# The Zero Forcing Numbers of Peony Graphs and Web Graphs

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**Abstract** - The concept of zero forcing involves a dynamic coloring process by which blue vertices cause white vertices to become blue, with the goal of forcing the entire graph blue while choosing as few as possible vertices to be initially blue. Past research in this area has focused on structural arguments, with approaches varying from graph substructures to the interplay between local and global graph structures. This paper explores the use of these structural concepts when determining the zero forcing number of complex classes of graphs, specifically two infinite classes of graphs each defined on multiple parameters.

Keywords: graph classes; zero forcing; forts; relaxed chronology; chain set

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# 1 Introduction

Graph theory, a branch of discrete mathematics, delves into the study of graphs mathematical structures comprising dots called vertices to represent elements or locations, and line segments called edges connecting vertices which are used to depict relationships between the vertices. The versatility of graphs enables their application in modeling and analyzing pairwise connections among objects, expanding their utility beyond mathematical boundaries to various real-world scenarios. In 2008, the AIM Minimum Rank - Special Graphs Work Group [1] first introduced a graph theory concept called zero forcing.

Zero forcing is a game in graph theory that involves coloring vertices blue with the aim of using the least number of initially blue vertices. In this game, each vertex starts as either blue or white. The rule permits a blue vertex to compel its lone white neighbor to turn blue. The game concludes when there are no more white vertices left to color. This method of analyzing graphs has played a role in the study of the inverse eigenvalue problem for graphs where it was introduced as an upper bound on the maximum nullity of graphs [1]. Its interdisciplinary applications extend beyond graph theory, finding utility in physics as quantum control theory [3] and in power grid monitoring as power domination [6] (with the role of zero forcing evident in [4]). These applications highlight the versatility and importance of this method across diverse fields.

A recent area of focus in research on the zero forcing numbers of graphs has been the exploration and introduction of new graph substructures called forts (first introduced in [5]), and the interactions these substructures have with the zero forcing number of a graph. Another recent strategy introduced in [7] restricts a forcing process occurring in a graph to one of its induced subgraphs. This provides one a way to transfer information concerning zero forcing between the local structure given by the induced subgraph and the global structure of the parent graph. These concepts were introduced in the abstract setting, and thus initial research focused more on the theory than the practice of using it. Due to this, a natural direction for additional research is to explore the usefulness of these substructures when determining the zero forcing numbers of relatively complex classes of graphs. To test this utility, this paper considers two classes of graphs defined on multiple parameters. The first graph class, peony graphs, is a new graph class defined on three parameters, and is a derivative of wheel graphs. The second graph class, web graphs, is defined on two parameters and is the result of adding pendant vertices to grid graphs. In Section 2, we determine the zero forcing number of peony graphs, making extensive use of the concept of forts. Later, in Section 3, we determine the zero forcing number for web graphs, and in various ways take advantage of the concept of restricting forcing processes to subgraphs. Before diving into the new results we provide the following preliminary definitions to ensure clear communication of the necessary concepts.

A graph G is a collection of vertices V(G) and a set E(G) of pairs of vertices, called edges. If two vertices u and v are members of the same edge, then we denote this by  $uv \in E(G)$  and we say that u is adjacent to v or that u and v are neighbors. The degree of a vertex v denoted deg(v) is the number of neighbors that v has. A pendant vertex is a vertex with degree 1. If H is a graph such that  $V(H) \subseteq V(H)$  and  $E(H) \subseteq E(G)$ , then H is a subgraph of G. Furthermore, if H is a subgraph of G such that given two vertices  $u, v \in V(H)$  we have that  $uv \in E(H)$  if and only if  $uv \in E(G)$ , then H is a vertex-induced subgraph of G. Given  $S \subseteq V(G)$ , the subgraph of G induced by S denoted G[S] is the vertex-induced subgraph of G with vertex set V(G[S]) = S.

Two graphs G and H are said to be *isomorphic* denoted  $G \cong H$  if there exists a bijective function  $\sigma : V(G) \to V(H)$ , called an *isomorphism*, such that for  $u, v \in V(G)$ , we have  $\sigma(u)\sigma(v) \in E(H)$  if and only if  $uv \in E(G)$ . A graph is said to be *rotationally isomorphic* if there exists an isomorphism  $\sigma : V(G) \to V(G)$  such that the application of  $\sigma$  can be represented geometrically as a rotation  $\theta$  with  $0 < \theta < 2\pi$  of V(G) onto itself. That is, a graph is rotationally isomorphic if an appropriate drawing of it has nontrivial rotational symmetry. The graph classes we discuss in this paper, peony graphs and web graphs, are both classes of rotationally isomorphic graphs. With this in mind, modular arithmetic can be used to streamline definitions and arguments. Specifically, let it be understood that given a vertex  $v_M \in \{u_i\}_{i=1}^n$ , if either M < 1 or M > n, then  $v_M = u_N$ where  $N \equiv M \pmod{n}$ . Also for the compactness of the notations, given an integer n, define [n] to denote  $\{1, 2, \ldots, n\}$ .

A path is a sequence of vertices  $(v_1, v_2, \ldots, v_k)$  such that for each i with  $1 \leq i \leq k-1$ we have  $v_i v_{i+1} \in E(G)$ . Likewise, a cycle is a sequence of vertices  $(v_1, v_2, \ldots, v_k)$  such that for each i with  $1 \leq i \leq k-1$  we have  $v_i v_{i+1} \in E(G)$  and also  $v_1 v_k \in E(G)$ . One can define path graphs and cycle graphs in the natural way. Specifically, the path graph on n vertices denoted  $P_n$  is the graph with vertex set  $V(P_n) = \{v_i\}_{i=1}^n$  and edge set  $E(P_n) = \{v_i v_{i+1}\}_{i=1}^{n-1}$ , and the cycle graph on n vertices denoted  $C_n$  is the graph with vertex set  $V(C_n) = \{v_i\}_{i=1}^n$  and edge set  $E(C_n) = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\}$ . A path cover of a graph G is a collection of vertex-induced path subgraphs  $\mathcal{Q} = \{Q_i\}_{i=1}^k$  such that  $\{V(Q_i)\}_{i=1}^k$  is a partition of the vertex set of G. The path cover number of G denoted P(G) is the minimum cardinality among the path covers of G. A path cover  $\mathcal{Q}$  is a minimum path cover of G if  $|\mathcal{Q}| = P(G)$ .

Zero forcing is a dynamic coloring process on graphs. Prior to initiating the process, the vertices of a graph G must first be each colored either blue or white. Once this is done, the zero forcing process is governed by the zero forcing color change rule, which states:

Zero forcing color change rule: If u is blue and v is the only white neighbor of u, then u can force v to be colored blue. If a vertex u forces v, then we denote this by  $u \to v$ .

To initiate the process, and for the duration of the process, the zero forcing color change rule is applied until either every vertex of G is blue or in its current state G contains no blue vertices with a unique white neighbor. Note that once a vertex is blue, it will remain blue. Further note that during each application of the zero forcing color change rule, multiple vertices may become blue, and that the process may include multiple such applications, with each application forming a step in the process and these steps being referred to as *time-steps*. If B is the set of vertices initially colored blue and after a sufficient number of time-steps every vertex in G is blue, then B is said to be a *zero forcing set* of G. The *zero forcing number* of G denoted Z(G) is the minimum cardinality among all zero forcing sets of G. A zero forcing set B of G is a *minimum zero forcing set* if |B| = Z(G).

In order to be able to construct rigorous arguments for proofs, it is necessary to introduce some additional terminology for discussing the specifics of the zero forcing process. We will be using the terminology introduced in [7]. The fundamental object in this discussion is the ordered collection of forces chosen during a zero forcing process called a relaxed chronology of forces. First define S(G, B') to be the set of valid forces in G when B' is the set of vertices which is currently blue. A relaxed chronology of forces  $\mathcal{F}$  is a collection of sets of forces  $\{F^{(k)}\}_{k=1}^{K}$  such that at each time-step k,  $F^{(k)} \subseteq S(G, E_{\mathcal{F}}^{[k-1]})$  where  $E_{\mathcal{F}}^{[k-1]}$  is the set of blue vertices after time-step k-1, with  $E_{\mathcal{F}}^{[0]} = B$  the set of vertices which were initially colored blue, and each vertex of  $v \in V(G) \setminus B$  occurring in exactly one force  $u \to v \in F^{(k)} \in \mathcal{F}$ . For a zero forcing set B and a relaxed chronology of forces  $\mathcal{F}$ , the sequence of sets  $\{E_{\mathcal{F}}^{[k]}\}_{k=0}^{K}$  with  $B = E_{\mathcal{F}}^{[0]} \subseteq E_{\mathcal{F}}^{[1]} \subseteq \cdots \subseteq E_{\mathcal{F}}^{[K-1]} \subseteq E_{\mathcal{F}}^{[K]} = V(G)$ , is called the *expansion sequence* of B induced by  $\mathcal{F}$  and for each time-step  $k, E_{\mathcal{F}}^{[k]}$  is called the *k*-th *expansion* of B induced by  $\mathcal{F}$ .

For a graph G, a zero forcing set B, and a relaxed chronology  $\mathcal{F}$ , a forcing chain induced by  $\mathcal{F}$  is a sequence of vertices  $(v_0, v_1, v_2, \ldots, v_N)$  such that the vertex  $v_0 \in B$ , the vertex  $v_N$  does not perform a force during  $\mathcal{F}$ , and for each i with  $0 \leq i \leq N-1$  we have  $v_i v_{i+1} \in F^{(k)} \in \mathcal{F}$ . The collection of vertices which do not perform a force during  $\mathcal{F}$  are the terminus of  $\mathcal{F}$ , denoted Term $(\mathcal{F})$ . The collection of forcing chains induced by  $\mathcal{F}$  is called a *chain set*, and since B is a zero forcing set, the forcing chains induced by  $\mathcal{F}$  form a partition of V(G). Furthermore, since vertices can only perform a force when they have a single white neighbor, it follows that each vertex of  $V(G) \setminus B$  is forced exactly once during  $\mathcal{F}$  and each vertex of  $V(G) \setminus \text{Term}(\mathcal{F})$  performs exactly one force during  $\mathcal{F}$ . Due to this, it follows that a chain set forms a path cover, providing the following result.

**Theorem 1.1** [2] Let G be a graph. Then  $P(G) \leq Z(G)$ .

Since forcing chains form a path cover, and paths are reversible, as a natural but very important result we have the following lemma concerning the terminus originally given in [2] and generalized in [7].

**Theorem 1.2** [2, 7] Let G be a graph, B be a zero forcing set of G, and  $\mathcal{F}$  be a relaxed chronology of forces of B on G. Then  $\text{Term}(\mathcal{F})$  is a zero forcing set of G.

#### 2 The Zero Forcing Numbers of Peony Graphs

In this section, we determine the zero forcing numbers of a class of graphs we call peony graphs. Intuitively, a peony graph is constructed by taking the star graph  $K_{1,m}$ and between each consecutive pair of pendant vertices in the star graph adding r paths of length s. The following is the formal definition of peony graphs.





Figure 1: The peopy graph Py(6,3,4)

Figure 2: A station  $S_1$  of the peony graph Py(6,3,4)

**Definition 2.1 (Peony Graph)** A peony graph, denoted as Py(m, r, s), is a graph with a vertex set  $V(Py(m, r, s)) = \{c\} \cup \{u_i\}_{i=1}^m \cup \{v_{i,j,k}\}_{i=1,j=1,k=1}^m$ . Given  $w, z \in V(Py(m, r, s))$  distinct, it follows that  $wz \in E(Py(m, r, s))$  if one of the following is true.

- $\{w, z\} = \{c, u_i\}$  for some  $i \in \{1, 2, ..., m\}$ ,
- $\{w, z\} = \{u_i, v_{i,j,1}\}$  for some  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, r\}$ ,

The pump journal of undergraduate research  $\mathbf{8}$  (2025), 258–272

- $\{w, z\} = \{u_i, v_{i-1,j,s}\}$  for some  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, r\}$ .
- $\{w, z\} = \{v_{i,j,k}, v_{i,j,k+1}\}$  for some  $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, r\}$ , and  $k \in \{1, 2, \dots, s-1\}$ .

To facilitate our proofs, it is useful to define certain substructures of peony graphs. For every  $i \in \{1, 2, ..., m\}$ , the *i*-th station, denoted  $S_i$ , is the collection of vertices  $\{u_i\} \cup \{v_{i,j,k}\}_{j=1,k=1}^r$ . For each i, j with  $i \in \{1, 2, ..., m\}$  and  $j \in \{1, 2, ..., r\}$ , the i, j-th layer, denoted  $S_{i,j}$ , is the collection of vertices  $\{v_{i,j,k}\}_{k=1}^s$ .

The concept of forts were introduced in [5] along with the following theorem. Forts can be considered as obstacles to the zero forcing processes. So they are often studied alongside zero forcing. For the purpose of this paper, forts will help us to determine the zero forcing number of peony graphs.

**Definition 2.2** [5] Let G be a graph and  $S \subseteq V(G)$ . If for all  $u \in V(G) \setminus S$ ,  $|N_G(u) \cap S| \neq 1$ , then we refer to S as a fort of G.

**Theorem 2.3** [5] Let G be a graph and  $S \subseteq V(G)$ . S is a zero forcing set of G if and only if for each fort F of G,  $S \cap F \neq \emptyset$ .

We now establish several families of forts that are present in peony graphs. We do so in order to calculate a lower bound on the zero forcing numbers of peony graphs. For the sake of clarity, we use the term type-x fort, with  $x \in \{2.4, 2.5, 2.6, 2.7\}$  to refer to a fort of the type identified in Lemmas 2.4, 2.5, 2.6, 2.7 respectively.



Figure 3: Type-2.4 fort

Figure 4: Type-2.5 fort

**Lemma 2.4** For each  $i \in \{1, 2, ..., m\}$  and each distinct  $j_1, j_2 \in \{1, 2, ..., r\}$ ,  $S_{i,j_1} \cup S_{i,j_2}$  is a fort of Py(m, r, s).

**Proof.** Let  $w \in V(Py(m, r, s)) \setminus (S_{i,j_1} \cup S_{i,j_2})$ . If  $w \notin \{u_i, u_{i+1}\}$ , then  $|N_{Py(m,r,s)}(w) \cap (S_{i,j_1} \cup S_{i,j_2})| = 0$ . If  $w \in \{u_i, u_{i+1}\}$ , then  $|N_{Py(m,r,s)}(w) \cap (S_{i,j_1} \cup S_{i,j_2})| = 2$ . Therefore,  $S_{i,j_1} \cup S_{i,j_2}$  is a fort.

**Lemma 2.5** For every  $i \in \{1, 2, ..., m\}$ , let  $j_i \in \{1, 2, ..., r\}$ . Then  $S = \bigcup_{i=1}^m S_{i,j_i}$  is a fort of Py(m, r, s).

**Proof.** Let  $w \in V(Py(m,r,s)) \setminus S$ . If  $w \notin \{u_i\}_{i=1}^m$ , then  $|N_{Py(m,r,s)}(w) \cap S| = 0$ . If  $w \in \{u_i\}_{i=1}^m$ , then  $|N_{Py(m,r,s)}(w) \cap S| = 2$ , because for any  $i \in \{1, 2, ..., m\}, |N_{Py(m,r,s)}(u_i) \cap S| = |N_{Py(m,r,s)}(u_i) \cap (S_{i,k_i} \cup S_{i+1,k_{i+1}})| = 2$ . Therefore, S is a fort.



Figure 5: Type-2.6 fort

Figure 6: One example of the set B constructed in Lemma 2.7

**Lemma 2.6** Let  $i_0 \in [m]$ . For each  $i \in \{1, 2, \dots, m\} \setminus \{i_0\}$  choose  $j_i \in \{1, 2, \dots, r\}$ . Then  $S = \left(\bigcup_{i \in [m] \setminus \{i_0\}} S_{i,j_i} \cup \{c\}\right) \text{ is a fort of } Py(m, r, s).$ 

**Proof.** Let  $w \in V(Py(m,r,s)) \setminus S$ . If  $w \notin \{u_i\}_{i=1}^m$ , then  $|N_{Py(m,r,s)}(w) \cap S| = 0$ . Given  $w \in \{u_i\}_{i=1}^m$ , if  $w \in \{u_{i_0}, u_{i_0+1}\}$ , then  $|N_{Py(m,r,s)}(w) \cap S| = 2$ , and otherwise  $|N_{Py(m,r,s)}(w) \cap S| = 3$ . Therefore, S is a fort.

**Lemma 2.7** Let  $T = \{(i, j) | i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., r\}\}$ . For each  $p = (i, j) \in T$ , let  $v_{i,j,k_p} \in S_{i,j}$ . If  $B = \{c\} \cup \{v_{i,j,k_p}\}_{i=1,j=1}^{m}$ , then  $V(Py(m,r,s)) \setminus B$  is a fort of Py(m,r,s).

**Proof.** Let  $w \in B$ .

Case 1. w = cIt follows that  $N_{Py(m,r,s)}(w) \cap (V(Py(m,r,s)) \setminus B) = \{u_i\}_{i=1}^m$  and  $|\{u_i\}_{i=1}^m| = m > 1$ . Case 2.  $w \in \{v_{i,j,k_p}\}_{1=1,j=1}^m$ 

First fix  $w = v_{i_0,j_0,k_0}$  where  $i_0 \in [m]$ ,  $j_0 \in [r]$ , and  $k_0$  denotes  $k_p$  with  $p = (i_0, j_0)$ .

- If s = 1, then  $|N_{Py(m,r,s)}(w) \cap (V(Py(m,r,s)) \setminus B)| = |N_{Py(m,r,s)}(w) \cap \{u_i\}_{i=1}^m| = 2$ .
- If s = 2, then  $|N_{Py(m,r,s)}(w) \cap \{u_i\}_{i=1}^m| = 1$  and  $|N_{Py(m,r,s)}(w) \cap \{v_{i_0,j_0,k}\}_{k \in [s] \setminus \{k_0\}}| = 1$ . Since  $\{u_i\}_{i=1}^m \cap \{v_{i_0,j_0,k}\}_{k \in [s] \setminus \{k_0\}} = \emptyset$ , it follows that  $|N_{Py(m,r,s)}(w) \cap (V(Py(m,r,s)) \setminus B)| = 2$ .
- Now suppose  $s \ge 3$ . If  $k_p \in \{1, s\}$ , then  $|N_{Py(m,r,s)}(w) \cap \{u_i\}_{i=1}^m| = 1$  and  $|N_{Py(m,r,s)}(w) \cap \{v_{i_0,j_0,k}\}_{k \in [s] \setminus \{k_0\}}| = 1$ . So  $|N_{Py(m,r,s)}(w) \cap (V(Py(m,r,s)) \setminus B)| = 2$ . Otherwise,  $|N_{Py(m,r,s)}(w) \cap \{v_{i_0,j_0,k}\}_{k \in [s] \setminus \{k_0\}}| = 2$ .

Therefore,  $V(Py(m, r, s)) \setminus B$  is a fort.

From Theorem 2.3 we see that if  $Py(m, r, s) \setminus B$  contains a fort, then B cannot be a zero forcing set of Py(m, r, s). Using this fact, we prove the following theorem.

**Theorem 2.8** Let Py(m, r, s) be a peoply graph. Then  $Z(Py(m, r, s)) \ge m(r-1) + 3$ .

**Proof.** To prove this result, we will construct a set of minimum cardinality  $B \subseteq V(Py(m, r, s))$  which intersects all type-*x* forts for  $x \in \{2.4, 2.5, 2.6\}$ . Noting that |B| = m(r-1)+2, we will then identify a type-2.7 fort *F* for which  $B \cap F = \emptyset$ , completing the proof.

First, we construct a set,  $B_1$ , which intersects all type-2.4 forts. Since type-2.4 forts are comprised of two layers from the same station, to intersect all of them, one vertex must be taken from each of r-1 layers at every station. Since the collection of stations  $\{S_i\}_{i=1}^m$ is pairwise disjoint, and furthermore the collection of layers  $\{S_{i,j}\}_{j=1}^r$  at the *i*-th station is also pairwise disjoint, this will require at least m(r-1) vertices, specifically, r-1 vertices chosen from distinct layers at each of the *m* stations. To construct an arbitrary set  $B_1$  of this type, for each station *i*, choose  $j_i \in [r]$  and for each  $S_{i,j}$  with  $j \neq j_i$ , choose  $k_{i,j} \in [s]$ . Let  $B_1 = \{v_{i,j,k_{i,j}}\}_{i=0,j\neq j_i}^m$ . Note that  $B_1$  is of the minimum cardinality necessary to hit all of the type-2.4 forts.

Second, we construct a set,  $B_2$ , containing  $B_1$ , which intersects all type-2.4 and type-2.5 forts. Since type-2.5 forts take a layer from every station, and for each station i, there is a layer  $S_{i,j_i}$  for which  $B_1 \cap S_{i,j_i} = \emptyset$ , it follows that  $\bigcup_{i=1}^m S_{i,j_i}$  is a type-2.5 fort for which  $B_1 \cap \bigcup_{i=1}^m S_{i,j_i} = \emptyset$ . With this in mind, choose  $i_2 \in [m]$  and  $k_2 \in [s]$ , and let  $B_2 = B_1 \cup \{v_{i_2,j_{i_2},k_2}\}$ . Note, the choice of  $B_1$  was arbitrary and we have shown that for each choice of  $B_1$ , a type-2.5 fort disjoint from  $B_1$  can be identified. Thus, it follows that to hit all type-2.4 and type-2.5 forts, will require at least m(r-1) + 1 vertices.

Next, we construct a set,  $B_3$ , containing  $B_2$ , which intersects all type-2.4, type-2.5, and type-2.6 forts. Since the center vertex and a layer from each of m-1 distinct stations form a type-2.6 fort, and for each station i with  $i \neq i_2$  there is a layer  $S_{i,j_i}$  for which  $B_2 \cap S_{i,j_i} = \emptyset$ , it follows that  $\bigcup_{i\neq i_2} S_{i,j_i} \cup \{c\}$  is a type-2.6 fort for which  $B_2 \cap \left(\bigcup_{i\neq i_2} S_{i,j_i} \cup \{c\}\right) = \emptyset$ . Thus, the possible alternatives are either  $B_3 = B_2 \cup \{c\}$  or  $B_3 = B_2 \cup \{v_{i_3,j_{i_3},k_3}\}$  for arbitrary chosen  $i_3 \in [m] \setminus \{i_2\}$  and  $k_3 \in [s]$ . Note that the choice of  $B_2$  was arbitrary and for any choice of  $B_2$ , a type-2.6 fort disjoint from  $B_2$  can be identified. Either way, to not miss any type-2.4, type-2.5, or type-2.6 forts,  $B_3$  must contain at least m(r-1) + 2vertices.

The pump journal of undergraduate research 8 (2025), 258–272

Finally, since  $B_3$  has been constructed by taking at most one vertex from each layer (as well as possibly the center vertex), by Lemma 2.7,  $V(Py(m, r, s)) \setminus B_3$  contains a type-2.7 fort F. Thus, any set intersecting all type-x forts for  $x \in \{2.4, 2.5, 2.6, 2.7\}$  must contain at minimum m(r-1) + 3 vertices.

**Theorem 2.9** Py(m, r, s) be a peoply graph. Then Z(Py(m, r, s)) = m(r - 1) + 3.

**Proof.** Let  $B = \{v_{i,j,1}\}_{i=2,j=2}^{m} \cup \{v_{m,j,s}\}_{j=1}^{r} \cup \{c\} \cup \{u_1\}$  be the set of vertices initially colored blue. Note that  $|\{v_{i,j,1}\}_{i=2,j=2}^{m} \cup \{v_{m,j,s}\}_{j=1}^{r}| = (m-1)(r-1) + r = m(r-1) + 1$ , so |B| = m(r-1) + 3.

We now construct a relaxed chronology of forces  $\mathcal{F}$  of B on Py(m, r, s). On the first time-step, the vertex  $v_{1,1,1}$  is the unique white neighbor of  $u_1$ , so we can let  $u_1$  force  $v_{1,1,1}$ , and after the first time-step  $v_{1,1,1}$  will be blue. On time-step p with  $2 \leq p \leq s$ , because  $\{v_{1,j,p-1}\}_{j=1}^r$  is a collection of blue vertices and for each  $j \in [r], v_{1,j,p}$  is the unique white neighbor of  $v_{1,j,p-1}$ , we can perform the forces  $\{v_{1,j,p-1} \rightarrow v_{1,j,p}\}_{j=1}^r$ . Thus, after time-step s, all vertices in the first station are blue. Similar forces occur in the m-th station. On time-step p with  $2 \leq p \leq s$ , because  $\{v_{m,j,s-p+2}\}_{j=1}^r$  is a collection of blue vertices and for each  $j \in [r], v_{m,j,s-p+1}$  is the unique white neighbor of  $v_{m,j,s-p+2}$ , we can perform the forces  $\{v_{m,j,s-p+2} \rightarrow v_{m,j,s-p+1}\}_{j=1}^r$ . On time-step s + 1, the vertex  $u_m$  is the unique white neighbor of  $v_{m,1,1}$ , so we can let  $v_{m,1,1}$  force  $u_m$ . Note that after time-step s + 1, both the first and m-th stations are blue.

To color the entire second station,  $u_2$  first must be made blue. On time-step s + 2, since the vertex  $u_2$  is the unique white neighbor of  $v_{1,1,s}$ ,  $v_{1,1,s}$  forces  $u_2$ . The remaining process of coloring the second station blue is identical to that which occurred in the first station. Now, repeat the aforementioned process m - 3 times, thus forcing stations 3 through m - 1 to become blue, and at this point no white vertices remain in Py(m, r, s).

Since B is a zero forcing set of Py(m,r,s) and |B| = m(r-1) + 3, it follows that  $Z(Py(m,r,s)) \leq m(r-1) + 3$ . Thus, by Theorem 2.8, we can conclude that Z(Py(m,r,s)) = m(r-1) + 3.

#### 3 The Zero Forcing Numbers of Web Graphs

We now introduce a class of graphs which we call web graphs. It is worth noting that the definition we provide here is more generalized than another definition which is commonly used, and sometimes the parameter r used to define our more generalized version is fixed such that r = 2.

**Definition 3.1** We define the web graph Wb(m, r) to be the graph with vertex set  $V(Wb(m, r)) = \{v_{i,j}\}_{i=1,j=1}^{m} \cup \{p_i\}_{i=1}^{m}$  such that for  $u, w \in V(Wb(m, r))$  distinct,  $uw \in E(Wb(m, r))$  if and only if one of the following is true:

•  $\{u, w\} = \{p_i, v_{i,1}\}$ 

• 
$$i_1 = i_2$$
 and  $|j_1 - j_2| = 1$ 

The pump journal of undergraduate research  $\mathbf{8}$  (2025), 258–272

•  $j_1 = j_2$ , and either  $\{i_1, i_2\} = \{1, m\}$  or  $|i_1 - i_2| = 1$ .



Figure 7: The web graph Wb(5,3)

Alternatively this can be viewed as a result of a small modification to the graph  $V(C_m \Box P_r)$ , where the *Cartesian product* of two graphs G and H denoted  $G \Box H$  is defined to be the graph with vertex set  $V(G) \times V(H)$  such that two vertices  $(g_1, h_1), (g_2, h_2) \in V(G \Box H)$  are adjacent in  $G \Box H$  if either  $g_1 = g_2$  and  $h_1 h_2 \in E(H)$  or  $h_1 = h_2$  and  $g_1 g_2 \in E(G)$ . Specifically, Wb(m, r) is the result of adding the set  $\{p_i\}_{i=1}^m$  of pendant vertices to the graph  $C_m \Box P_r$ , with vertex set  $V(C_m \Box P_r) = \{v_{i,j}\}_{i=1,j=1}^m$ , such that for each  $i, p_i$  is adjacent to  $v_{i,1}$ . To determine the zero forcing numbers of web graphs, we first identify a pair of commonly cited results which will be key to our argument.

**Lemma 3.2** [1]  $Z(C_m \Box P_r) = \min\{m, 2r\}.$ 

**Lemma 3.3** [2]  $P(G) \leq Z(G)$ . Furthermore, if C is a chain set for some relaxed chronology of forces  $\mathcal{F}$  for some zero forcing set B of G, then C is a path cover of G.

Recent work in the area has been focused on identifying and studying the interaction between local properties of a graph (those confined to a particular subgraph) and global properties of a graph and the affect this interaction has on zero forcing. One such concept introduced in [7] involves restricting relaxed chronologies of forces to those forces in a given subgraph. Specifically, given a graph G and H a vertex-induced subgraph of G, if  $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$  is a relaxed chronology of forces of some zero forcing set B of G, then the restriction of  $\mathcal{F}$  to H denoted  $\mathcal{F}|_{H} = \{F|_{H}^{(k)}\}_{k=1}^{K}$  is the relaxed chronology of forces such that at each time-step k we have  $F_{H}^{(k)} = \{u \to v : u, v \in V(H) \text{ and } u \to v \in F^{(k)}\}$ . Letting  $\mathcal{C}$  be the chain set induced by  $\mathcal{F}$ , the forcing subpaths of  $\mathcal{C}$  in H are the subgraphs of H induced by the vertices in each member of  $\mathcal{C}$ . A vertex  $u \in V(H)$  is an initial vertex of a forcing subpath of  $\mathcal{C}$  in H if either  $u \in B$  or  $v \to u$  is a force which occurs during  $\mathcal{F}$  but  $v \in V(G) \setminus V(H)$ . The following result concerning initial vertices helps us to prove a lower bound on the zero forcing numbers of web graphs.

**Lemma 3.4** [7] Let G be a graph, H be a vertex-induced subgraph of G, B be a standard zero forcing set of G, and  $\mathcal{F}$  be a relaxed chronology of standard forces of B on G with chain set C. Let B' be the set of initial vertices in the forcing subpaths of C in H. Then B' is a standard zero forcing set of H, and  $\mathcal{F}|_H$  defines a relaxed chronology of standard forces for B' in H.

Lemma 3.5  $Z(Wb(m,r)) \ge \max\left\{ \left\lceil \frac{m}{2} \right\rceil, \min\{m, 2r\} \right\}.$ 

**Proof.** The graph Wb(m, r) contains m pendant vertices. If  $\mathcal{Q}$  is a path cover of Wb(m, r), then since  $\mathcal{Q}$  forms a partition of V(Wb(m, r)) each of these m pendant vertices must be contained in some  $Q \in \mathcal{Q}$ . Furthermore, for each  $Q \in \mathcal{Q}$ , Q contains at most 2 pendant vertices as the vertices which are not endpoints of Q each have two distinct neighbors in Q. So by 3.3,  $Z(Wb(m, r)) \geq P(Wb(m, r)) \geq \left\lceil \frac{m}{2} \right\rceil$ .

We will now show, by way of contradiction, that  $Z(Wb(m,r)) \ge \min\{m,2r\}$ . Suppose B is a minimum zero forcing set of Wb(m,r) with  $|B| < \min\{m,2r\}$ . Let  $\mathcal{F}$  be a relaxed chronology of forces of B on Wb(m,r) and let  $\mathcal{C}$  be the chain set induced by  $\mathcal{F}$ . Now let H be the subgraph of Wb(m,r) with  $H \cong C_m \Box P_r$ , let  $\mathcal{F}|_H$  be the restriction of  $\mathcal{F}$  to H, and let  $\mathcal{C}_H$  be the collection of forcing subpaths of  $\mathcal{C}$  contained in H. By 3.4, if  $B_H$  is the set of initial vertices of  $\mathcal{C}_H$  in H, then  $B_H$  is a zero forcing set of H. Since pendant vertices can only be the endpoints of paths, and thus the endpoints of forcing chains, each forcing chain  $C \in \mathcal{C}$  contains at most one member of  $\mathcal{C}_H$  as a forcing subpath. So it follows that  $B_H$  is a zero forcing set of  $H \cong C_m \Box P_r$  with  $|B_H| = |\mathcal{C}_H| \le |\mathcal{C}| = |B| < \min\{m, 2r\}$ , a contradiction of Lemma 3.2. Thus  $Wb(m,r) \ge \min\{m, 2r\}$ .

#### **Lemma 3.6** If $m \leq 2r$ , then $Z(Wb(m, r)) \leq m$ .

**Proof.** Let  $B = \{p_i\}_{i=1}^m$ . We will construct a relaxed chronology of forces  $\mathcal{F}$  for B on Wb(m, r). On time-step 1,  $F^{(1)} = \{p_i \to v_{i,1}\}_{i=1}^m$ . On time-step k with  $2 \le k \le r$ ,  $F^{(k)} = \{v_{i,k-1} \to v_{i,k}\}_{i=1}^m$ . So after time-step r, we have that  $E_{\mathcal{F}}^{[r]} = V(Wb(m, r))$ . Thus,  $Z(Wb(m, r)) \le |B| = m$ 

**Lemma 3.7** If  $\lceil \frac{m}{2} \rceil < 2r < m$ , then  $Z(Wb(m, r)) \leq 2r$ .

**Proof.** Let  $B = \{p_i\}_{i=1}^{2r}$ . We will construct a relaxed chronology of forces  $\mathcal{F}$  for B on Wb(m, r). On time-step 1,  $F^{(1)} = \{p_i \rightarrow v_{i,1}\}_{i=1}^{2r}$ . Moving forward, for k with  $2 \leq k \leq 2r$ , at each successive time-step k, fewer vertices in the set  $\{v_{i,k-1}\}_{i=1}^{2r}$  will be able to force as an increasing number of vertices in these sets will have more than one white neighbor. However, forcing will continue. Specifically, for k with  $2 \leq k \leq r$ , during time-step k the forces in the set  $\{v_{i,k-1} \rightarrow v_{i,k}\}_{i=k}^{2r-k+1}$  can be performed because  $v_{i,k}$  is the only white neighbor of  $v_{i,k-1}$ . On the other hand, once these forces are completed, an increasing number of vertices will be able to force during each successive time-step k, with  $r+1 \leq k \leq 2r-1$ . Specifically, for k with  $r+1 \leq k \leq 2r-1$ , during time-step k the forces in



$p_1$	$p_2$	$p_3$	$p_4$	$p_5$	$p_6$
$v_{1,1}$	$v_{2,1}$	$v_{3,1}$	$v_{4,1}$	$v_{5,1}$	$v_{6,1}$
$v_{1,2}$	$v_{2,2}$	v <sub>3,2</sub>	$v_{4,2}$	$v_{5,2}$	$v_{6,2}$
$v_{1,3}$	$v_{2,3}$	$v_{3,3}$	$v_{4,3}$	$v_{5,3}$	$v_{6,3}$

Figure 8: Wb(9,3) and an example of the subgraph H of Wb(9,3) discussed in Lemma 3.7



Figure 9: The first 2r - 1 = 5 time-steps of  $\mathcal{F}$  in Wb(9,3)

the set  $\{v_{2r-k+1,j} \rightarrow v_{2r-k,j}, v_{k,j} \rightarrow v_{k+1,j}\}_{j=2r-k+1}^r$  can be performed because  $v_{2r-k,j}$  is the

only white neighbor of  $v_{2r-k+1,j}$  and  $v_{k+1,j}$  is the only white neighbor of  $v_{k,j}$ . Now identify that  $E_{\mathcal{F}}^{[2r-1]} = \{v_{i,j}\}_{i=1,j=1}^{2r} \cup \{p_i\}_{i=1}^{2r}$ , and let H be the subgraph of Wb(m,r) induced by the vertices which are blue after time-step 2r-1, specifically, H = 1 $Wb(m,r)[E_{\mathcal{F}}^{[2r-1]}]$ . Since every vertex in H is blue after time-step 2r-1, B is a zero forcing set of H. Furthermore,  $\mathcal{F}|_H$  is a relaxed chronology of forces of B on H, so by Theorem 1.2,  $\operatorname{Term}(\mathcal{F}|_H)$  is also a zero forcing set of H.

Now let  $H' = Wb(m, r)[\{v_{i,j}\}_{i=m-2r+2, j=1}^r \cup \{v_{1,j}\}_{j=1}^r \cup \{p_i\}_{i=m-2r+2}^m \cup \{p_1\}]$  and note that  $H \cong H'$  under some isomorphism  $\sigma$ . Due to this,  $\sigma(\operatorname{Term}(\mathcal{F}|_H))$  is a zero forcing set of H'. Furthermore, since  $\left\lceil \frac{m}{2} \right\rceil < 2r$ , it follows that  $m - 2r + 2 \leq 2r$  and the vertex sets of H and H' overlap in such a way that  $\sigma(\operatorname{Term}(\mathcal{F}|_H)) \subseteq E_{\mathcal{F}}^{[2r-1]}$ . It then follows that  $E_{\mathcal{F}}^{[2r-1]}$  contains a zero forcing set of H'. Since Wb(m,r) contains no blue vertices outside of H' after time-step 2r - 1, the forcing process can continue and B must be a

zero forcing set of Wb(m, r). Thus,  $Z((Wb(m, r)) \le |B| = 2r$ .



Figure 10: The zero forcing set B of Wb(16,3)

**Lemma 3.8** If  $2r \leq \lceil \frac{m}{2} \rceil$ , then  $Z(Wb(m, r)) \leq \lceil \frac{m}{2} \rceil$ .

**Proof.** Construct the zero forcing set B using the following pendant vertices. First take  $\{p_i\}_{i=1}^{2r}$  and then also include every second pendant vertex starting at  $p_{2r+2}$  up until vertex  $p_{m-2r}$ , making sure to include vertex  $p_{m-2r}$  even in the case where m is odd. Specifically, in the case that m is even  $B = \{p_i\}_{i=1}^{2r} \cup \{p_{2r+2i}\}_{i=1}^{(m-4r)/2}$ , and in the case that m is odd  $B = \{p_i\}_{i=1}^{2r} \cup \{p_{2r+2i}\}_{i=1}^{(m-1-4r)/2} \cup \{p_{m-2r}\}$ ; and in either case  $|B| = \lceil \frac{m}{2} \rceil$ . As in Lemma 3.7, chain sets beginning at the vertices  $\{p_i\}_{i=1}^{2r}$  can force  $\{v_{i,j}\}_{i=1,j=1}^{2r}$  blue and will finish doing so at the conclusion of time-step 2r - 1.



Figure 11: Time-steps 2r = 6 through m - 2r - 1 = 9 of  $\mathcal{F}$  in Wb(16, 3)

Now, during time-step 2r the forces  $\{v_{2r,j} \to v_{2r+1,j}\}_{j=1}^r$  as well as  $p_{2r+2} \to v_{2r+2,1}$  can occur. At which point, the only white neighbor of  $v_{2r+1,1}$  will be  $p_{2r+1}$ , so the force

269

 $v_{2r+1,1} \rightarrow p_{2r+1}$  as well as the set of forces  $\{v_{2r+1,j} \rightarrow v_{2r+2,j}\}_{j=2}^r$  can occur during timestep 2r + 1. Time-steps analogous to Time-steps 2r and 2r + 1 will alternate (with one additional time-step during which  $\{v_{m-2r-1,j} \rightarrow v_{m-2r,j}\}_{j=1}^r$  in the case where m is odd) until time-step m - 2r - 1, at which point  $\{v_{i,j}\}_{i=1, j=1}^{m-2rr} \cup \{p_i\}_{i=1}^{m-2r} \subseteq E_{\mathcal{F}}^{[m-2r-1]}$ . Finally, during time-step m - 2r, the sets of forces  $\{v_{1,j} \rightarrow v_{m,j}\}_{j=1}^r$  and  $\{v_{m-2r,j} \rightarrow v_{m-2r+1,j}\}_{j=1}^r$ can occur.



Figure 12: Subgraph H' of Wb(16,3) discussed in Lemma 3.8



Figure 13: The last 2r - 1 = 5 time-steps of  $\mathcal{F}$  in Wb(16,3)

Next, in a way similar to Lemma 3.7, define  $H = Wb(m, r) \left[ \{ v_{i,j} \}_{i=1,j=1}^{2r} \cup \{ p_i \}_{i=1}^{2r} \right]$ and  $H' = Wb(m, r) \left[ \{ v_{i,j} \}_{i=m-2r+1,j=1}^{m} \cup \{ p_i \}_{i=m-2r+1}^{m} \right]$ . Note that again H and H' are isomorphic by some isomorphism  $\sigma$ , and as before the fact that  $\operatorname{Term}(\mathcal{F}|_H)$  is a zero forcing set of H implies that  $\sigma(\operatorname{Term}(\mathcal{F}|_H))$  is a zero forcing set of H'. Since  $\sigma(\operatorname{Term}(\mathcal{F}|_H)) =$  $\{ v_{m,j}, v_{m-2r+1,j} \}_{j=1}^{r} \subseteq E_{\mathcal{F}}^{[m-2r]}$ , it follows that  $E_{\mathcal{F}}^{[m-2r]}$  contains a zero forcing set of H'. Finally, since there are no white vertices outside of H', B is a zero forcing set of Wb(m, r)and  $Z(Wb(m, r)) \leq \left\lceil \frac{m}{2} \right\rceil$ .

Combining Lemmas 3.5, 3.6, 3.7, and 3.8, one obtains the following theorem.

**Theorem 3.9**  $Z(Wb(m,r)) = \max\left\{ \left\lceil \frac{m}{2} \right\rceil, \min\{m, 2r\} \right\}, \text{ or equivalently}$ 

$$Z(Wb(m,r)) = \begin{cases} m & \text{if } m \le 2r, \\ 2r & \text{if } \lceil \frac{m}{2} \rceil < 2r < m, \\ \lceil \frac{m}{2} \rceil & \text{if } 2r \le \lceil \frac{m}{2} \rceil. \end{cases}$$

The pump journal of undergraduate research 8 (2025), 258–272

# 4 Conclusion

This wraps up our exploration of peony graphs and web graphs. In this paper we have seen that recent discoveries concerning the interactions of graph substructures and the zero forcing numbers of graphs can provide not only new abstract concepts but also concrete tools for determining the zero forcing numbers of relatively complex graphs classes. Since new graph substructures related to zero forcing are an active topic of research in this area, it seems that further exploration of the applications of these new substructures warrants additional study.

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The pump journal of undergraduate research  ${f 8}$  (2025), 258–272

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