The Optimal Ratio of a Generalized Chaos Game in Regular Polytopes

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Abstract - This paper investigates the concept of an optimal ratio for regular polytopes in n-dimensional space within the framework of the Generalized Chaos Game. The optimal ratio, r_{opt}, is defined as the value at which the self-similar regions of the resulting fractal touch but do not overlap. Using a series of Python simulations, we explore how the optimal ratio varies across different polytopes, from two-dimensional polygons to three-dimensional polyhedra and beyond. The results, visualized through plots generated for various polytopes and values of the scaling factor r, demonstrate that the optimal ratio is not universal but rather depends on each polytope's specific properties. A formula is then derived to determine the optimal ratio for any regular polytope in any dimension. The formula is then experimentally verified using multiple Python programs designed to search and find the optimal ratio iteratively.

Keywords : Chaos Game; fractals; regular polytopes; self-similarity; Chaos Game Representation

Mathematics Subject Classification (2020) **:** 28A80; 51M20; 37D45

1 Introduction

Fractals have captivated mathematicians, scientists, and artists alike for their self-similar structures and often intricate, seemingly infinite detail. Fractals have come to represent a class of geometric objects defined not by smooth boundaries, but by recursive, repeating patterns that remain complex under magnification. These structures appear throughout nature, from the branching of trees to the contours of coastlines, and have led to numerous applications in fields as diverse as biology, physics, and computer graphics.

One notable method for generating fractals is the Chaos Game, a stochastic process in which points are plotted iteratively, converging to reveal fractal structures within a given shape. Originally applied to two-dimensional polygons, the Chaos Game has traditionally yielded classic fractals such as the Sierpinski Triangle. In this paper, we extend the Chaos Game into higher dimensions, investigating its behavior within regular polytopes, including those beyond three-dimensional polyhedra. By defining and exploring an optimal ratio—a specific scaling factor where self-similar regions of the fractal touch without overlapping—our work offers insights into the conditions that generate distinct fractal patterns in various polytopes.

The goal of this study is twofold: to generalize the Chaos Game to higher-dimensional regular polytopes and to develop a formula that determines the optimal ratio for any regular polytope. Through a combination of theoretical analysis and Python-based simulations, we aim to offer a framework for understanding fractals in higher-dimensional spaces, shedding light on both their underlying mathematics and their potential applications.

2 Chaos Game

Generally, a chaos game is described as a method to generate fractals through an iterative random process. The rules of such a process are as follows:

1. Pick a ratio $r \in (0, 1)$. This ratio will remain fixed.
2. Pick a random point c inside a regular n-gon and plot it.
3. Select a random vertex of the n-gon.
4. Calculate the vector v from the current point c to the randomly selected vertex.
5. Plot the next point $c + rv$.
6. Set the point $c + rv$ as the current point and repeat from step 3 [6].

Once the select number of iterations is completed, the initial few points are discarded. This process is known to generate fractals depending on the polygon and the value of r [6].

2.1 Generalized Chaos Game

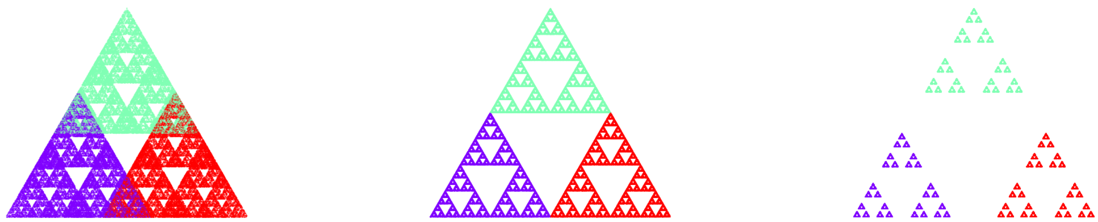
In this paper, we will modify the rules above in order to include higher-dimensional polytopes. Let P_n be an n-dimensional regular polytope with vertices $v_1, v_2, \ldots, v_V$. Then, the Generalized Chaos Game (GCG) can be described as generating a sequence of points in $\mathbb{R}^n$ using the iterative function:

$$x_{c+1} = (1 - r)x_c + rv_i \tag{1}$$

where x_0 is a randomly selected initial point, and v_i is one of the polytope's V vertices with $i \in \{1, 2, \ldots, V\}$, chosen at random in each iteration. To visually represent the GCG, we plot the points generated by this iteration. Each point is assigned a color based on which vertex v_i was used in that iteration. The reader should note that the iterative function (1), given a 2-dimensional polytope (i.e. a polygon), is equivalent to the procedure discussed in Section 2. However, the power of the GCG as defined is that it can be played in any n-dimensional regular polytope.

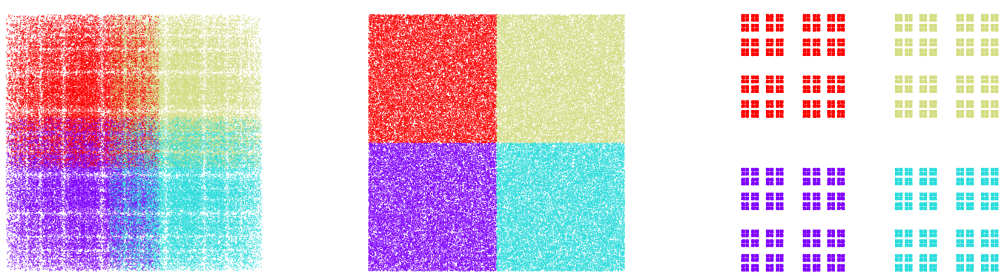
2.2 Plots of GCG in Two and Three-Dimensional Regular Polytopes

The following plots were generated from a GCG played inside two-dimensional polygons and three-dimensional polyhedra, using different values of r. These plots were created using the Python programs provided in [4] where the initial 6 points were discarded.



(a) $r = 0.4$. (b) $r = 0.5$. (c) $r = 0.6$.

Figure 1: Equilateral triangles with different values of r.

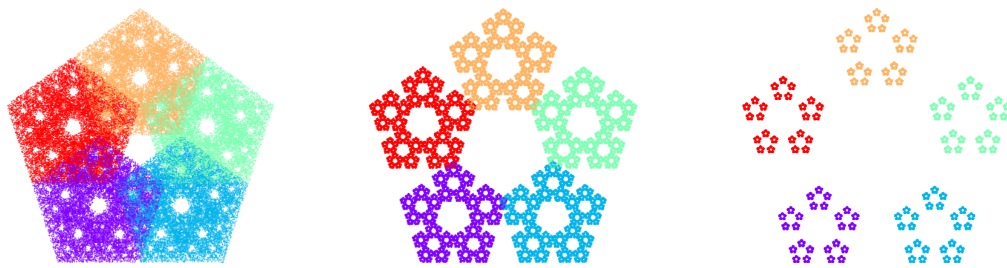


(a) $r = 0.4$ (b) $r = 0.5$ (c) $r = 0.6$

Figure 2: Squares with different values of r.

Definition 2.1 *The optimal ratio of a polytope, r_{opt}, is the ratio at which the self-similar regions of the fractal touch without overlapping. In other words, these regions may share vertex coordinates, but no point from one region appears within another region of a different color.*

This paper will examine the optimal ratio for regular polytopes in n dimensions. As shown in Figures 1 through 5, the optimal ratio can vary depending on the specific polytope.

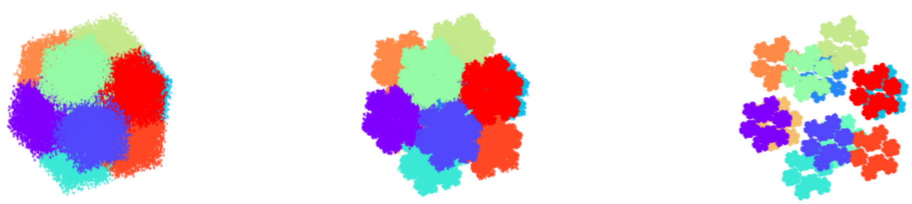


(a) $r = 0.5$ (b) $r = 0.6$ (c) $r = 0.7$

Figure 3: Regular pentagons with different values of r.

(a) $r = 0.4$ (b) $r = 0.5$ (c) $r = 0.6$

Figure 4: Regular tetrahedra with different values of r.



(a) $r = 0.5$ (b) $r = 0.6$ (c) $r = 0.7$

Figure 5: Regular icosahedra with different values of r.

3 Deriving the Optimal Ratio Formula for a Generalized Chaos Game in Regular Polytopes

To find a formula for the optimal ratio depending on the polytope, we start by describing the Generalized Chaos Game from the perspective of metric spaces.

Definition 3.1 *Let* $(\mathbf{X}, d)$ *be a complete metric space. Then* $\mathcal{H}(\mathbf{X})$ *denotes the space whose points are the compact subsets of* $\mathbf{X}$*, other than the empty set [1].*

Definition 3.2 *Let* $(\mathbf{X}, d)$ *be a complete metric space,* $x \in \mathbf{X}$*, and* $B \in \mathcal{H}(\mathbf{X})$*. Define*

$$d(x, B) = min\{d(x, y) : y \in B\}.$$

We call $d(x, B)$ *the distance from the point* x *to the set* B *[1].*

Definition 3.3 *Let* $(\mathbf{X}, d)$ *be a complete metric space. Let* $A, B \in \mathcal{H}(\mathbf{X})$*. Define*

$$d(A, B) = max\{d(x, B) : x \in A\}.$$

We call $d(A, B)$ *the distance from the set* $A \in \mathcal{H}(\mathbf{X})$ *to the set* $B \in \mathcal{H}(\mathbf{X})$ *[1].*

Definition 3.4 *Let* $(\mathbf{X}, d)$ *be a complete metric space. Then the Hausdorff distance between points* $A, B \in \mathcal{H}(\mathbf{X})$ *is defined by*

$$h(A, B) = \max\{d(A, B), d(B, A)\}$$

[1].

Definition 3.5 *A transformation* $w : \mathbf{X} \to \mathbf{X}$ *on a metric space* $(\mathbf{X}, d)$ *is called contractive or a contraction mapping if there is a constant* $0 \leq s < 1$ *such that*

$$d(w(x), w(y)) \leq s \cdot d(x, y) \quad \forall x, y \in \mathbf{X}$$

Any such number s *is called a contractivity factor for* w *[1].*

For each vertex of the polytope, we can interpret a point being plotted towards that vertex as a contraction mapping. Let P_n be a regular n-dimensional polytope with V vertices denoted $v_1, v_2, \ldots, v_V \in \mathbb{R}^n$. Define the metric space $(\mathbf{X}, d)$ by setting $\mathbf{X} = \mathbb{R}^n$ and letting d be the Euclidean metric. Let $w_i : \mathbb{R}^n \to \mathbb{R}^n$ for each vertex v_i, where each w_i maps points closer to v_i via

$$w_i(x) = (1 - r)x + rv_i$$

where $r \in (0, 1)$ and $i \in \{1, 2, \ldots, V\}$.

To verify that each w_i is a contraction mapping, we compute

$$\begin{aligned} d(w_i(x), w_i(y)) &= \|w_i(x) - w_i(y)\| \\ &= \|(1 - r)x + rv_i - ((1 - r)y + rv_i)\| \\ &= (1 - r) \cdot \|x - y\|. \end{aligned}$$

Since $(1-r) \in (0,1)$, each w_i is in fact a contraction mapping of $\mathbb{R}^n$ with contractivity factor $1-r$.

In order to study the global behavior of the Generalized Chaos Game, especially the emergence of self-similar patterns, it is helpful to consider how these mappings act on regular polytopes in $\mathbb{R}^n$. Let $\mathcal{H}(\mathbb{R}^n)$ denote the space of all nonempty compact subsets of $\mathbb{R}^n$, as introduced earlier. Since every regular polytope in $\mathbb{R}^n$ is both closed and bounded, the Heine–Borel theorem ensures that it is compact, and hence an element of $\mathcal{H}(\mathbb{R}^n)$. This allows us to apply the theory of contraction mappings on $\mathcal{H}(\mathbb{R}^n)$ using the Hausdorff distance.

Each contraction $w_i : \mathbb{R}^n \to \mathbb{R}^n$ induces a mapping on $\mathcal{H}(\mathbb{R}^n)$ by acting on each point of the polytope P_n:

$$w_i(P_n) = \{w_i(x) : x \in P_n\} \quad \text{for all } P_n \in \mathcal{H}(\mathbb{R}^n).$$

It is a known result (see Barnsley [1]) that if w_i is a contraction on $(\mathbf{X}, d)$ with contractivity factor s, then this induced mapping is also a contraction on $(\mathcal{H}(\mathbf{X}), h)$ with the same contractivity factor.

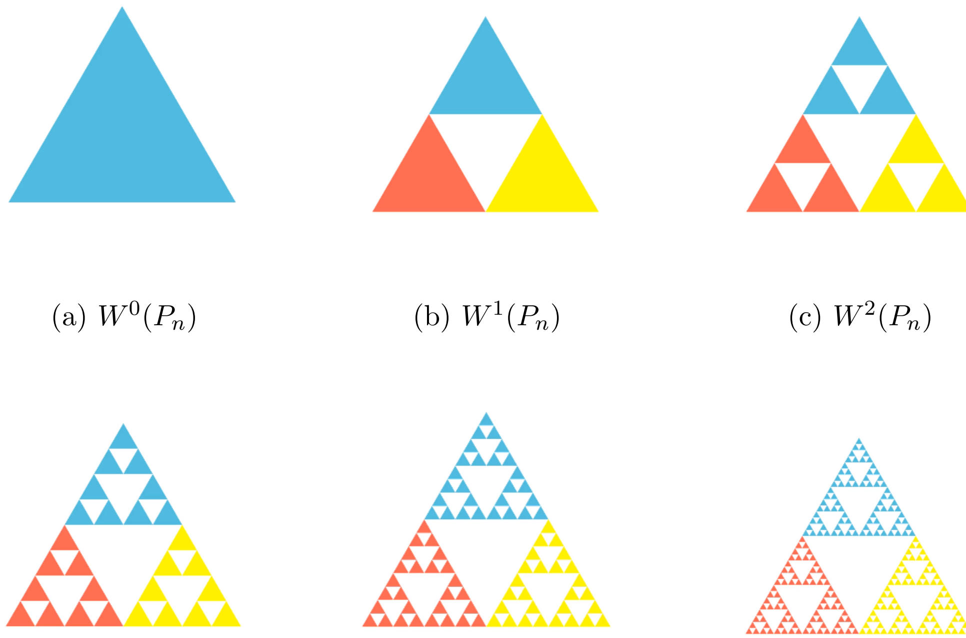
We now define the set-valued function $W : \mathcal{H}(\mathbb{R}^n) \to \mathcal{H}(\mathbb{R}^n)$ by

$$W(P_n) = \bigcup_{i=1}^{V} w_i(P_n),$$

which applies all of the w_i transformations simultaneously and unifies their images. Since each w_i is a contraction on $\mathcal{H}(\mathbb{R}^n)$ with the same contractivity factor $1-r$, it follows from Lemma 7.5 in Barnsley [1] that W is also a contraction with contractivity factor $1-r$.

Let W^n denote the nth iteration of W. Each iteration of W produces a union of V scaled copies of the previous set, resulting in V^n subsets after n iterations. These represent all possible locations where a point may land after n steps of the Generalized Chaos Game, depending on the sequence of vertices selected. In this sense, W^n captures the full space of reachable points in n iterations of the stochastic process.

Figure 6 shows the first five iterations of applying W to an equilateral triangle. Each stage consists of three affine transformations: three identical copies of W^{n-1} is scaled by a factor of $1-r$ and are then translated to each vertex so that the original vertices remain fixed. As n increases, the resulting set approaches the self-similar structure seen in Figure 1b.



(a) $W^0(P_n)$ (b) $W^1(P_n)$ (c) $W^2(P_n)$

(d) $W^3(P_n)$ (e) $W^4(P_n)$ (f) $W^5(P_n)$

Figure 6: First Iterations of $W(P_n)$ where P_n is an Equilateral Triangle using $r = 0.5$ [4].

We are interested in the optimal ratio of the Generalized Chaos Game. We want to determine the value of r at which the differently colored self-similar regions, produced by applying W to the regular n-dimensional polytope, touch each other without overlapping. Note that the boundaries of the self-similar regions remain constant through all iterations of W. Thus, the optimal ratio will also remain constant through all iterations.

We will use the notation $\vec{u} \parallel \vec{v}$ to denote two nonzero parallel vectors, $\vec{u}, \vec{v}$. That is, $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} = \lambda\vec{v}$, for some constant λ.

Let $\mathcal{A}$ denote the set of all vectors between pairs of distinct vertices of the polytope, with each pair contributing exactly one vector (represented by the black and red vectors in Figure 7). For any pair of vertices, the vector included in $\mathcal{A}$ is chosen to ensure that no two vectors in $\mathcal{A}$ are opposites.

Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ consisting of edge vectors (the red vectors in Figure 7). Note that the magnitudes of all vectors in $\mathcal{B}$ are equal to the polytope's edge length, ℓ.

Definition 3.6 *The maximum magnitude of one of the vectors in $\mathcal{A}$ that is parallel to at least one of the vectors in $\mathcal{B}$ is denoted $\delta_{\parallel}$ and is defined by*

$$\delta_{\parallel} := \max\{|\vec{u}| : \vec{u} \parallel \vec{v},\ \vec{u} \in \mathcal{A} \text{ and } \vec{v} \in \mathcal{B}\}.$$

Note that $\delta_{\parallel}$ is well-defined since $\mathcal{B} \subseteq \mathcal{A}$. Also, since all edge vectors are included in $\mathcal{A}$, $\delta_{\parallel} \geq \ell$. For example, $\delta_{\parallel} = \ell$ for a square and $\delta_{\parallel} > \ell$ for a pentagon. Figure 8 shows

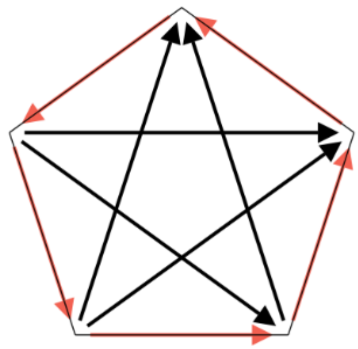


Figure 7: Visual representation of the vectors in $\mathcal{A}$ and $\mathcal{B}$ for a regular pentagon [4].

the boundary of two self-similar regions (obtained from W) when the GCG is played in a pentagon using the optimal ratio, r_{opt}.

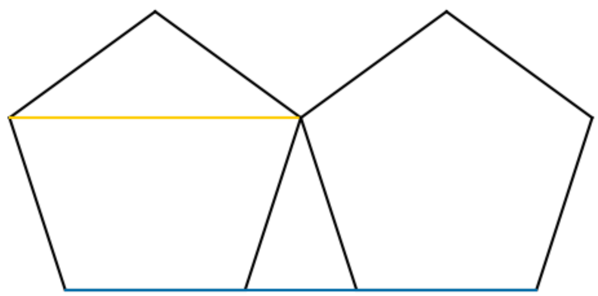


Figure 8: Two adjacent scaled-down pentagons showing $(1 - r_{\text{opt}})\delta_{\parallel}$ (yellow) and ℓ (blue) [4].

Theorem 3.7 *The optimal ratio, r_{opt}, when playing the Generelized Chaos Game in any regular n-dimensional polytope is given by*

$$r_{opt} = \frac{\delta_{\parallel}}{\delta_{\parallel} + \ell}.$$

Proof. Case 1: Suppose $\delta_{\parallel} > \ell$. In this case, the situation is depicted in Figure 9, where A and H are vertices of the original polytope, and C is the midpoint of $\overline{AH}$. Additionally, B is the scaled-down vertex corresponding to H when it is scaled and translated towards vertex A.

The points D and G represent the scaled-down vertices corresponding to those that form the vector with the maximum magnitude in $\mathcal{A}$, parallel to a vector in $\mathcal{B}$, which defines $\delta_{\parallel}$. The points F and E lie on $\overline{GD}$ and are perpendicular to the line joining A and B. Note that Figure 9 depicts a projection of our situation into two dimensions. In reality, the line segments $\overline{GD}$ and $\overline{AH}$ need not have the same depth (or the n-dimensional equivalent).

By the definition of $\delta_{\parallel}$, the line segments $\overline{GD}$, with length $|GD| = (1 - r_{\text{opt}})\delta_{\parallel}$, and $\overline{AH}$, with length $|AH| = \ell$, are parallel. Thus, we have the following relationships:

- $|AB| = (1 - r_{\text{opt}})\ell$
- $|FD| = |GD| - |GF| = |GD| - \frac{1}{2}(|GD| - |AB|) = \frac{1}{2}(1 - r_{\text{opt}})(\delta_{\parallel} + \ell)$

The optimal ratio ensures that $|FD| = |AC|$, so we equate:

$$\frac{1}{2}(1 - r_{\text{opt}})(\delta_{\parallel} + \ell) = \frac{1}{2}\ell.$$

Simplifying this expression and solving for r_{opt}, we get:

$$r_{\text{opt}} = \frac{\delta_{\parallel}}{\delta_{\parallel} + \ell}.$$

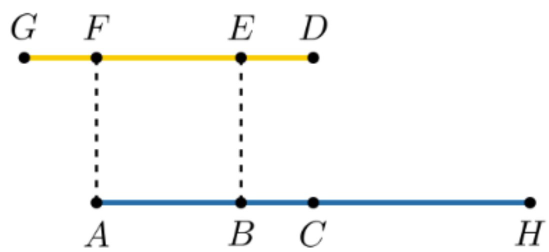


Figure 9: Sketch of the lines $(1 - r_{\text{opt}})\delta_{\parallel}$ and ℓ if $\delta_{\parallel} > \ell$ [4].

Case 2: Suppose $\delta_{\parallel} = \ell$, where ℓ is the edge length of the polytope. In this case, a projection of the situation is depicted in Figure 10, where A and C are vertices of the original polytope, and B is the midpoint of $\overline{AC}$. The points D and E represent the scaled-down vertices corresponding to the vertices that form the vector with the maximum magnitude in $\mathcal{A}$, parallel to a vector in $\mathcal{B}$, which defines $\delta_{\parallel}$.

By the definition of $\delta_{\parallel}$, the line segments $\overline{ED}$, with length $|ED| = (1 - r_{\text{opt}})\delta_{\parallel}$, and $\overline{AC}$, with length $|AC| = \ell$, are parallel. The optimal ratio ensures that $|ED| = |AB|$, where $|AB| = \frac{1}{2}\ell$ since B is the midpoint of $\overline{AC}$. Thus, we have the equation:

$$(1 - r_{\text{opt}})\delta_{\parallel} = \frac{1}{2}\ell.$$

Solving for r_{opt}, we get:

$$r_{\text{opt}} = 1 - \frac{\ell}{2\delta_{\parallel}}.$$

Since $\delta_{\parallel} = \ell$, this simplifies to:

$$r_{\text{opt}} = \frac{1}{2} = \frac{\delta_{\parallel}}{\delta_{\parallel} + \ell}.$$

□

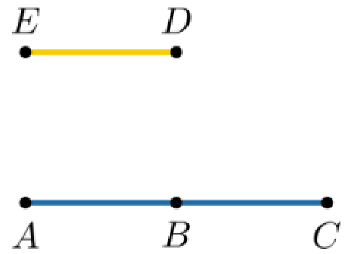


Figure 10: Sketch of the lines of length $(1 - r_{\text{opt}})\delta_{\|}$ and ℓ if $\delta_{\|} = \ell$ [4].

3.1 Finding $\delta_{\|}$ in Practice

In practice, we find $\delta_{\|}$ by first orienting the polytope so that at least one of the edges is parallel to one of the n axes. Without loss of generality, say the polytope is oriented such that one edge is parallel to the x-axis. In that case, $\delta_{\|}$ will be the largest distance between two vertices' x-coordinates. For example, say we have a regular icosahedron with edge length $\ell = 2$ with coordinates:

$$(0, \pm\phi, \pm 1)$$
$$(\pm 1, 0, \pm\phi)$$
$$(\pm\phi, \pm 1, 0),$$

where ϕ is the golden ratio [2].

Then, the polyhedron is oriented such that one edge is parallel to the x-axis since $(-1, 0, \phi)$ and $(1, 0, \phi)$ constitute an edge. Thus, $\delta_{\|}$ can be found by looking at the largest difference between any two vertices' x-coordinates. Hence, $\delta_{\|} = 2\phi$. To find the optimal ratio for the icosahedron we use the formula derived above to obtain

$$r_{\text{opt}} = \frac{2\phi}{2\phi + 2} = \frac{\phi}{\phi + 1} = \frac{1}{\phi}.$$

4 Tables for the Optimal Ratio in Various Polytopes

The following tables give the optimal ratio together with $\delta_{\|}$ for regular polytopes in 2, 3, 4, and 5 dimensions with

$$\phi = \frac{\sqrt{5} + 1}{2}.$$

The values of r_{opt} and $\delta_{\|}$ have been experimentally verified using the Overlap Testing Python programs provided in [4].

Polytope (2D)	$\delta_{\parallel}$	$\mathbf{r}_{\text{opt}}$
Triangle	ℓ	$\frac{1}{2}$
Square	ℓ	$\frac{1}{2}$
Pentagon	$\phi\ell$	$\frac{1}{\phi}$
Hexagon	2ℓ	$\frac{2}{3}$

Table 1: Optimal Ratio for Regular 2D Polytopes.

Polytope (3D)	$\delta_{\parallel}$	$\mathbf{r}_{\text{opt}}$
Tetrahedron	ℓ	$\frac{1}{2}$
Cube	ℓ	$\frac{1}{2}$
Octahedron	ℓ	$\frac{1}{2}$
Icosahedron	$\phi\ell$	$\frac{1}{\phi}$
Dodecahedron	$(\phi+1)\ell$	$\frac{\phi+1}{\phi+2}$

Table 2: Optimal Ratio for Regular 3D Polytopes.

Polytope (4D)	$\delta_{\parallel}$	$\mathbf{r}_{\text{opt}}$
5-cell (4-simplex)	ℓ	$\frac{1}{2}$
8-cell (4-cube)	ℓ	$\frac{1}{2}$
16-cell	ℓ	$\frac{1}{2}$
24-cell	2ℓ	$\frac{2}{3}$

Table 3: Optimal Ratio for Regular 4D Polytopes.

Polytope (5D)	$\delta_{\parallel}$	$\mathbf{r}_{\text{opt}}$
5-simplex	ℓ	$\frac{1}{2}$
5-cube	ℓ	$\frac{1}{2}$
5-orthoplex	ℓ	$\frac{1}{2}$

Table 4: Optimal Ratio for Regular 5D Polytopes.

5 Applications

The applications of this paper are primarily in the field of biology. In [5], Thomas discusses a three-dimensional Chaos Game representation of protein sequences. In the paper, a GCG is played on a regular icosahedron using a ratio of $r = 0.5$. The resulting graph (Figure 11) reveals no visible structure.

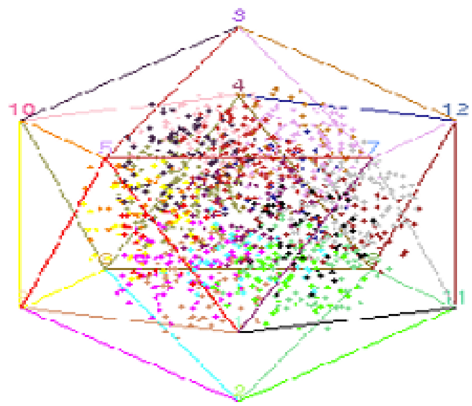


Figure 11: GCG played on a regular icosahedron with $r = 0.5$ [5].

However, if Thomas had used the optimal ratio of $r = \frac{1}{\phi}$, a discernible structure would have emerged, resembling the one shown in Figure 12.

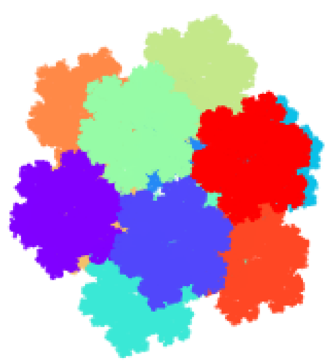


Figure 12: GCG played on a regular icosahedron with $r = \frac{1}{\phi}$.

Similarly, in [3], Sun et al. play a GCG on a regular dodecahedron using $r = \frac{\sqrt{5}-1}{\sqrt{5+2\sqrt{3}}-1} \approx 0.7370$. The authors correctly state that this ratio ensures the scaled-down dodecahedra do not intersect. However, for the dodecahedra to also touch, the ratio should be $r = \frac{\phi+1}{\phi+2} \approx 0.7236$.

Acknowledgments

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