathbb{C} : |z - (1 - b)a| = b, b \geq 0\}$. When necessary, we express the $a$ dependence of $A_b$ as $A_b^a$. Observe that $A_1$ is the unit circle and $A_0 = \{a\}$. Furthermore, for $a \in [0, 1)$, a given $z$ in the unit disk lies on a unique $A_b$ with $b \geq 0$. When $a = 1$, all the circles $A_b^1$ contain the point $z = 1$. In this case, each $z \neq 1$ in the unit disk lies on a unique $A_b^1$ with $b > 0$.

A polynomial of the form $p(z) = (z - a)(z - r_1)(z - r_2)$ has two critical points. To characterize these critical points, we investigate how they are related to their associated roots. If $c$ is a critical point of $p(z)$, then

$$0 = p'(c) = 3c^2 - 2(a + r_1 + r_2)c + r_1r_2 + ar_1 + ar_2$$

and it follows that

$$0 = r_2[r_1 - (2c - a)] - [(2c - a)r_1 - (3c^2 - 2ac)].$$

(1)

If $r_1 \neq 2c - a$, (1) becomes

$$r_2 = \frac{(2c - a)r_1 - (3c^2 - 2ac)}{r_1 - (2c - a)}$$

(2)

and when $r_1 = 2c - a$, (1) implies $c = a$.

Definition 2.1 Given $c \neq a$, we define

$$f_c(z) = \frac{(2c - a)z - (3c^2 - 2ac)}{z - (2c - a)}$$

(3)

and $S_c = f_c(A_1)$. 
Observe that $f_c$ is a linear fractional transformation with $f_c(r_1) = r_2$. Furthermore, for $c \neq a$, direct calculation shows $(f_c)^{-1} = f_c$. Therefore, $f_c(r_2) = r_1$ and we have established the following result.

**Theorem 2.2** A polynomial $p(z) = (z - a)(z - r_1)(z - r_2)$ has a critical point at $c$ if and only if $f_c(r_1) = r_2$.

To gain intuition, we recall Gauss’s physical interpretation relating roots and critical points of a polynomial [4]. The critical points of a polynomial are the equilibrium points of a force field. The field is generated by particles placed at the roots of the polynomial, the particles having masses equal to the multiplicity of the roots and attracting with a force inversely proportional to the distance from the particle. With this in mind, when a critical point of a polynomial in $P_a$ occurs at a repeated root on the unit circle, the second critical point is forced to be as close as possible to the root at $z = a$, and hence lie on the boundary of the desert region.

![Figure 2](image_url)

**Figure 2:** The polynomial $p(z) = (z - a)(z - c)^2 \in P_a$ has critical points $c \in A_1$ and $c_2 \in A_{1/3}$. Observe that $c_2$ is in the convex hull of $\{c, a\}$.

**Example 2.3** Suppose $p \in P_a$ has a critical point $c \in A_1$. The Gauss-Lucas Theorem implies that $c$ is a repeated root of $p$. Therefore, $p(z) = (z - a)(z - c)^2$ is the only polynomial in $P_a$ with a critical point at $c \in A_1$. Furthermore,

$$p'(z) = (z - c)(3z - (2a + c))$$

so the other critical point of $p(z)$ satisfies $3c_2 - (2a + c) = 0$. Therefore, as $3c_2 - 2a = c$ implies $|c_2 - \frac{2}{3}| = \frac{1}{3}$, it follows that $c_2 \in A_{1/3}$. See Figure 2.

This example provides us with a conjecture. We hypothesize that no polynomial in $P_a$ will have a critical point strictly inside $A_{1/3}$ (See Theorem 4.1). Proving this hypothesis requires a better understanding of $S_c$. 

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[1] Gauss's physical interpretation

[2] Figure 2

[4] Gauss’s physical interpretation
Figure 3: If \( c \) is a critical point of \( p(z) = (z-a)(z-r_1)(z-r_2) \in P_a \), then \( \{r_1, r_2\} \subseteq S_c \cap A_1 \).

To further explore \( S_c \), recall that \( \{r_1, r_2\} \subseteq A_1 \), \( f_c(r_1) = r_2 \), \( f_c(r_2) = r_1 \), and \( S_c = f_c(A_1) \). These facts imply \( \{r_1, r_2\} \subseteq S_c \cap A_1 \). That is, if \( c \) is a critical point of \( p(z) = (z-a)(z-r_1)(z-r_2) \in P_a \), then \( \{r_1, r_2\} \subseteq S_c \cap A_1 \). See Figure 3. Lemma 2.4 investigates the cardinality of \( S_c \cap A_1 \) and is the direct extension of a result in [3, Lemma 4].

**Lemma 2.4** Suppose \( c \neq a \). The circles \( S_c \) and \( A_1 \) coincide, are disjoint, or intersect in one or two distinct points.

1. If \( S_c \cap A_1 = \emptyset \), then no polynomial in \( P_a \) has a critical point at \( c \).

2. If \( S_c = A_1 \), then for each \( r \in A_1 \) there is a unique polynomial \( (z-a)(z-r)(z-f_c(r)) \in P_a \) with a root at \( r \) and a critical point at \( c \).

3. Otherwise, \( S_c \cap A_1 = \{r, f_c(r)\} \) consists of two points, or one point if \( f_c(r) = r \). In this case, \( (z-a)(z-r)(z-f_c(r)) \) is the unique polynomial in \( P_a \) with a critical point at \( c \).

**3 Properties of \( S_c \)**

Suppose \( c \neq a \). As

\[
f_c(z) = \frac{(2c-a)z - (3c^2 - 2ac)}{z - (2c-a)}
\]

is a linear fractional transformation and \( A_1 \) is a circle, \( S_c \) will be a circle or a line. Furthermore, \( S_c \) will be a line whenever there exists a \( z \in A_1 \) with \( z - (2c-a) = 0 \). As \( |z| = 1 \), this occurs when \( \frac{1}{2} = |c - \frac{a}{3}| \). Therefore, \( S_c \) is a line if and only if \( c \in A_{1/2} \).

To determine when \( S_c = A_1 \), we need an additional fact related to linear fractional transformations.
Theorem 3.1 A linear fractional transformation $T$ sends the unit circle to the unit circle if and only if
\[ T(z) = \frac{\alpha z - \beta}{\beta z - \alpha} \]
for some $\alpha, \beta \in \mathbb{C}$ with $|\frac{\alpha}{\beta}| \neq 1$.

Example 3.2 When $a = 0$, it is shown in [3] that $S_c = A_1$ if and only if $|c| = \sqrt{1/3}$ and when $a = 1$, it is shown in [1] that $S_c = A_1$ if and only if $c = -\frac{1}{3}$.

Suppose $a \in (0, 1)$. Applying Theorem 3.1 to (3) implies $S_c = A_1$ whenever
\[ 2c - a = 2c - a \quad \text{and} \quad 3c^2 - 2ac = 1. \]

The left equality implies $c$ is real, and for a real-valued $c$ the right equation gives $c = \frac{a}{3} \pm \frac{\sqrt{a^2 + 4}}{3} \in \mathbb{R}$. That is, if $c = \frac{a}{3} \pm \frac{\sqrt{a^2 + 4}}{3}$, then $S_c = A_1$.

Conversely, if $S_c = A_1$, then Theorem 3.1 implies
\[ f_c(z) = \frac{(2c - a)z - (3c^2 - 2ac)}{z - (2c - a)} = \frac{\alpha z - \beta}{\beta z - \alpha}. \]

In this case, there exists a nonzero complex number $v$ such that
\[ v((2c - a)z - (3c^2 - 2ac)) = \overline{\alpha z - \beta} \quad \text{and} \quad v(z - (2c - a)) = \beta z - \alpha. \]

Therefore,
\[ v(2c - a) = \overline{v(2c - a)} \quad \text{and} \quad v(3c^2 - 2ac) = \overline{v} \]
so
\[ 3c^2 - 2ac = \frac{2c - a}{2c - a} \quad (4) \]

Setting $c = x + iy$ in (4) and equating real and imaginary parts gives
\[ 2x - a = (2x - a)(3x^2 - 3y^2 - 2ax) + 4y^2(3x - a) \quad (5) \]
and
\[ 2y = 2y \left[(2x - a)(3x - a) - (3x^2 - 3y^2 - 2ax)\right]. \quad (6) \]

If $y \neq 0$, (6) implies
\[ y^2 = \frac{1 - 3x^2 + 3ax - a^2}{3} \quad (7) \]
Substituting (7) into (5) eventually gives $0 = a(a^2 + 2 - 3ax)$ so $x = \frac{a^2 + 2}{3a}$. Substituting into (7) implies
\[ y^2 = -\frac{a^4 - 5a^2 + 4}{9a^2}. \]
a contradiction. It follows that \( y = 0 \). In this case we have that \( c = x \) and from (5) we get

\[
0 = (2x - a)(3x^2 - 2ax - 1).
\]

If \( 2x - a = 0 \), then \( c = \frac{a}{2} \). In this case we have \( f_c(z) = \frac{-3x^2+2ac}{2} = \frac{a^2}{4z} \) so that \( 1 = |f_c(1)| = \frac{a^2}{4} \), a contradiction. Therefore \( 3x^2 - 2ax - 1 = 0 \) and

\[
c = x = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3}.
\]

To summarize, \( S_c = A_1 \) if and only if \( c = \frac{a}{3} \pm \frac{\sqrt{a^2+3}}{3} \).

For \( c \notin A_{1/2} \), \( S_c \) is a circle. By the definition of \( S_c \), \( z \in S_c \) if and only if there exists some \( w \in A_1 \) with \( f_c(w) = z \). Equivalently, \( f_c(z) = w \) implies \( |f_c(z)| = 1 \), so that

\[
\left| \frac{(2c - a)z - (3c^2 - 2ac)}{z - (2c - a)} \right| = 1.
\]

Thus, \( z \in S_c \) if and only if

\[
|z - (2c - a)| = |2c - a| \left| z - \frac{3c^2 - 2ac}{2c - a} \right|.
\]

(8)

For \( k \neq 1 \), an introductory complex analysis result [6] states that the solution set of \( |z - u| = k|z - v| \) is a circle with center \( C \) and radius \( R \) satisfying

\[
C = \frac{k^2v - u}{k^2 - 1} \quad \text{and} \quad R = |v - u| \left| \frac{k}{k^2 - 1} \right|.
\]

When \( k = |2c - a| = 1 \) in [6], \( c \in A_{1/2} \) and \( S_c \) is a line. This leads to the following lemma.

**Lemma 3.3** Suppose \( c \notin A_{1/2} \). Then \( S_c \) is a circle with center \( \gamma \) and radius \( r \) given by

\[
\gamma = \frac{(2c - a)(3c^2 - 2ac) - (2c - a)}{|2c - a|^2 - 1} \quad \text{and} \quad r = \left| \frac{3c^2 - 2ac}{2c - a} - (2c - a) \right| \left| \frac{|2c - a|}{|2c - a|^2 - 1} \right|.
\]

For future use, we illustrate Lemma 3.3 with an example.

**Example 3.4** For \( c = \frac{2}{3}a \), the center and radius of \( S_c \) are

\[
\gamma = \frac{3a}{9 - a^2} \quad \text{and} \quad r = \frac{a^2}{9 - a^2}.
\]

As \( |\gamma + r| = \left| \frac{3a+a^2}{9-a^2} \right| = \left| \frac{a}{3-a} \right| < 1 \), \( S_c \) is contained inside \( A_1 \). Therefore, by Lemma 2.4, no polynomial in \( P_a \) has a critical point at \( c = \frac{2}{3}a \).
According to Lemma 2.4 if $S_c$ is tangent to $A_1$ at $r$, then $c$ is the critical point of the unique polynomial

$$p(z) = (z - a)(z - r)^2 \in P_a.$$ 

Furthermore, as seen in Example 2.3, the critical points of $p$ are $r \in A_1$ and $\frac{r + 2a}{3} \in A_{1/3}$. That is, if $S_c$ is tangent to $A_1$, then $c \in A_1 \cup A_{1/3}$. Example 3.5 investigates the converse of this statement.

**Example 3.5** Suppose $c \in A_1$. Then $c = e^{i\theta}$ and by Lemma 3.3 the center of $S_c$ is

$$\gamma = \frac{(2e^{-i\theta} - a)(3e^{2i\theta} - 2ae^{i\theta}) - (2e^{i\theta} - a)}{(2e^{-i\theta} - a)(2e^{i\theta} - a) - 1}$$

$$= \frac{-3ae^{2i\theta} + (4 + 2a^2)e^{i\theta} - 3a}{3 + a^2 - 2ae^{-i\theta} - 2ae^{i\theta}}$$

$$= \frac{-3ae^{i\theta} - 3ae^{-i\theta} + 4 + 2a^2}{3 + a^2 - 4a \cos(\theta)}e^{i\theta}$$

$$= \frac{4 + 2a^2 - 6a \cos(\theta)}{3 + a^2 - 4a \cos(\theta)}c.$$ 

Additionally, for $a \in (0, 1)$, one can verify that $\frac{4 + 2a^2 - 6a \cos(\theta)}{3 + a^2 - 4a \cos(\theta)} > 1$. Since $c \in A_1$ and $f_c(c) = c \in S_c$, $c$ is a point of intersection of $S_c$ and $A_1$ that lies on the line connecting their centers. Therefore, $S_c$ is externally tangent to $A_1$ at $c$.

Similar calculations show that if $c \in A_{1/3}$, then $S_c$ is internally tangent to $A_1$ at $3c - 2a$. See Figure 4.

![Figure 4: Circles $S_{c_1}$ and $S_c$ for $c_1 \in A_{1/3}$ and $c = 3c_1 - 2a \in A_1$.](image)

We have established the following result.

**Lemma 3.6** Suppose $c \in \mathbb{C}$.

1. $S_c$ is internally tangent to $A_1$ if and only if $c \in A_{1/3}$. 

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2. \( S_c \) is externally tangent to \( A_1 \) if and only if \( c \in A_1 \)

For \( c \in A_{1/3} \cup A_1 \), \( S_c \) is tangent to \( A_1 \) and it follows that \( |S_c \cap A_1| = 1 \). Lemmas 3.7 and 3.8 determine \( |S_c \cap A_1| \) when \( c \notin A_{1/3} \cup A_1 \).

**Lemma 3.7** If \( c \in A_b \) with \( b \in (\frac{1}{3}, 1) \) and \( c \neq \frac{a+\sqrt{a^2+3}}{3} \), then \( |S_c \cap A_1| = 2 \).

**Proof.** Let \( c \in A_b \) with \( b \in (\frac{1}{3}, 1) \) and \( c \neq \frac{a+\sqrt{a^2+3}}{3} \). Without loss of generality, suppose to the contrary that \( |S_c \cap A_1| = 0 \) and \( S_c \) is contained inside \( A_1 \). As we drag \( c \) to \( A_1 \) along a line segment avoiding \( A_{1/3} \), \( S_c \) is continuously transformed into a circle externally tangent to \( A_1 \). By the Intermediate Value Theorem, there exists a \( c_0 \) on the line segment with \( S_{c_0} \) internally tangent to \( A_1 \). However, as \( c \) does not cross \( A_{1/3} \), this contradicts Lemma 3.6 and it follows that \( |S_c \cap A_1| = 2 \).

A similar ‘dragging’ argument shows that \( S_c \cap A_1 = \emptyset \) whenever \( c \) is contained inside \( A_{1/3} \).

**Lemma 3.8** If \( c \) is contained inside \( A_{1/3} \), then \( S_c \cap A_1 = \emptyset \).

**Proof.** Suppose to the contrary that \( c \) is contained inside \( A_{1/3} \) with \( |S_c \cap A_1| = 2 \). As we drag \( c \) to \( \frac{2}{3}a \), the center of \( A_{1/3} \), along a line segment, it follows from Example 3.4 that \( S_c \) is continuously transformed into a circle contained inside \( A_1 \). By the Intermediate Value Theorem, there must exist a \( c_0 \) on the line segment with \( S_{c_0} \) internally tangent to \( A_1 \). However, as \( c \) does not cross \( A_{1/3} \), this contradicts Lemma 3.6 and it follows that \( S_c \cap A_1 = \emptyset \).

### 4 Main Result

We are now ready to characterize the critical points of polynomials in \( P_a \).

**Theorem 4.1** Let \( c \in \mathbb{C} \).

1. If \( c \in A_b \) with \( b \in [0, \frac{1}{3}) \cup (1, \infty) \), then no polynomial in \( P_a \) has a critical point at \( c \).

2. If \( c \in A_b \) with \( b \in [\frac{1}{3}, 1] \) and \( c \neq \frac{a+\sqrt{a^2+3}}{3} \), then \( c \) is the critical point of exactly one polynomial in \( P_a \).

3. If \( c = \frac{a+\sqrt{a^2+3}}{3} \), then infinitely many polynomials in \( P_a \) have a critical point at \( c \).

**Proof.** Let \( c \in \mathbb{C} \).

1. If \( c \in A_b \) with \( b \in [0, \frac{1}{3}) \), then Lemmas 3.8 and 2.4 imply that no polynomial in \( P_b \) has a critical point at \( c \). Furthermore, if \( c \in A_b \) with \( b > 1 \), the Gauss-Lucas Theorem implies that no polynomial in \( P_a \) has a critical point outside of the unit disk.
2. If \( c \in A_b \) with \( b \in [\frac{1}{3}, 1] \) and \( c \neq \frac{a \pm \sqrt{a^2 + 3}}{3} \), then Example 3.2 and Lemmas 3.6 and 3.8 imply \( |S_c \cap A_1| \in \{1, 2\} \). Therefore, by Lemma 2.4, \( c \) is the critical point of exactly one polynomial in \( P_a \).

3. If \( c = \frac{a \pm \sqrt{a^2 + 3}}{3} \), then Example 3.2 implies \( S_c = A_1 \) and by Lemma 2.4, 
\[
p(z) = (z - a)(z - r)(z - f_c(r)) \in P_a
\]
has a critical point at \( c \) for each \( r \in A_1 \).

This completes the characterization of critical points of polynomials in \( P_a \). The unit disk contains a single desert region, \( \{z \in A_b : b \in [0, \frac{1}{3})\} \), in which critical points do not occur, and almost every \( c \) inside the unit disk and outside the desert region is the critical point of a unique polynomial in \( P_a \).

5 Structure of Critical Points

Having determined where the critical points of a polynomial in \( P_a \) are located, we now investigate how they are related to each other. This relationship involves inversion over a circle. Recall that if \( C \) is a circle centered at \( O \) with radius \( r \) and \( X \) is a point distinct from \( O \), then the inversion of \( X \) over \( C \) is the point \( Y \) on the ray \( \overrightarrow{OX} \) such that \( |OX| \cdot |OY| = r^2 \).

For motivation, we revisit the results of [1]. Polynomials of the form
\[
p(z) = (z - 1)(z - r_1)(z - r_2)
\]
with \( |r_1| = |r_2| = 1 \) have two critical points. If one critical point, \( c_1 \), lies on the circle \( A_{1/2} \), then the other critical point, \( c_2 \), also lies on \( A_{1/2} \) with \( c_2 = \overline{c_1} \). Otherwise, the critical points lie on opposite sides of \( A_{1/2} \). In this case, \( A_1 \), the unit circle, is the inversion of \( A_{1/3} \), the boundary of the desert region, across \( A_{1/2} \).

For \( a = 0 \), [3] establishes similar results with \( A^0_{\sqrt{1/3}} \) being the circle of inversion. As this structure is present for the extreme values of \( a = 0 \) and \( a = 1 \), one might expect similar results for \( a \in (0, 1) \).

**Definition 5.1** For \( a \in [0, 1] \), we define \( C_I \) to be the circle
\[
\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{1}{3} - \frac{a^2}{12}.
\]
Observe that when \( a = 1 \), we have \( A_{1/2} = C_I \), and when \( a = 0 \), we have \( A^0_{\sqrt{1/3}} = C_I \). However, when \( a \notin \{0, 1\} \), \( C_I \) is not an \( A_b \) circle. We claim that \( A_1 \) is the inversion of \( A_{1/3} \) across \( C_I \).

**Lemma 5.2** The circles \( A_1 \) and \( A_{1/3} \) are inversions with respect to \( C_I \).
Proof. Since $A_1$, $A_{1/3}$, and $C_I$ are symmetric across the real axis, it suffices to show that the two pairs of real numbers $\frac{2}{3}a + \frac{1}{3} \in A_{1/3}$ and $1 \in A_1$, and $\frac{2}{3}a - \frac{1}{3} \in A_{1/3}$ and $-1 \in A_1$ are inversions across $C_I$. Setting $X = \frac{2}{3}a - \frac{1}{3}$, $Y = -1$ and denoting the center and radius of $C_I$ as $O = \left(\frac{a}{2}, 0\right)$ and $r = \sqrt{\frac{1}{3} - \frac{a^2}{12}}$ gives

$$|OX| \cdot |OY| = \left| \frac{a}{2} - \left(\frac{2}{3}a - \frac{1}{3}\right) \right| \cdot \left| \frac{a}{2} + 1 \right|$$

$$= \left(\frac{1}{3} - \frac{a}{6}\right) \left(1 + \frac{a}{2}\right)$$

$$= \frac{1}{3} - \frac{a^2}{12}$$

$$= r^2.$$  

It follows that $\frac{2}{3}a - \frac{1}{3} \in A_{1/3}$ and $-1 \in A_1$ are inversions across $C_I$. Similar calculations verify the claim for the other set of points. Therefore, the circles $A_1$ and $A_{1/3}$ are inversions with respect to $C_I$. \qed

We are now ready to describe the structure relating critical points of polynomials in $P_a$.

![Diagram](image)

Figure 5: The circles $A_1$ and $A_{1/3}$ are inversions with respect to $C_I$.

**Theorem 5.3** Let $c_1$ and $c_2$ be the critical points of $p \in P_a$. If $c_1 \in C_I$, then $c_2 = \overline{c_1} \in C_I$. Otherwise, $c_1$ and $c_2$ lie on opposite sides of $C_I$.

**Proof.** Let $p \in P_a$ have critical points $c_1$ and $c_2$. 

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Suppose \( c_1 = x + iy \in C_I \). For \( \cos(\theta) = \frac{3x-a}{2}, r = e^{i\theta} \), and
\[
p_r(z) = (z - a)(z - r)(z - \overline{r}) \in P_a,
\]
we will verify that \( c_1 \) is a critical point of \( p_r \). Direct calculations give
\[
p_r(z) = 3z^2 - 2(a + r + \overline{r})z + r\overline{r} + ar + a\overline{r}
= 3z^2 - 2(a + 2\cos(\theta))z + 1 + 2a\cos(\theta)
= 3z^2 - 6xz + 1 + 3ax - a^2.
\]
As \( c_1 \in C_I, c_1\overline{c_1} = ax + \frac{1}{3} - \frac{1}{3}a^2 \) and it follows that
\[
p_r(z) = 3z^2 - 3(c_1 + \overline{c_1})z + 3c_1\overline{c_1} = 3(z - c_1)(z - \overline{c_1}).
\]
Therefore, \( p_r \) has critical points at \( c_1 \) and \( \overline{c_1} \in C_I \). By uniqueness, \( p_r(z) = p(z) \), and if \( c_1 \in C_I \), then \( c_2 = \overline{c_1} \in C_I \).

Suppose \( c_1 \notin C_I \). We use a dragging argument to show that \( c_1 \) and \( c_2 \) lie on opposite sides of \( C_I \). Without loss of generality, suppose to the contrary that \( c_1 \) and \( c_2 \) lie strictly inside \( C_I \). As we drag \( c_1 \) along a line segment, contained strictly inside \( C_I \), to a point on \( A_{1/3} \), Example [2,3] and the uniqueness from Theorem [4,1] implies that \( c_2 \) travels along a continuous path to \( A_1 \). By the Intermediate Value Theorem, \( c_2 \) must cross \( C_I \) at some point \( c_{2,t_0} \). By the first part of this theorem, \( c_{2,t_0} = c_{1,t_0} \in C_I \) which contradicts the fact that \( c_1 \) is strictly inside \( C_I \). Therefore, \( c_1 \) and \( c_2 \) lie on opposite sides of \( C_I \).

As an interesting observation, we note that for \( r = e^{i\theta}, p_r(z) = (z - a)(z - r)(z - \overline{r}) \) will not always have critical points on \( C_I \). As \( r \) moves around the unit circle, there are threshold values of \( \theta \) where the critical points move from \( C_I \) to the real axis. For
\[
\theta_{\pm} = \arccos \left( \frac{a \pm \sqrt{12 - 3a^2}}{4} \right),
\]
the set of points \( \{a, e^{i\theta+}, e^{-i\theta+}\} \) and \( \{a, e^{i\theta-}, e^{-i\theta-}\} \) form equilateral triangles. With this observation in mind, direct calculations give: If \( \theta \in [\theta_-, \theta_+] \), then \( p_r(z) \) has critical points on \( C_I \); If \( \theta \in (0, \theta_-) \cup (\theta_+, \pi) \), then \( p_r(z) \) has real-valued critical points not on \( C_I \).

This completes the characterization of critical points of polynomials in \( P_a \). Our results can be extended to polynomials of the form
\[
p(z) = (z - a)^k(z - r_1)^m(z - r_2)^n
\]
with \( |r_1| = |r_2| = 1 \) and \( \{k, m, n\} \subseteq \mathbb{N} \) with \( m = n \). Similar to \( P_a \), the unit disk contains a desert region bounded by \( A_{\frac{k}{2m+n}} \) and a critical point almost always determines a polynomial uniquely. However, many open questions remain. For example, what happens when \( m \neq n \)?
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