A Class of Polynomials From Enumerating Queen Paths

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Abstract - We study a class of polynomials obtained from an enumeration of the number of queen paths. In particular, we find the generating function for the diagonal sequence of this table and the zero distribution of a sequence of related polynomials.

Keywords : zero distribution; generating function; rook and queen paths

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1 Introduction

In this paper, we study a class of polynomials arising from a combinatorial problem of counting the number of rook and queen paths. We let $a_{m,n}$ and $b_{m,n}$ be the numbers of paths a rook and queen (respectively) can move from $(0,0)$ to (m,n) on an infinite 2D chess board. These numbers have been studied in [7] and they satisfy the recurrences

$$a_{m,n} = 2a_{m-1,n} + 2a_{m,n-1} - 3a_{m-1,n-1} \tag{1}$$

and

$$\begin{aligned} b_{m,n} &= 2b_{m-1,n} + 2b_{m,n-1} - b_{m-1,n-1} - 3b_{m-2,n-1} \\ &\quad - 3b_{m-1,n-2} + 4b_{m-2,n-2}. \end{aligned} \tag{2}$$

Motivated by these recurrences, we define a table of polynomials by replacing one of the coefficients in the equations by a variable z. In particular, we define the table of rook polynomials $\{P_{m,n}(z)\}_{m,n=0}^{\infty}$ by the recurrence

$$P_{m,n}(z) = 2P_{m-1,n}(z) + 2P_{m,n-1}(z) + zP_{m-1,n-1}(z)$$

for $m, n \in \mathbb{N}$ and $(m,n) \neq (0,0)$. For simplicity, we use the standard initial condition $P_{0,0}(z) = 1$ and $P_{m,n}(z) = 0$ if $m < 0$ or $n < 0$. In the definition above, we replace the coefficient of $a_{m-1,n-1}$ in (1) by z since we will see below that the main diagonal polynomials, $P_{m,m}(z)$, have a connection with the famous sequence of Legendre polynomials. To see this connection, we note from the given recurrence relation and the initial condition that the polynomials $\{P_{m,n}(z)\}_{m,n=0}^{\infty}$ are generated by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{m,n}(z)s^m t^n = \frac{1}{1-2s-2t-zst}.$$

With the substitutions $s \to s/(-2)$, $t \to t/(-2)$, and $z \to -4z$, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{P_{m,n}(-4z)}{(-2)^{m+n}} s^m t^n = \frac{1}{1+s+t+zst}.$$

From [10, Lemma 4], we conclude that when $m = n$

$$\frac{P_{m,m}(-4z)}{2^{2m}} = z^m L_m\left(\frac{2}{z}-1\right)$$

or equivalently

$$P_{m,m}(z) = (-z)^m L_m\left(-\frac{8}{z}-1\right) = z^m L_m\left(\frac{8}{z}+1\right) \tag{3}$$

where $L_m(z)$ is the sequence of Legendre polynomials generated by

$$\sum_{m=0}^{\infty} L_m(z)t^m = \frac{1}{(1-2zt+t^2)^{1/2}}.$$

The sequence of Legendre polynomials is a special case of the sequence of Gegenbauer polynomials whose generating function is ([12, IV.2])

$$\frac{1}{(1-2zt+t^2)^{\alpha}}.$$

In the case $\alpha = 1$, the function above generates the sequence of Chebyshev polynomials of the second kind. For $\alpha > -1/2$, the sequence of Gegenbauer polynomials are orthogonal on $[-1, 1]$ ([1, page 302]) with respect the weight function $(1-z^2)^{\alpha-1/2}$. As a consequence, the zeros of Gegenbauer polynomials lie on this interval for $\alpha > -1/2$. We deduce from (3) that the zeros of $P_{m,m}(z)$ lie on the interval $(-\infty, -4]$.

Understanding the zero distribution of polynomials is an active area of research and is of interest to many mathematicians. Given the unsolvability of the general quintic equation, many approaches have been developed to understand the zeros of high-degree polynomials. One such approach is the analysis of connections between the coefficients of a polynomial and its zeros. This approach has a long history, tracing all the way from Descartes' rule of signs to the modern theory of linear operators preserving zeros on circular domains by J. Borcea and P. Brändén ([4]). Another approach is to study the zeros of polynomials in relation to recurrence relations, generating functions, and applications of complex analysis. For some papers in this direction, see [3, 5, 8]. Motivated by this approach and and (2), we define the table of Queen polynomials by the recurrence

$$\begin{aligned}Q_{m,n}(z) = 2Q_{m-1,n}(z) + 2Q_{m,n-1}(z) - Q_{m-1,n-1}(z) - 3Q_{m-2,n-1}(z)\\ - 3Q_{m-1,n-2}(z) + zQ_{m-2,n-2}(z).\end{aligned} \tag{4}$$

and the standard initial condition $Q_{0,0}(z) = 1$ and $Q_{m,n}(z) = 0$ if $m < 0$ or $n < 0$. Equivalently, this table is generated by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Q_{m,n}(z)s^n t^m = \frac{1}{1 - 2(s + t + st) + 3(st + s^2t + st^2) - zs^2t^2}. \tag{5}$$

This means that for each $z \in \mathbb{C}$, there is a sufficiently small $\delta > 0$ so that (4) holds for all $|s| < \delta$ and $|t| < \delta$. Similar to the table of rook polynomials above, we seek to understand the generating function for the diagonal sequence $Q_{m,m}(z)$ and the zero distribution of related polynomials. To achieve this goal, we first make the substitution $s = x/t$ in (5) to obtain the following

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Q_{m,n}(z)x^n t^{m-n} &= \frac{1}{1 - 2(x/t + t + x) + 3(x + x^2/t + xt) - zx^2}\\ &= \frac{t}{t^2(3x - 2) + t(-zx^2 + x + 1) + 3x^2 - 2x}.\end{aligned} \tag{6}$$

Since $|s| < \delta$ and $|t| < \delta$, we have $\frac{|x|}{\delta} < |t| < \delta$. We deduce from the equation above that the generating function

$$\sum_{m=0}^{\infty} Q_{m,m}(z)x^m \tag{7}$$

is the t^0-coefficient of the Laurent series of (6) in the annulus

$$\frac{|x|}{\delta} < |t| < \delta. \tag{8}$$

To compute this coefficient, we apply partial fraction decomposition to write (6) as

$$\frac{\tau_1}{(3x - 2)(\tau_1 - \tau_2)} \cdot \frac{1}{t - \tau_1} + \frac{\tau_2}{(3x - 2)(\tau_2 - \tau_1)} \cdot \frac{1}{t - \tau_2}. \tag{9}$$

where τ_1 and τ_2 are the two zeros (in t) of the denominator of (6). The quadratic formula gives

$$\tau_1 = \frac{zx^2 - x - 1 + \sqrt{(-zx^2 + x + 1)^2 - 4x(3x - 2)^2}}{2(3x - 2)}$$

and

$$\tau_2 = \frac{zx^2 - x - 1 - \sqrt{(-zx^2 + x + 1)^2 - 4x(3x - 2)^2}}{2(3x - 2)}.$$

From (8), we have $|x| < \delta^2$. Thus for small δ, x is small and $\tau_1 \sim x/4$ and $\tau_2 \sim 1/2$. We conclude that for sufficiently small δ,

$$\tau_1 < |t| < \tau_2$$

for any t in the annulus (8). With these inequalities, we apply the Laurent series expansions

$$\frac{1}{t-\tau_1} = \frac{1}{t}\frac{1}{1-\tau_1/t} = \sum_{n=0}^{\infty} \frac{\tau_1^n}{t^{n+1}},$$
$$\frac{1}{t-\tau_2} = \frac{1}{\tau_2}\frac{1}{t/\tau_2-1} = -\sum_{n=0}^{\infty} \frac{t^n}{\tau_2^{n+1}},$$

to conclude that (7), which equals the t^0-coefficient in the Laurent series expansion of (9), is

$$-\frac{1}{(3x-2)(\tau_2-\tau_1)} = \frac{1}{\sqrt{(-zx^2+x+1)^2-4x(3x-2)^2}}$$
$$= \frac{1}{\sqrt{x^4z^2+x^3(-2z-36)+x^2(49-2z)-14x+1}}.$$

Similar to the idea that Gegenbauer polynomials are generalization of Legendre polynomial, the generating function above motivates us to define a sequence of polynomials

$$\sum_{m=0}^{\infty} P_m(z)t^m = \frac{1}{(t^4z^2+t^3(-2z-36)+t^2(49-2z)-14t+1)^\alpha}. \tag{10}$$

We conjecture that for any $\alpha > 0$, the zeros of $P_m(z)$ lie on the interval $(-\infty, -9/4)$. In the next section, we will show that the conjecture holds for $\alpha = 1$. The method of the proof in this paper provides us a direction in tackling the problems of finding the zero distribution of sequence of polynomials whose denominator of the generating function is nonlinear in z. For studies on the case this denominator is linear in z in its variations, see [13, 8].

2 Zero Distribution of $P_m(z)$

The main goal of this section is to prove the theorem below.

Theorem 2.1 *The zeros of the polynomials $P_m(z)$ generated by*

$$\sum_m P_m(z)t^m = \frac{1}{t^4z^2+t^3(-2z-36)+t^2(49-2z)-14t+1} \tag{11}$$

lie on the interval $(-\infty, -9/4)$.

To prove Theorem 2.1, we will count the number of zeros of $P_m(z)$ on $(-\infty, -9/4)$ and show that this number is at least the degree of $P_m(z)$. Theorem 2.1 will follow directly from the Fundamental Theorem of Algebra. The theorem below provides an upper bound for the degree of $P_m(z)$.

Lemma 2.2 *The degree of $P_m(z)$ is at most $\lfloor \frac{m}{2} \rfloor$.*

Proof. From (11), the sequence $\{P_m(z)\}_{m=0}^{\infty}$ satisfies the recurrence relation

$$P_m(z) - 14P_{m-1}(z) + (49 - 2z)P_{m-2}(z) + (-2z - 36)P_{m-3}(z) + z^2 P_{m-4}(z) = 0$$

for $m \geq 1$ with initial condition $P_0(z) = 1$ and $P_{-m} = 0$. From the recurrence above and the induction hypothesis, we have

$$\deg(P_m(z)) \leq \max\left(\left\lfloor \frac{m-1}{2} \right\rfloor, \left\lfloor \frac{m-2}{2} \right\rfloor + 1, \left\lfloor \frac{m-3}{2} \right\rfloor + 1, \left\lfloor \frac{m-4}{2} \right\rfloor + 2\right) = \left\lfloor \frac{m}{2} \right\rfloor.$$

□

It remains show that number of zeros of $P_m(z)$ on $(-\infty, -9/4)$ is at least $\lfloor \frac{m}{2} \rfloor$. For this reason, we assume $z \in (-\infty, -9/4)$. For each $z \in (-\infty, -9/4)$, let t_1, t_2, t_3, and t_4 be the zeros (in t) of

$$D(t, z) := t^4 z^2 + t^3(-2z - 36) + t^2(49 - 2z) - 14t + 1. \tag{12}$$

We will show that for $z \in (-\infty, -9/4)$, these zeros are not real. A useful concept in proving this is the discriminant of a polynomial defined below.

Definition 2.3 *The discriminant of a polynomial $P(z)$ with lead coefficient p and degree n is*

$$\operatorname{Disc}_z P(z) = p^{2n-2} \prod_{1 \leq i < j \leq n} (z_i - z_j)^2$$

where z_i, $1 \leq i \leq n$, are the zeros of $P(z)$.

From this definition, $\operatorname{Disc}_z P(z) = 0$ if and only if $P(z)$ has a multiple zero. In the case, the degree of $P(z)$ is 4, $\operatorname{Disc}_z P(z) > 0$ if and only if either all the zeros of $P(z)$ are real or none of these zeros are real [10]. For further studies of discriminants of various polynomials, see [2, 6, 9].

Lemma 2.4 *For each $z \in (-\infty, -9/4)$, the four zeros of $D(t, z)$ are distinct and not real.*

Proof. From a computer algebra system, the discriminant of $D(t, z)$ as a polynomial in t is

$$\operatorname{Disc}_t D(t, z) = -256(z - 4)(4z - 15)^2(4z + 9)^2 > 0$$

for $z \in (-\infty, -9/4)$. Thus either (i) all zeros of $D(t, z)$ are real for all $z \in (-\infty, -9/4)$ or (ii) none of zeros of $D(t, z)$ is real for all $z \in (-\infty, -9/4)$. When $z = -3$, from (12) we can check that the following polynomial

$$D(t, -3) = 9t^4 - 30t^3 + 55t^2 - 14t + 1$$

has four non-real zeros. Thus all the zeros of $D(t, z)$ are non-real for all $z \in (-\infty, -9/4)$. $\square$

Since $D(t, z)$ is a real polynomial (for $z \in (-\infty, -9/4)$), their non-real zeros, denoted by t_1, t_2, t_3, and t_4, form conjugate pairs. Without loss of generality, we let $t_1 = \overline{t_2}$, $t_3 = \overline{t_4}$, $|t_1| \leq |t_3|$, and t_1 and t_3 lie on the upper half plane. We can write these zeros as $t_1 = re^{i\theta}$, $t_2 = re^{-i\theta}$, $t_3 = \varrho e^{i\phi}$, and $t_4 = \varrho e^{-i\phi}$ where $0 < r \leq \rho$. These zeros satisfy Vieta's formulas

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= \frac{2z + 36}{z^2}, \\ t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 &= \frac{-2z + 49}{z^2}, \\ t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 &= \frac{14}{z^2}, \\ t_1t_2t_3t_4 &= \frac{1}{z^2}. \end{aligned}$$

The first equation is the same as

$$2r\cos\theta + 2\varrho\cos\phi = \frac{2z + 36}{z^2}. \tag{13}$$

Similarly the second equation is equivalent to

$$r^2 + \varrho^2 + r\varrho e^{-i(\theta+\phi)} + r\varrho e^{i(\theta-\phi)} = \frac{-2z + 49}{z^2}$$

where the left side is

$$r^2 + \varrho^2 + 2r\varrho\cos(\theta + \phi) + 2r\varrho\cos(\theta - \phi) = r^2 + \varrho^2 + 4r\varrho\cos\theta\cos\phi.$$

The third equation gives

$$2r^2\varrho\cos\phi + 2r\varrho^2\cos\theta = \frac{14}{z^2}.$$

Using

$$r\varrho = -\frac{1}{z}$$

from the fourth equation (as $z < 0$) we can rewrite this equation as

$$2r\cos\phi + 2\varrho\cos\theta = \frac{-14}{z}. \tag{14}$$

Recall that $|t_1| \leq |t_3|$. From these elementary symmetric equations, we can show that this inequality is strict in the lemma below.

Lemma 2.5 *For any $z \in (-\infty, -9/4)$, we have $|t_1| = |t_2| < |t_3| = |t_4|$.*

Proof. Assume, by way of contradiction, that $r = \rho$, then (13) and (14) yield

$$\frac{2z+36}{z^2} = \frac{-14}{z}$$

or equivalently

$$16z^2 + 36z = 0$$

which is a contradiction since $z \in (-\infty, -9/4)$. □

Recall that for each $z \in (-\infty, -9/4)$, θ is the principal angle of $t_1 = t_1(z)$. Thus we can view θ as a function of $z \in (-\infty, -9/4)$.

Lemma 2.6 *If $t_1 = re^{i\theta}$, then $\theta(z)$ is a decreasing function on $z \in (-\infty, -9/4)$.*

Proof. By the chain rule,

$$\frac{d\theta}{dz} = \frac{dt_1}{dz} \cdot \frac{d\theta}{dt_1}.$$

We will show that $d\theta/dz \neq 0$ by showing that each term on the right side is nonzero. We differentiate both sides of

$$t_1^4 z^2 + t_1^3(-2z - 36) + t_1^2(49 - 2z) - 14t_1 + 1 = 0$$

with respect to z and obtain

$$\frac{dt_1}{dz} = -\frac{D_z(t_1, z)}{D_t(t_1, z)} = -\frac{2zt_1^4 - 2t_1^3 - 2t_1^2}{4t_1^3 z^2 + 3t_1^2(-2z - 36) + 2t_1(49 - 2z) - 14}.$$

We note that the denominator of the last expression is nonzero since $D_t(t_1, z) \neq 0$ as t_1 is a simple zero of $D(t, z)$ by Lemma 2.4. Thus dt_1/dz is a continuous function in $z \in (-\infty, -9/4)$. We claim that the numerator of this expression,

$$2t_1^2(zt_1^2 - t_1 - 1),$$

is also nonzero. Indeed, if by contradiction $zt_1^2 - t_1 - 1 = 0$. Then, $z = \frac{t_1+1}{t_1^2}$. We substitute this value of z into $D(t_1, z) = 0$ and conclude

$$t_1^4 z^2 + t_1^3(-2z - 36) + t_1^2(49 - 2z) - 14t_1 + 1 = -4t_1(-2 + 3t_1)^2 = 0$$

which is a contradiction since $t_1 \notin \mathbb{R}$. Thus, we can conclude $\frac{dt_1}{dz} \neq 0$ for $z \in (-\infty, -9/4)$.

We will show that $d\theta/dz$ is also nonzero for z in this interval. We differentiate both sides of $t_1 = re^{i\theta}$ with respect to θ and conclude that

$$\frac{dt_1}{d\theta} = e^{i\theta}\left(\frac{dr}{d\theta} + ir\right),$$

or equivalently

$$\frac{d\theta}{dt_1} = \frac{1}{e^{i\theta}\left(\frac{dr}{d\theta} + ir\right)} \neq 0.$$

Note that the denominator of the last expression is nonzero since $dr/d\theta \in \mathbb{R}$ and thus $d\theta/dt_1$ is continuous on $z \in (-\infty, -9/4)$.

Since

$$\frac{d\theta}{dz} = \frac{dt_1}{dz} \cdot \frac{d\theta}{dt_1}$$

is a continuous function in $z \in (-\infty, -9/4)$ and it has no zero on this interval, $\theta(z)$ is monotone on $z \in (-\infty, -9/4)$. Thus, to complete this lemma, we compare two values of $\theta(z)$ at two different values of z. From a simple computer algebra, we have $\theta(-10) = 0.55491..$ and $\theta(-3) = 0.206599...$ from which the lemma follows. □

Now that we know $\theta(z)$ is decreasing, the lemma below provides the image of the interval $(-\infty, -9/4)$ under this map.

Lemma 2.7 *The function $\theta(z)$ maps $(-\infty, -9/4)$ onto $(0, \pi/2)$.*

Proof. Recall that $t_1(z)$ is a zero of

$$D(t, z) = t^4z^2 + t^3(-2z - 36) + t^2(49 - 2z) - 14t + 1.$$

A simple evaluation yields

$$\lim_{z \to -\frac{9}{4}} D(t, z) = \frac{1}{16}(4 - 28t + 9t^2)^2$$

where the zeros of the last expression are positive real. Thus

$$\lim_{z \to -9/4} \theta(z) = \lim_{z \to -9/4} \operatorname{Arg} t_1 = 0.$$

Next, we consider the case $z \to -\infty$. First, we note that as z approaches $-\infty$, t_1 must approach 0, since if otherwise we will have

$$|t_1^4z^2| > |t_1^3(-2z - 36) + t_1^2(49 - 2z) - 14t_1 + 1|$$

when $|z|$ is large. This contradicts $D(t_1, z) = 0$ as t_1 is a zero of $D(t, z)$.

We next show that $\lim_{z \to -\infty} |t_1^3z|$ exists and equals to 0 by showing that

$$\limsup_{z \to -\infty} |t_1^3z| = 0$$

where the left side of the expression above is defined as

$$\lim_{x \to -\infty} \sup\{|t_1^3z| : z \in (-\infty, x)\}.$$

If, by contradiction,

$$\limsup_{z \to -\infty} |t_1^3z| \neq 0$$

then we have

$$\limsup_{z \to -\infty} |t_1z| = \limsup_{z \to -\infty} \left|\frac{t_1^3z}{t_1^2}\right| = \infty$$

since $\lim_{z\to-\infty} t_1 = 0$. So

$$\limsup_{z\to-\infty} |t_1^4 z^2| = \limsup_{z\to-\infty} |(t_1^3 z)(t_1 z)| = \infty.$$

We factor $t_1^4 z^2$ out of $D(t_1, z)$ and utilize $\limsup_{z\to-\infty} |t_1 z| = \limsup_{z\to-\infty} |t_1^2 z| = \infty$ to get

$$\begin{aligned}\limsup_{z\to-\infty} |D(t_1, z)| &= \limsup_{z\to-\infty} |t_1^4 z^2| \cdot \left|1 - \frac{2}{t_1 z} - \frac{36}{t_1 z^2} + \frac{49}{t_1^2 z^2} - \frac{2}{t_1^2 z} - \frac{14}{t_1^2 z(t_1 z)} + \frac{1}{t_1^4 z^2}\right| \\ &= \limsup_{z\to-\infty} |t_1^4 z^2| = \infty.\end{aligned}$$

This contradicts $D(t_1, z) = 0$ as t_1 is a zero of $D(t, z)$.

The equation $\lim_{z\to-\infty} |t_1^3 z| = 0$ allows us to rewrite $\lim_{z\to-\infty} D(t_1, z)$ as

$$0 = \lim_{z\to-\infty} D(t_1, z) = \lim_{z\to-\infty} t_1^4 z^2 - 2z t_1^2 + 1.$$

If $u = \limsup_{z\to-\infty} t_1^2 z$, then

$$0 = u^2 - 2u + 1 = (u - 1)^2$$

from which we deduce that $u = 1$. Similarly $\liminf_{z\to-\infty} t_1^2 z = 1$. Thus $\lim_{z\to-\infty} t_1^2 z = 1$ from which we have

$$\mathrm{Arg}\left(\lim_{z\to-\infty} t_1^2 z\right) = 0.$$

Since z is a negative real number, the equation above gives $\lim_{z\to-\infty} \mathrm{Arg}(t_1) = \pi/2$ (recall that $\mathrm{Arg}(t_1) > 0$ as t_1 lies in the upper half plane). We conclude the proof of this lemma. $\square$

Remark 2.8 From the proof of Lemma 2.7, we have

$$\lim_{z\to-\infty} t_1^2 z = 1$$

from which and the fact that t_1 lies on the upper-half plane, we deduce that as $z \to -\infty$

$$t_1 \sim \frac{i}{\sqrt{-z}}.$$

We will obtain a more precise asymptotic approximation of t_1, which will be useful later in the proof of Lemma 2.11. We let

$$t_1 = \frac{i}{\sqrt{-z}} + \epsilon$$

where

$$\epsilon = o\left(\frac{1}{\sqrt{-z}}\right)$$

and substitute this equation to $D(t_1,z)=0$ to conclude

$$\begin{aligned}0&=\epsilon^4z^2-4i\epsilon^3\sqrt{-z}z-2\epsilon^3z-36\epsilon^3+4\epsilon^2z+6i\epsilon^2\sqrt{-z}+\frac{108i\epsilon^2\sqrt{-z}}{z}\\&+49\epsilon^2-\frac{98i\epsilon\sqrt{-z}}{z}-\frac{108\epsilon}{z}-20\epsilon+\frac{36i\sqrt{-z}}{z^2}+\frac{16i\sqrt{-z}}{z}+\frac{49}{z}.\end{aligned}$$

We apply $\epsilon=o(1/\sqrt{-z})$ and reduce this identity to

$$0=4\epsilon^2z+\frac{16i\sqrt{-z}}{z}+o\left(\epsilon^2|z|+\frac{1}{\sqrt{|z|}}\right)$$

from which we conclude

$$\epsilon=\pm\frac{2e^{3i\pi/4}}{(-z)^{3/4}}+o\left(\frac{1}{|z|^{3/4}}\right).$$

Consequently

$$t_1=\frac{i}{\sqrt{-z}}\pm\frac{2e^{3i\pi/4}}{(-z)^{3/4}}+o\left(\frac{1}{|z|^{3/4}}\right).$$

Since t_1 lies in the first quadrant, we conclude that

$$t_1=\frac{i}{\sqrt{-z}}-\frac{2e^{3i\pi/4}}{(-z)^{3/4}}+o\left(\frac{1}{|z|^{3/4}}\right). \tag{15}$$

Recall that we want to find the number of zeros of $H_m(z)$ on the interval $(-\infty,-9/4)$. To achieve this goal, we will provide a closed formula for the polynomial $H_m(z)$. The following lemma provides such a formula in terms of $t_1(z)$, $t_2(z)$, $t_3(z)$, and $t_4(z)$. For the ease of notations, we suppress the parameter z in these variables.

Lemma 2.9 *For any $z\in(-\infty,-9/4)$, if t_1, t_2, t_3 and $t_4$are the zeros of (12), then*

$$-z^2P_m(z)=\frac{A}{t_1^{m+1}}+\frac{B}{t_2^{m+1}}+\frac{C}{t_3^{m+1}}+\frac{D}{t_4^{m+1}} \tag{16}$$

where

$$\begin{aligned}A&=\frac{1}{(t_1-t_2)(t_1-t_3)(t_1-t_4)},\\B&=\frac{1}{(t_2-t_1)(t_2-t_3)(t_2-t_4)},\\C&=\frac{1}{(t_3-t_1)(t_3-t_2)(t_3-t_4)},\\D&=\frac{1}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}.\end{aligned}$$

Proof. Since t_1, t_2, t_3, and t_4 are the zeros of the denominator on the right side of (11), we can factor this denominator and write our generating function as

$$\sum_{m=0}^{\infty} P_m(z)t^m = \frac{1}{z^2(t-t_1)(t-t_2)(t-t_3)(t-t_4)}.$$

As t_1, t_2, t_3, and t_4 are distinct by Lemma 2.4, partial fraction decomposition yields

$$\sum_{m=0}^{\infty} P_m(z) = \frac{1}{z^2}\left(\frac{A}{t-t_1} + \frac{B}{t-t_2} + \frac{C}{t-t_3} + \frac{D}{t-t_4}\right)$$

where A, B, C, and D are given in the statement of the lemma. We express each term on the right side as a power series in t as follow:

$$\frac{A}{t-t_1} = \frac{A}{t_1(1-\frac{t}{t_1})} = \sum_{m=0}^{\infty} -\frac{At^m}{t_1^{m+1}}.$$

From similar computations for the remainder three terms, we conclude

$$-z^2 P_m(z) = \frac{A}{t_1^{m+1}} + \frac{B}{t_2^{m+1}} + \frac{C}{t_3^{m+1}} + \frac{D}{t_4^{m+1}},$$

from which our lemma follows. □

We note that the first two terms on the right side of (16) are complex conjugates and the same statement holds for the last two terms of this expression. To count the number of real zeros of $P_m(z)$, we will find the dominant term on the right side of (16), which is provided by the lemma below.

Lemma 2.10 *For any $m \in \mathbb{N}$ and any $z \in (-\infty, -9/4)$, let A, B, C, D be defined as in Lemma 2.9. Then*

$$\left|\frac{A}{t_1^{m+1}}\right| > \left|\frac{C}{t_3^{m+1}}\right|.$$

Proof. It is equivalent to show

$$\left|\frac{t_3}{t_1}\right|^{m+1} > \left|\frac{C}{A}\right|.$$

From the definition of A and C in (16), the right side is

$$\left|\frac{C}{A}\right| = \left|\frac{(t_1-t_2)(t_1-t_4)}{(t_3-t_2)(t_3-t_4)}\right|.$$

Since $\overline{t_1} = t_2$ and $\overline{t_3} = t_4$, we have $|t_1 - t_4| = |t_3 - t_2|$ and consequently the right hand side becomes

$$\left|\frac{t_1-t_2}{t_3-t_4}\right| = \left|\frac{\mathrm{Im}(t_1)}{\mathrm{Im}(t_3)}\right|.$$

Thus it remains to prove that
$$\left|\frac{t_3}{t_1}\right|^{m+1} > \left|\frac{\mathrm{Im}(t_1)}{\mathrm{Im}(t_3)}\right|.$$
Since $|t_3| > |t_1|$ by Lemma 2.5, it suffices to show
$$\mathrm{Im}(t_3) > \mathrm{Im}(t_1).$$
From the fact that $|t_3| > |t_1|$ and t_1 and t_3 lie in the upper-half plane, it remains to show $\sin(\mathrm{Arg}(t_3)) > \sin(\mathrm{Arg}(t_1))$. We recall that $\theta = \mathrm{Arg}(t_1)$ and $\phi = \mathrm{Arg}(t_3)$. By the continuity of θ and ϕ as functions of z, it suffices to show $\sin\phi \neq \sin\theta$ for all $z \in (-\infty, -9/4)$ and verify this inequality at one value of z. It is easy to check that with a computer algebra system that at $z = -5,$, $\theta = 0.37..$ and $\phi = 1.299...$

To finish the proof of this lemma we will prove $\sin(\phi) \neq \sin(\theta)$. Assuming by contradiction that $\sin(\phi) = \sin(\theta)$, which is equivalent to either $\phi = \theta$ or $\phi = \pi - \theta$.In the first case when $\phi = \theta$, Equations (13) and (14) give
$$-\frac{14}{z} = \frac{2z+36}{z^2}$$
which implies $z = -\frac{9}{4}$. This contradicts $z \in (-\infty, -\frac{9}{4})$.

Similarly, in the the case $\phi = \pi - \theta$ or equivalently $\cos\theta = -\cos\phi$, we have
$$\frac{14}{z} = \frac{2z+36}{z^2},$$
which implies $z = 3$, a contradiction to the fact that $z \in (-\infty, -\frac{9}{4})$. □

Recall that we want to find the number of zeros of $P_m(z)$ on the interval $(-\infty, -9/4)$ and compare that number with the degree of this polynomial which is at most $\lfloor m/2 \rfloor$ by Lemma 2.2. The lemma below gives a lower bound of the number of real zeros of $P_m(z)$ on the given interval. Theorem 2.1 follows from this lemma and the Fundamental Theorem of Algebra.

Lemma 2.11 *$P_m(z)$ has at least $\lfloor \frac{m}{2} \rfloor$ zeros on the interval $(-\infty, -9/4)$.*

Proof. Recall from (16) that for $z \in (-\infty, -9/4)$
$$-z^2 P_m(z) = 2\Re\left(\frac{A}{t_1^{m+1}}\right) + 2\Re\left(\frac{C}{t_3^{m+1}}\right) \tag{17}$$
where, by Lemma 2.10,
$$\left|\frac{A}{t_1^{m+1}}\right| > \left|\frac{C}{t_3^{m+1}}\right|.$$
Let
$$f(t_1) = \frac{A}{t_1^{m+1}}. \tag{18}$$

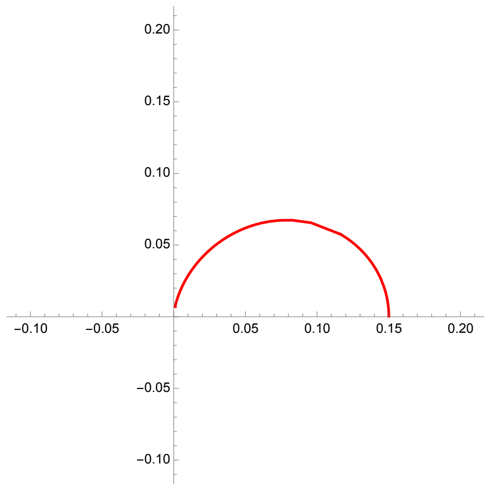

Figure 1: The curve $t_1(z), -\infty < z < -9/4$.

From the Implicit Function Theorem, the function $t_1(z)$, $z \in (-\infty, -9/4)$ produces a smooth curve in the complex plane (see Figure 1) and hence so is $f(t_1(z))$. We deduce from (17) that at the value of z where $f(t_1(z)) \in \mathbb{R}^+$ or $f(t_1(z)) \in \mathbb{R}^-$ we have $-z^2 P_m(z) > 0$ or $-z^2 P_m(z) < 0$ respectively. By the Intermediate Value theorem, there is a zero of $P_m(z)$ when there is a change in sign of $-z^2 P_m(z)$. Thus, to count the number of zeros of $P_m(z)$ on $(-\infty, -9/4)$, we can count number of times the curve $f(t_1(z))$, $z \in (-\infty, -9/4)$, intersects the real axis. To count this number, we compute the change in argument of this curve. From (18) we have

$$\Delta \arg_{-\infty<z<-9/4} f(t_1(z)) = \Delta \arg_{-\infty<z<-9/4} A - (m+1)\Delta \arg_{-\infty<z<-9/4} t_1(z). \tag{19}$$

We deduce from Lemma 2.7 that

$$\Delta \arg_{-\infty<z<-9/4} t_1(z) = -\pi/2.$$

To measure the change in argument of A, we claim that $A \notin i\mathbb{R}^+$ for all $z \in (-\infty, -9/4)$ from which we can conclude that the change in argument is

$$\lim_{z\to -9/4} \operatorname{Arg} A - \lim_{z\to-\infty} \operatorname{Arg} A. \tag{20}$$

Indeed, we write

$$\begin{aligned} t_1 &= a + bi, \\ t_2 &= a - bi, \\ t_3 &= c + di, \\ t_4 &= c - di, \end{aligned}$$

and obtain from procedural computations that

$$A = \frac{1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} = \frac{1}{(2bi)(a^2 - b^2 + c^2 + d^2 - 2ac + 2abi - 2bci)}.$$

If by contradiction $A \in i\mathbb{R}^+$, then the fact that $b > 0$ (as t_1 lies in the first quadrant) gives

$$a^2 - b^2 + c^2 + d^2 - 2ac + 2abi - 2bci \in \mathbb{R}^-.$$

which implies $b(a-c) = 0$. In the first case when $b = 0$, the real part of the expression above is $(a-c)^2 + d^2 > 0$. In the second case when $a - c = 0$, this real part is $d^2 - b^2 > 0$ by Lemma 2.5. Thus we obtain a contradiction in both cases.

We now compute (20). To compute the first term in this expression, we note from

$$D(t, -9/4) = \frac{1}{16}\left(9t^2 - 28t + 4\right)^2,$$

the equation $D(t_1(z), z) = 0$, and Lemma 2.5, that

$$\lim_{z\to -9/4} t_1(z) = \lim_{z\to -9/4} t_2(z) = \frac{2}{9}\left(7 - 2\sqrt{10}\right)$$

and

$$\lim_{z\to -9/4} t_3(z) = \lim_{z\to -9/4} t_4(z) = \frac{2}{9}\left(7 + 2\sqrt{10}\right).$$

We recall from Lemma 2.10 that

$$A = \frac{1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}.$$

As $z \to -9/4$ we have

$$(t_1 - t_3)(t_1 - t_4) \to \mathbb{R}^+$$

while

$$\operatorname{Arg}(t_1 - t_2) = \pi/2$$

since $t_2 = \overline{t_1}$ and t_1 lies in the upper half plane. These equations imply that

$$\lim_{z\to -9/4} \operatorname{Arg} A = -\frac{\pi}{2}. \tag{21}$$

We will next compute the second term of (20). We note that the denominator of A is $D_t(t_1, z)/z^2$ since

$$D(t, z) = z^2(t - t_1)(t - t_2)(t - t_3)(t - t_4).$$

Thus

$$\lim_{z\to -\infty} \operatorname{Arg}(A) = -\lim_{z\to -\infty} \operatorname{Arg} D_t(t_1, z)$$

where from (12)

$$D_t(t_1, z) = 4t_1^3 z^2 - 6t_1^2 z - 108t_1^2 - 4t_1 z + 98t_1 - 14.$$

We substitute (15) to the right side of this equation to obtain the following asymptotics as $z \to \infty$

$$D_t(t_1, z) \sim 16(-z)^{1/4} e^{3i\pi/4}$$

and consequently

$$\lim_{z\to-\infty} \operatorname{Arg}(A) = -3\pi/4.$$

We conclude from (21) and the fact that $A \notin i\mathbb{R}$ that

$$\Delta \arg_{-\infty<z<-9/4} A = \pi/4.$$

Consequently, from (19)

$$\Delta \arg_{-\infty<z<-9/4} f(t_1(z)) = \frac{\pi}{4} + \frac{(m+1)\pi}{2} = \frac{m\pi}{2} + \frac{3\pi}{4}.$$

We recall that that at the values of $z \in (-\infty, -9/4)$ where $f(t_1(z)) \in \mathbb{R}^+$ or $f(t_1(z)) \in \mathbb{R}^-$ we have $-z^2H_m(z) > 0$ or $-z^2H_m(z) < 0$ respectively. By the Intermediate Value Theorem, there is at least a zero of $H_m(z)$ between two consecutive values of z where $f(t_1(z))$ changes from $\mathbb{R}^+$ to $\mathbb{R}^-$ or vice versa. Since the change of argument of $f(t_1(z))$ is $m\pi/2 + 3\pi/4$ there are at least $\lfloor \frac{m}{2} \rfloor$ such changes. Thus we obtain at least $\lfloor \frac{m}{2} \rfloor$ zeros of $P_m(z)$ on $(-\infty, -9/4)$ and the lemma follows. □

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