Firefighting on Trees and Infinite Grids

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Abstract - We apply Hartnell's model for firefighting on graphs. For rooted trees, we propose an Unburning Algorithm, a greedy algorithm starting from the leaves and working back towards the root. We show that the algorithm saves at least half the vertices of the optimal solution and that this bound is sharp. We confirm a conjecture of Hartke about integrality gaps when comparing linear and integer program solutions. For general graphs, we propose a Containment Protocol, which looks ahead two time steps to decide where to place firefighters. We show that the protocol performs near optimally on four well studied infinite grids. The protocol is available for any graph and we realize this flexibility by investigating an infinite pentagonal graph.

Keywords : graphs; firefighting; integrality gap; rooted trees; infinite grids

Mathematics Subject Classification (2020) **:** 05C57; 05C85; 92D30

1 Introduction

In 1995, Hartnell [5] proposed a model for fire spread on a graph through discrete time steps. Each vertex is in one of three states: susceptible, burned, or protected. Once a susceptible vertex is burned, the vertex remains burned and similarly for the protected state. In other words, the only available response is that a finite number n of susceptible vertices may be protected by a firefighter at each time step.

The fire begins with a single vertex burned. At each time step, we place n firefighters and then the conflagration spreads from the burned vertices to any neighbors that remain susceptible. The goal is to use the firefighters to contain the fire with as few vertices as possible burned. The fire is contained when no burned vertex has a susceptible neighbor. A secondary and related goal is to contain the fire in the least possible number of time steps.

In this paper we use this model to investigate firefighting on two types of graphs, rooted trees and infinite grids. For rooted trees, the fire begins with the root vertex burned and we place one (i.e., $n = 1$) firefighter per time step. It is known that the greedy algorithm of Hartnell and Li [6] will protect at least half as many vertices as an optimal solution. We propose the Unburning Algorithm, which is another greedy algorithm, but working backwards from the leaves towards the root. This algorithm also performs quite well.

Theorem 1.1 *The Unburning Algorithm saves at least half as many vertices as saved by an optimal placement of firefighters.*

We prove this in Section 2.1 where we also present an infinite family of trees that show it is sharp. It turns out Hartnell and Li's greedy algorithm is optimal for this infinite family. We remark that there is a similar infinite family due to Hartke [3] with the roles reversed: the Unburning Algorithm is optimal and the vertices saved by Hartnell and Li's algorithm become arbitrarily close to half; see the discussion at the end of Section 2.1.

Hartke [4] made some modifications to an integer program of MacGillivray and Wang (see [7]) for firefighting in a rooted tree. This led him to a linear program that is a good approximation to the optimal solution found by the integer program. Indeed, Hartke conjectured that the linear program realizes the optimal solution for all trees of 12 or fewer vertices. In Section 2.2 we verify this conjecture.

For infinite grids, we propose a Containment Protocol, which is a strategy for placing firefighters based on looking ahead two time steps. The Containment Protocol is quite robust and in Section 3 we show that it either matches or nearly matches the optimal solution for four well studied infinite grids, the hexagonal, square, triangular, and strong grids. The Containment Protocol could be applied to any graph. As an example, in Section 4 we investigate an infinite pentagonal grid. We report on the best solutions found by the protocol that, we conjecture, are optimal.

2 Trees

In this section we investigate firefighting on a rooted tree. At time step 0, the fire begins with the root vertex burned. At each time step we place one firefighter. We will first contrast Hartnell and Li's greedy algorithm with our Unburning Algorithm. Then, in Section 2.2, we verify a conjecture of Hartke [4].

2.1 Greedy Versus Unburning Algorithm

In a rooted tree with root v_0, the *level* $l(v)$ of vertex v is the distance or number of edges between v_0 and v: $l(v) = d(v_0, v)$. Let T_v denote the subtree rooted at vertex v. Let a_i be the vertex that is protected by a firefighter at time step i. An optimal firefighting sequence $a_1, a_2, \ldots$ is one that saves the maximum number of vertices. Hartnell and Li, and, independently, MacGillivray and Wang provided an important basic observation.

Theorem 2.1 ([6, 7]) *If $a_1, a_2, \ldots$ is optimal, then $l(a_i) = i$ for all i.*

In Hartnell and Li's greedy algorithm, at each time step i, we protect the vertex at level i that is the root of a biggest remaining subtree. This protects the vertex along with all the vertices in its subtree. That is, if we define the weight of vertex v as the number of vertices in the subtree rooted at v: $\mathrm{wt}(v) = |V(T_v)|$, then at time i, Hartnell and Li's algorithm protects a vertex v that maximizes $wt(v)$ among the remaining, unprotected vertices with $l(v) = i$.

Hartnell and Li showed that their algorithm performs quite well, in general.

Theorem 2.2 (Hartnell and Li [6]) *Hartnell and Li's algorithm saves at least half as many vertices as saved by an optimal placement of firefighters.*

We next propose a new algorithm that we call the Unburning Algorithm. We imagine a fire sweeping through the tree and then attempt to 'unburn' it starting from the leaves.

In the Unburning Algorithm, we work backwards from the maximum level n, choosing the sequence $u_n, u_{n-1}, u_{n-2}, \dots$ in turn with $l(u_i) = i$. To avoid some obvious poor choices, the weight of vertices at level i are updated to reflect vertices protected below level i. Suppose v has $l(v) = i$ and that $\{u_1^v, u_2^v, \dots, u_{k_v}^v\}$ is the (possibly empty) set of vertices in T_v included in the sequence $u_n, u_{n-1}, \dots, u_{i+1}$. Let U_v be the vertices in T_v protected by those firefighters:

$$U_v = \bigcup_{k=1}^{k_v} V(T_{u_k^v}).$$

We define

$$\tilde{\mathrm{wt}}(v) = |V(T_v)| - |U_v|.$$

and choose u_i among the vertices v with $l(v) = i$ to maximize $\tilde{\mathrm{wt}}(v)$.

Our main observation is that, similar to Hartnell and Li's Algorithm, the Unburning Algorithm performs quite well, in general.

Theorem 1.1 *The Unburning Algorithm saves at least half as many vertices as saved by an optimal placement of firefighters.*

Proof. We adapt Hartke's [3] charging argument for Hartnell and Li's algorithm. Fix an optimal sequence of firefighters $b_1, b_2, \dots, b_k$ that saves the largest number of vertices and let $u_1, u_2, \dots, u_\ell$ be the vertices selected by the Unburning Algorithm. Whenever a vertex u_i is in the subtree T_{b_j} for some b_j we will assign or 'charge' u_i to b_j. This means $i \geq j$. We can assume that no b_i is in the subtree T_{b_j} with $j < i$ as we would then save the same number of vertices by removing b_i from the sequence. Then each u_i is charged to at most one b_j. Let $C_{b_j} = \{u_i \mid u_i \text{ is charged to } b_j\}$ denote the set of vertices charged to b_j.

Let $S_o = \sum_{i=1}^{k} \mathrm{wt}(b_i)$ denote the number of vertices saved by the optimal sequence. Similarly, let $S_u = \sum_{i=1}^{\ell} \tilde{\mathrm{wt}}(u_i)$ denote the number of vertices saved by the Unburning Algorithm. If, for some i, $\tilde{\mathrm{wt}}(u_i) < \mathrm{wt}(b_i)$, it must be that $C_{b_i} \neq \emptyset$ as otherwise the Unburning Algorithm would have chosen $u_i = b_i$. Since the Unburning Algorithm chooses u_i rather than b_i, we know that the $\tilde{\mathrm{wt}}(u_i)$ together with the weights of the u_j assigned to b_i is at least as big as $\mathrm{wt}(b_i)$. That is,

$$\mathrm{wt}(b_i) \leq \tilde{\mathrm{wt}}(u_i) + \sum_{u \in C_{b_i}} \tilde{\mathrm{wt}}(u).$$

On the other hand, this inequality remains valid in case $\tilde{\mathrm{wt}}(u_i) \geq \mathrm{wt}(b_i)$.

Summing over the vertices in the optimal sequence,

$$\begin{aligned} S_o &= \sum_{i=1}^{k} \mathrm{wt}(b_i) \\ &\leq \sum_{i=1}^{k} \left(\tilde{\mathrm{wt}}(u_i) + \sum_{u \in C_{b_i}} \tilde{\mathrm{wt}}(u) \right). \end{aligned}$$

Fix an index i_0 in $1, \ldots, \ell$. There are at most two occurrences of u_{i_0} in the sum. It will appear in $\tilde{\mathrm{wt}}(u_i)$ when $i = i_0$. Also, u_{i_0} is the descendant of at most one b_j. So, u_{i_0} may occur again in C_{b_j} when $i = j$. Since each u_{i_0} occurs at most twice in the sum, we conclude that $S_o \leq 2S_u$, which means $S_u \geq \frac{1}{2} S_o$. $\square$

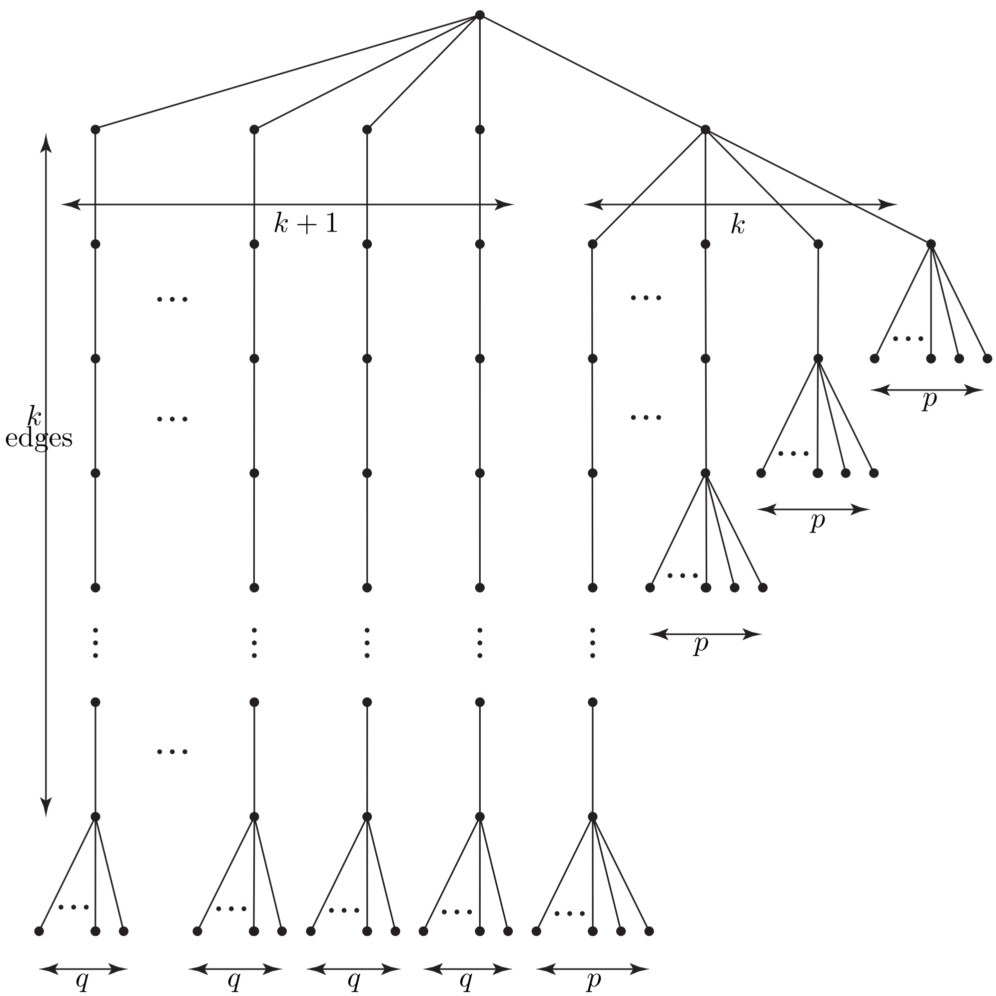

Figure 1: The (k, p, q) tree.

We next show that this bound is tight by providing a family of trees for which the number of vertices saved by the Unburning Algorithm can be made arbitrarily close to half of those saved optimally. Let (k, p, q) denote a tree of the type illustrated in Figure 1. The root vertex has degree $k+2$. The $k+1$ subtrees to the left are identical, each consisting of a path of length k with q pendant vertices as leaves. The remaining subtree to the right, includes k subtrees, each with p leaves at the ends of paths of length increasing from 0 to $k-1$.

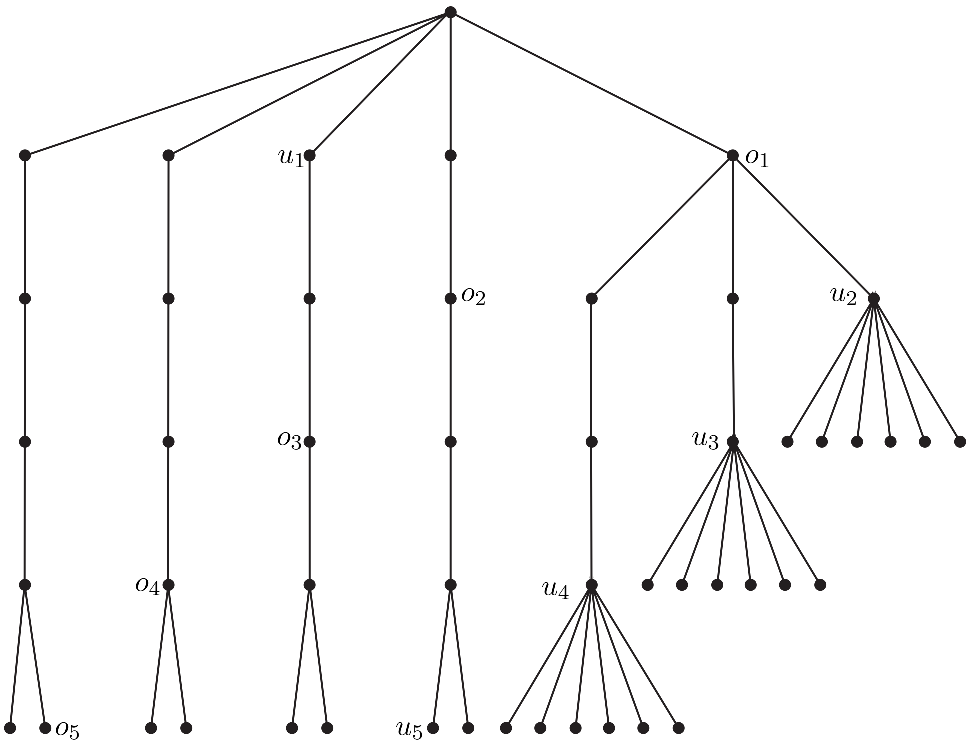

Figure 2: The $(3, 6, 2)$ tree.

For example, Figure 2 illustrates the situation when $(k, p, q) = (3, 6, 2)$. For this graph, Hartnell and Li's algorithm is optimal. The vertices for this optimal solution $o_1, o_2, \ldots, o_5$ are as shown in the graph with total weight $S_o = 25 + 5 + 4 + 3 + 1 = 38$. We have also labelled the vertices $u_1, u_2, \ldots, u_5$ of the vertices for the Unburning Algorithm with a total weight $S_u = 6 + 7 + 7 + 7 + 1 = 28$.

To generalize the $(3, 6, 2)$ example we will impose conditions on k, p, and q. Compare the vertices at level one. Note that $\tilde{\mathrm{wt}}(o_1) = 4 < 6 = \mathrm{wt}(u_1)$. In general, the subtree at the right in a (k, p, q) tree will have $\tilde{\mathrm{wt}}(o_1) = k(k-1)/2 + 1$ while the equivalent $k+1$ subtrees to the right have $\mathrm{wt}(u_1) = k + q + 1$. To maintain the relationship between these weights, we will require $k(k-1)/2 + 1 < k + q + 1$, or equivalently, $q > k(k-3)/2$. Next look at the vertices at level two where $\mathrm{wt}(u_2) = 7 > 5 = \mathrm{wt}(o_2)$. In a (k, p, q) tree we will have $\mathrm{wt}(u_2) = p + 1$ and $\mathrm{wt}(o_2) = k + q$. To maintain that relationship, we require that $p + 1 > k + q$ or $p > k + q - 1$.

Assuming these conditions on k, p, and q, we can determine formulas for the number of vertices saved by the optimal placement of firefighters (which agrees with Hartnell and

Li's algorithm) and the Unburning Algorithm. For the optimal solution,

$$\begin{aligned}\mathrm{wt}(o_1) &= 1 + (p+1) + (p+2) + \cdots + (p+k)\\ &= 1 + kp + k(k+1)/2.\end{aligned}$$

The weights of $o_2, o_3, o_4, \ldots o_{k+1}$ go down by one at each step:

$$\begin{aligned}\sum_{i=2}^{k+1} \mathrm{wt}(o_i) &= (q+k) + \cdots + (q+2) + (q+1)\\ &= kq + k(k+1)/2.\end{aligned}$$

Adding in $\mathrm{wt}(o_{k+2}) = 1$, we have

$$S_o = 1 + kp + k(k+1)/2 + kq + k(k+1)/2 + 1 = k(k+p+q+1) + 2.$$

For the Unburning Algorithm, we have $\tilde{\mathrm{wt}}(u_1) = \mathrm{wt}(u_1) = k+1+q$, $\tilde{\mathrm{wt}}(u_i) = \mathrm{wt}(u_i) = p+1$ for $2 \le i \le k+1$, and $\tilde{\mathrm{wt}}(u_{k+1}) = \mathrm{wt}(u_{k+1}) = 1$ so that

$$S_u = k + 1 + q + k(p+1) + 1 = k(p+2) + q + 2.$$

One way to satisfy our conditions on k, p, and q is by setting $q = k(k-3)/2 + 1$ and

$$p = k + q = k + k(k-3)/2 + 1 = k(k-1)/2 + 1.$$

With these choices,

$$\begin{aligned}S_u/S_o &= \frac{k(k(k-1)/2+3) + k(k-3)/2 + 3}{k\left(k + k(k-1)/2 + 1 + k(k-3)/2 + 2\right) + 2}\\ &= \frac{\frac{1}{2}k^3 + \frac{3}{2}k + 3}{k\left(k + 3 + k(k-2)\right) + 2}\\ &= \frac{\frac{1}{2}k^3 + \frac{3}{2}k + 3}{k^3 - k^2 + 3k + 2}.\end{aligned}$$

Taking the limit as k goes to infinity, the ratio becomes arbitrarily close to $\frac{1}{2}$.

We remark that, for the (k,p,q) trees, Hartnell and Li's algorithm is optimal and the Unburning Algorithm does worse, tending to saving half as many vertices as k increases. In contrast, in his thesis, Hartke [3] shows that the bound of Theorem 2.2 is tight using a family of trees that are the right subtrees of the (k,p,q) trees of Figure 1, that is, the analogs of the subtrees T_{o_1} of Figure 2. For those trees, it is the Unburning Algorithm that agrees with the optimal placement of firefighters, while Hartnell's and Li's approach saves half as many vertices, in the limit. This family of trees was part of our initial motivation in developing the Unburning Algorithm.

In general, Hartnell and Li's algorithm and the Unburning Algorithm are complementary in that if one performs poorly on a given tree, the other tends to correct for that weakness. While the two algorithms share strengths in that they are both straightforward to apply and perform at least half as well as the optimal firefighting sequence, an even better general approach would be to apply both algorithms and pick the one that saves more vertices.

Perhaps there are even better ways to combine the two approaches. In particular, the Unburning Algorithm sometimes protects subtrees that were already protected, meaning some u_i nodes are no longer needed in step i. Is there a way to redeploy these superfluous nodes as part of a new algorithm that combines the benefits of both?

2.2 Integrality Gap Conjecture

In this subsection we verify Conjecture 2.2 of Hartke [4], to which we refer the reader for further details. MacGillivray and Wang [7] formulated an integer program to find an optimal sequence $a_1, a_2, \ldots$ for firefighting in a tree T. For each vertex v, define $x(v) = 1$ if v is protected and 0 otherwise. Recall that $l(v) = d(v_0, v)$ is the number of edges between v and the root vertex v_0 and $\mathrm{wt}(v) = |V(T_v)|$ is the number of vertices in the subtree rooted at v. If vertex u is in T_v, we will write $v \succeq u$.

With this notation, we can state MacGillivray and Wang's integer program as follows.

$$\text{maximize } m = \sum_{v \in V(T)} wt(v)x(v)$$

$$\begin{aligned} \text{subject to: } & \sum_{l(v)=L} x(v) \leq 1, \text{ for each level } L, && (1)\\ & \sum_{v \succeq u} x(v) \leq 1, \text{ for each leaf vertex } u, \text{ and} && (2)\\ & x(v) \in \{0,1\}, \text{ for each vertex } v. && (3) \end{aligned}$$

By relaxing constraint (3) to $0 \leq x(v) \leq 1$, we instead obtain a linear program (LP) whose maximum m^* is an upper bound for the maximum m of the integer program (IP). We say there is an "integrality gap" when $m^* > m$. Since a linear program is much more efficient computationally, it is useful to understand the occurrence and size of integrality gaps.

Hartke [4] attempts to reduce the occurrence of integrality gaps by adding an additional linear constraint, constraint (6).

$$(6) \qquad \sum_{v \succeq u} x(v) + \sum_{u \succeq v, l(v)=i} x(v) \leq 1, \text{ for each vertex } u \text{ and } i > l(u).$$

Although this improves the situation, there remain trees with an integrality gap including the one on 13 vertices shown in Figure 3. The two black vertices give the optimal (IP) solution with $m = 7$ vertices saved. The nonzero $x(v)$ appear in the figure and give the LP optimum $m^* = 7.5$.

We can now state Hartke's conjecture.

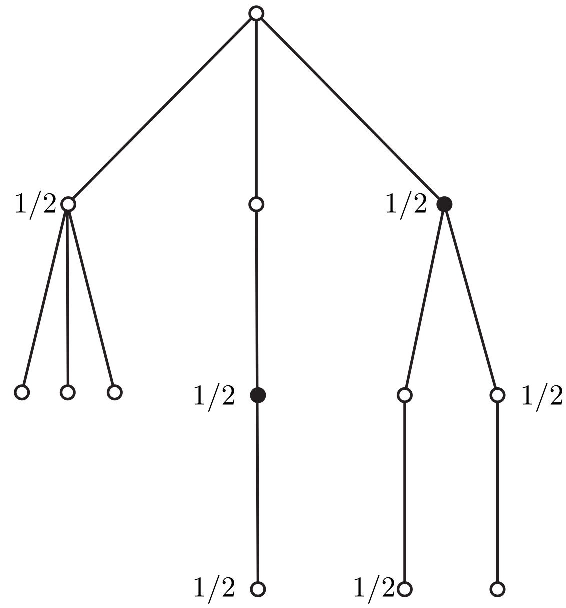

Figure 3: A tree, due to Hartke [4], on 13 vertices with integrality gap when using constraint (6).

Conjecture 2.3 (Conjecture 2.2 of [4]) The tree in Figure 3 is the smallest tree such that the LP optimal when using constraint (6) is not the IP optimal.

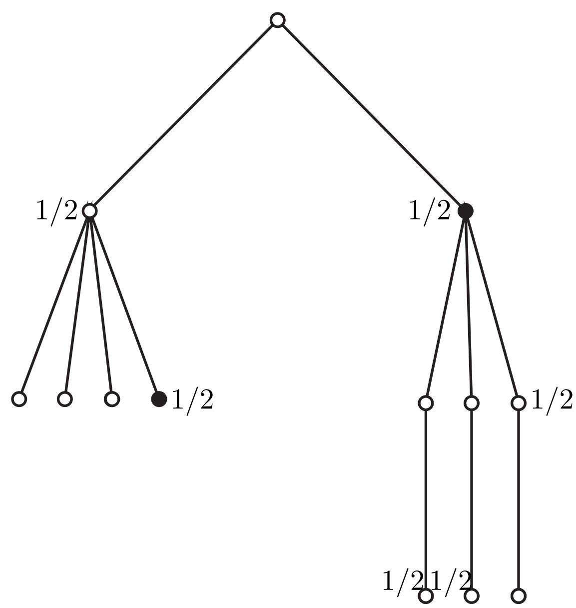

Figure 4: A second tree on 13 vertices with integrality gap when using constraint (6).

Here, "smallest" refers to the number of vertices. Hartke explained that he had verified the conjecture for trees of 11 or fewer vertices. To complete the proof of the conjecture then, we used Sage [10] to run through the 4766 trees on 12 vertices and check that none of those have an integrality gap using constraint (6). We also checked trees on 13 and 14 vertices. Among the 12486 trees on 13 vertices, there are exactly two with integrality gap. Figure 4 shows the second graph. The black vertices realize $m = 8$, while the given $x(v)$ show that $m^* = 8.5$. We found that of the 32973 trees on 14 vertices, there are ten trees that have an integrality gap. Moreover, each of these ten is formed from one of the two examples on 13 vertices by adding a leaf and its edge. There are six ways to add a leaf to the graph of Figure 3 and four ways for the graph of Figure 4. All 12 examples have $m^* - m = 0.5$.

To summarize, the integrability gap is zero for trees on 12 or fewer vertices and at most $\frac{1}{2}$ for those of 13 or 14 vertices. This suggests that there may be a way to bound the size of the integrability gap in terms of the tree size.

3 Infinite Regular Graphs

In this section we look at firefighting on four infinite regular graphs. Three correspond to tessellations of the plane by regular polygons: the hexagonal, square, and triangular grids. A fourth model is the strong grid, which is regular of degree eight. We describe a Containment Protocol to bring a fire on a graph under control in a finite number of steps. We show that this strategy matches, or nearly matches the best known results on these four grids.

We begin by describing the Containment Protocol. At each time step, we imagine allowing the fire to spread for two time steps. Based on this idea, we make the following definitions.

Definition 3.1 *At time step T, we denote by ABV,* already burned vertices*, the set of vertices that were burned at a time $t < T$, $BV1$,* burned vertices of type 1*, the vertices burned in time step T, and $BV2$,* burned vertices of type 2*, those burned in time step $T+1$ (assuming no firefighters are placed at time T). We refer to vertices where firefighters were placed at times $t < T$ as PPF,* previously placed firefighters*. For a $v \in PPF$ that is adjacent to a $BV1$ vertex, we say that v is a* good PPF *if at most half of its neighbors are in $ABV \cup BV1$. Otherwise, v is a* bad PPF*.*

In the Containment Protocol, we place n firefighters at each time step according to the following three criteria: 0) each firefighter is placed at a vertex v that is in neither ABV nor PPF, 1) each firefighter is placed adjacent to an ABV, and 2) each firefighter is at most distance two from either another firefighter placed at that time step, or a firefighter placed at an earlier time step, a PPF.

In rule CP5 below, we calculate the distance of the placement from the PPF vertices. For each v in a candidate placement, let $d(v, \mathrm{PPF})$ be the distance to the PPF set. That is,

$$d(v, \mathrm{PPF}) = \min_{u \in PPF} d(v, u),$$

where $d(v, u)$ is the number of edges in the shortest path from v to u.

We choose among the allowed firefighter placements at a given time step according to the following five rules.

CP1 Choose a placement that minimizes |BV2|. If there is more than one placement that minimizes |BV2| eliminate all placements that do not and continue to the next rule.

CP2 Choose placements that minimize the number of PPF adjacent to an BV2. If there is more than one that minimize this, continue with those placements that minimize.

CP3 If there is a placement that includes a good PPF, eliminate all placements that have no good PPF. If more than one remains, continue to the next rule.

CP4 If there remains a placement with no bad PPF, then eliminate all that have one. If more than one placement remains, continue to the next rule.

CP5 Use a placement that minimizes the total distance $\sum_v d(v, \mathrm{PPF})$. If there is more than one placement that minimizes the total distance, eliminate all that do not and choose one at random from the remaining placements.

In addition we have a rule regarding the initial placement of vertices:

CP0 If it is possible to place the initial firefighters in time step 0 such that each firefighter is adjacent to at least one other firefighter, then eliminate all placements that do not and continue to rule CP1.

3.1 Hexagonal Grid

The hexagonal grid is an infinite graph based on a tiling of the plane by regular hexagons. Vertices are vertices of the hexagons and edges are sides. The hexagon grid is a 3-regular infinite graph. It is conjectured [2, 8] that placing one firefighter per time step is not enough to contain the fire. On the other hand, it's easy to see that, with two firefighters per time step, the fire will be contained after two time steps.

Let's see how this plays out for the Containment Protocol. Suppose the fire starts at vertex x_0 at time $t = 0$. Let $N(x_0) = \{x_1, x_2, x_3\}$. As we must place firefighters adjacent to vertices in $\mathrm{ABV} = \{x_0\}$, the two firefighters at time $t = 1$ are two of the three in $N(x_0)$. By symmetry, all three choices are equivalent. Since there's only one possible placement (up to symmetry) we do not need to consider the five rules, CP0 to CP5, to make a choice. Let's say that the firefighters are placed at x_1 and x_2. In time step 1, the fire spreads to x_3. Let $N(x_3) = \{x_0, x_4, x_5\}$. Again, there is only one possible placement of the firefighters, namely, at vertices x_4 and x_5, which contains the fire. Thus, the Containment Protocol matches the best known containment strategy in the case of the hexagonal grid.

3.2 Square Grid

The square grid is an infinite 4-regular graph based on a tessellation of the plane by squares. It is sometimes called the rectangular or Cartesian grid and is the Cartesian product $P_\infty \square P_\infty$ of two infinite paths. It is known [1, 9] that one firefighter per time step is not enough to contain the fire. With $n = 2$ firefighters per time step, the best strategy requires eight time steps for containment with 18 vertices burned [3, 9]. We will see that the Containment Protocol matches this optimal performance.

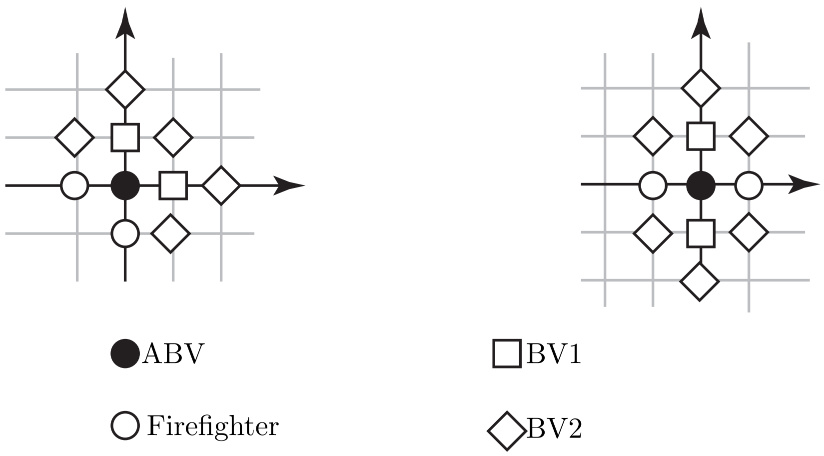

Figure 5: Two candidate placements at time 0.

Time step 0. Let's say the fire begins at vertex $(0, 0)$. Up to symmetry, there are two ways to place two firefighters: 1) at vertices $(-1, 0)$ and $(0, -1)$ or 2) at vertices $(-1, 0)$ and $(1, 0)$. Since neither has the two firefighters adjacent, following CP0, we continue with both. As in Figure 5, CP1 says we should choose the first placement for which $|\mathrm{BV2}| = 5$, in contrast to $|\mathrm{BV2}| = 6$ for the second placement.

Time step 1. Up to symmetry, there are two ways to place firefighters 1) $(-1, 1)$ and $(0, 2)$; or 2) $(-1, 1)$ and $(1, -1)$. We choose placement 1) which yields $|\mathrm{BV2}| = 5$ as opposed to 2) for which $|\mathrm{BV2}| = 6$.

Time step 2. There are three possible placements; 1) $(1, 2)$ and $(2, 1)$ with $|\mathrm{BV2}| = 6$; 2) $(1, 2)$ and $(1, -2)$ with $|\mathrm{BV2}| = 5$; and 3) $(1, -2)$ and $(2, -1)$ with $|\mathrm{BV2}| = 5$, see Figure 6. Using CP1, we eliminate the first placement as a candidate. Neither of the remaining candidates has adjacent PPF and BV2 vertices, so CP2 does not help us distinguish between them. However, placement 3) has a good PPF at $(0, 2)$ (adjacent to the BV1 vertex $(1, 2)$) while placement 2) has none. So, by CP3, we use placement 3.

Time step 3. The possible placements are 1) $(1, 3)$ and $(2, 2)$; 2) $(1, 3)$ and $(3, -1)$; 3) $(3, -1)$ and $(4, 0)$; and 4) $(2, 2)$ and $(3, 1)$, see Figure 7. All but the last (with $|\mathrm{BV2}| = 7$) yield $|\mathrm{BV2}| = 5$. By CP1, we eliminate the last placement and continue with the first three. Since arrangement 3) has the PPF at $(0, 2)$ adjacent to the BV2 vertex $(0, 3)$, we eliminate that one using CP2 and continue to CP3 with the other two. Neither of the

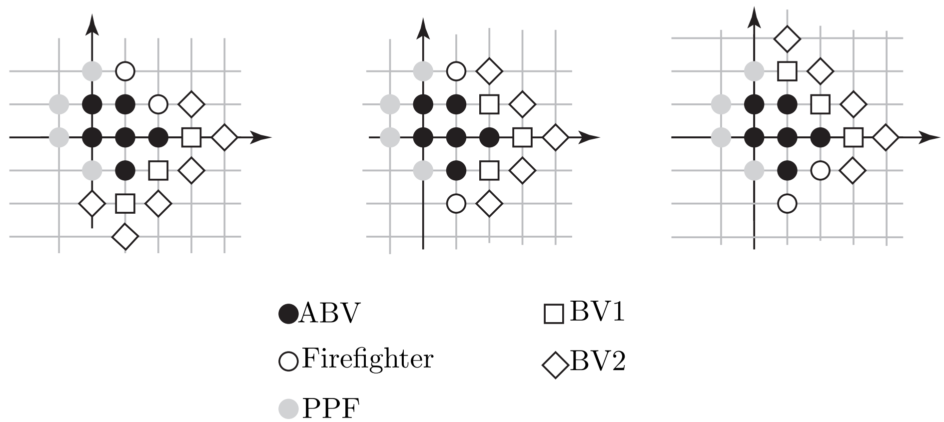

Figure 6: Three candidate placements at time 2.

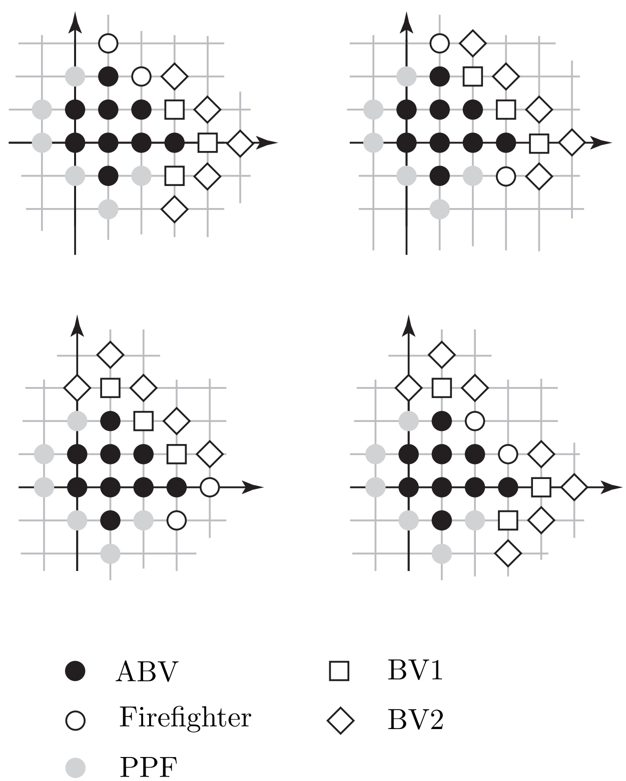

Figure 7: Four candidate placements at time 3.

remaining placements has a good PPF, so we move on to CP4. Since placement 1) has a bad PPF at $(2,-1)$, we will use placement 2).

Time step 4. 1) $(2,3)$ and $(3,2)$ with $|\mathrm{BV2}|=5$; 2) $(2,3)$ and $(4,-1)$ with $|\mathrm{BV2}|=5$; and 3) $(4,-1)$ and $(5,0)$ with $|\mathrm{BV2}|=4$. By CP1, we use arrangement 3).

Time step 5. 1) $(2,4)$ and $(3,3)$ with $|\text{BV2}|=3$; 2) $(2,4)$ and $(5,1)$ with $|\text{BV2}|=3$; 3) $(5,1)$ and $(4,2)$ with $|\text{BV2}|=4$; and 4) $(3,3)$ and $(4,2)$ with $|\text{BV2}|=5$. By CP1, we eliminate arrangements 3) and 4). CP2 does not allow us to eliminate either of the remaining two candidates. However CP3 says we should choose arrangement 1) due to the good PPF at $(5,0)$ (adjacent to the BV1 vertex $(5,1)$).

Time step 6. 1) $(4,3)$ and $(5,2)$ with $|\text{BV2}|=3$; 2) $(4,3)$ and $(6,1)$ with $|\text{BV2}|=2$; and 3) $(6,1)$ and $(5,2)$ with $|\text{BV2}|=2$. By CP1, we eliminate arrangement 1). CP2 and CP3 do not allow us to distinguish, but CP4 indicates arrangement 2) due to the bad PPF at $(3,3)$ in arrangement 3).

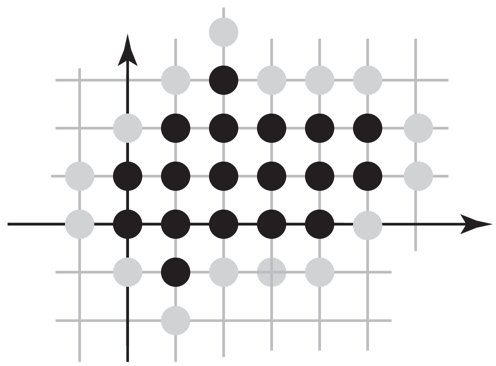

Figure 8: The containment protocol stops the fire in eight time steps.

By Time Step 7, there is only one placement of the two firefighters that satisfy our criteria and that choice ends the fire after eight steps and with 18 vertices burned, see Figure 8. For the square grid, the containment protocol matches the known best solution.

3.3 Triangular Grid

A triangular grid is the infinite 6-regular graph that corresponds to a tessellation of the plane by equilateral triangles. For convenience, we place this graph in the plane with vertices at lattice points $(m,n)\in\mathbb{Z}^2$ as in Figure 9. Although Fogarty [1] proposed an argument showing that two firefighters per time step do not suffice to contain the fire, there is a flaw (see [2]) and this remains a conjecture. It is known that the fire can be contained by placing three firefighters at each time step. In fact, Messinger [8] conjectures that the optimal solution sees 17 vertices burned with containment in six time steps. We now show that the Containment Protocol matches or nearly matches this best known result.

Time step 0. Let's say the fire begins at vertex $(0,0)$. Up to symmetry, there are three ways to place three firefighters: 1) at vertices $(-1,0)$, $(-1,-1)$, and $(0,-1)$; 2) at vertices $(-1,0)$, $(-1,-1)$, and $(1,0)$; and 3) at vertices $(-1,0)$, $(1,1)$, and $(0,-1)$, see Figure 9. By CP0, we use the first placement, the only one that has each of the three firefighters adjacent to at least one other firefighter.

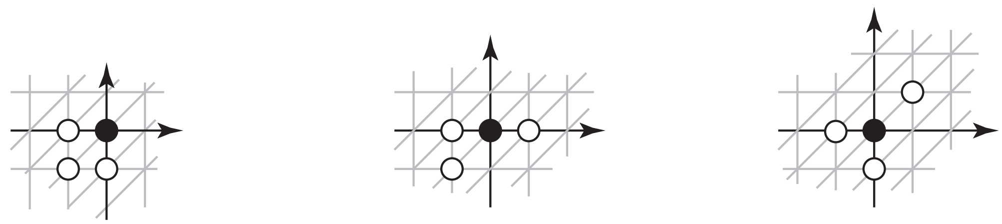

Figure 9: Three placements for time step 0.

Time step 1. Up to symmetry, there are 15 ways to place the three firefighters. Of these, only one, with firefighters at $(-1,1)$, $(0,2)$, and $(1,-1)$, realizes the minimum $|\text{BV2}| = 7$.

Time step 2. There are seven vertices adjacent to an ABV, meaning there are $\binom{7}{3} = 35$ placements. However, two of these involve placing a firefighter more than distance two from the other firefighter sites and PPF vertices, namely 1) $(1,3)$, $(2,-1)$, and $(3,2)$ (for which $(3,2)$ is too far from the other firefighters) and 2) $(2,-1)$, $(3,0)$, and $(3,3)$ (with the last one too distant). Of the remaining placements, only one, at $(1,3)$, $(2,-1)$, and $(3,0)$, realizes the minimum $|\text{BV2}| = 6$.

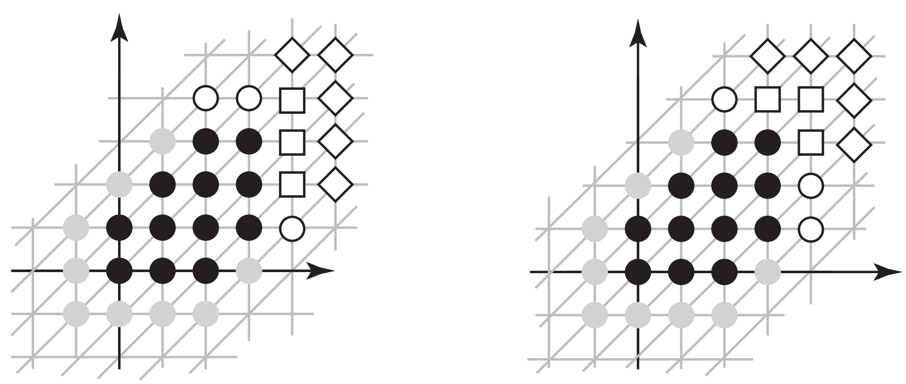

Figure 10: Two placements with $|\text{BV2}| = 5$ and total distance four.

Time step 3. There are six vertices adjacent to an ABV and $\binom{6}{3} = 20$ placements. Four of these realize the minimum $|\text{BV2}| = 5$. CP2 says we must remove one of the four, with firefighters at $(2,4)$, $(3,4)$, and $(4,4)$, due to the BV2 vertex $(4,0)$ adjacent to the PPF at $(3,0)$. The remaining three all survive CP3 and CP4, and two have the same minimal total distance of four. The one eliminated due to a larger total distance of five, has firefighters at $(2,4)$, $(4,4)$, and $(4,1)$.

Although the Containment Protocol says we should choose at random between the remaining two placements, we will continue with both to show that they both perform well. Going forward we will compare Solution 1, based on firefighters at $(2,4)$, $(3,4)$, and

$(4,1)$, with Solution 2, which uses the placement $(2,4)$, $(4,1)$, and $(4,2)$.

Solution 1: Time step 4. There are five vertices adjacent to an ABV with $\binom{5}{3} = 10$ placements. Of these, only one realizes the minimum $|\text{BV2}| = 3$: $(4,5)$, $(5,5)$, and $(5,2)$.

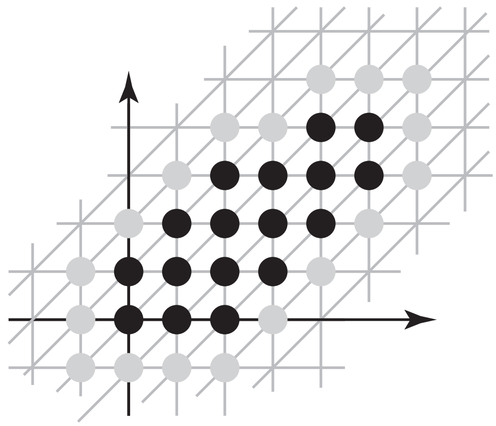

Figure 11: Solution one contains the fire in six time steps.

This also means we can contain the fire by placing three firefighters at the three BV2 vertices: $(6,3)$, $(6,4)$, and $(6,5)$. The complete solution is illustrated in Figure 11. With this solution, the Containment Protocol matches the best known result with 17 vertices burned and containment after six time steps.

Solution 2: Time step 4. As with solution 1, there are five available vertices and ten ways to place firefighters. Five of them realize the minimum $|\text{BV2}| = 4$. Two are eliminated by CP2. The placement $(3,5)$, $(4,5)$, and $(5,5)$ has a BV2 vertex $(5,2)$ adjacent to the PPF at $(4,2)$. The placement $(5,3)$, $(5,4)$, and $(5,5)$ has BV2 $(2,5)$ adjacent to PPF $(2,4)$. The remaining three survive CP3 and CP4 and two realize the minimum total distance four. The one with a larger total distance of five is eliminated by CP5: $(3,5)$, $(5,3)$, and $(5,5)$. Again, although the protocol indicates that we should choose one of the remaining two placements at random, we will continue with both to see that they preform equally well. For Solution 2a, we continue with the placement $(3,5)$, $(4,5)$, and $(5,3)$ while we will denote the placement $(3,5)$, $(5,3)$, and $(5,4)$ as Solution 2b.

Solution 2a: Time step 5. There are four vertices available, so $\binom{4}{3} = 4$ possible placements. One placement realizes the minimum $|\text{BV2}| = 2$: $(5,6)$, $(6,4)$, and $(6,6)$. This means we can contain the fire in one more time step. In fact, we only need two firefighters for the last step. In total, 18 vertices are burned, see Figure 12.

Solution 2b: Time step 5. Again, there are four vertices and four possible placements. One, $(4,6)$, $(6,5)$, and $(6,6)$, realizes the minimum $|\text{BV2}| = 2$. As in Solution 2a, we contain the fire in seven time steps (with only two firefighters needed in the last step) and 18 vertices burned, see Figure 12.

In summary, the Containment Protocol would have us choose at random between three possible solutions. One of these matches the best known solution with 17 vertices burned

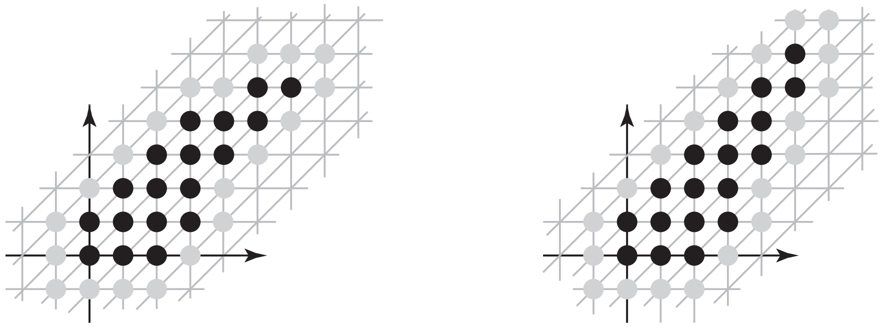

Figure 12: Solutions 2a (left) and 2b (right) contain the fire in seven time steps.

and containment in six time steps. The other two are nearly as good with 18 vertices burned and containment in seven time steps.

3.4 Strong Grid

The strong grid is an eight regular infinite graph that, similar to the square and triangular grid, we can realize by placing vertices at the lattice points $(m, n) \in \mathbb{Z}^2$. As in Figure 13, each (m, n) is adjacent to its eight nearest neighbors in the plane, those points that are at most $\sqrt{2}$ away. Messinger [8] shows that four firefighters per time step are required to contain a fire in the strong grid and conjectures that the best solution requires eight time steps. She demonstrates a solution in eight time steps for which 35 vertices are burned.

Let's see how the Containment Protocol handles a fire on the strong grid.

Time step 0. Let's say the fire begins at vertex $(0, 0)$. Up to symmetry, there are five ways to place four firefighters such that each is adjacent to at least one of the others. Of these, only one results in the minimum $|\text{BV2}| = 10$; $(-1,-1)$, $(-1, 0)$, $(0,-1)$, and $(1 - 1)$. By CP1, we reject the other three : $(-1,-1)$, $(0,-1)$, $(1, 0)$, and $(1, 1)$ with $|\text{BV2}| = 12$; $(-1,-1)$, $(-1, 0)$, $(0,-1)$, and $(1, 0)$ with $|\text{BV2}| = 13$; $(-1,-1)$, $(0,-1)$, $(0, 1)$, and $(1, 0)$ with $|\text{BV2}| = 14$; and $(-1, 0)$, $(0,-1)$, $(0, 1)$, and $(1, 0)$ with $|\text{BV2}| = 16$.

Time step 1. There are $\binom{10}{4} = 210$ placements of the four firefighters, but only one realizes the minimum $|\text{BV2}| = 10$: $(-2, 0)$, $(-2, 1)$, $(-2, 2)$, and $(2,-1)$.

Time step 2. Again, there are $\binom{10}{4} = 210$ placements of the four firefighters. Of these, five realize the minimum $|\text{BV2}| = 10$. Note that two others would also give a low BV2 count, but they are not allowed placements as there is a firefighter more than distance two from the others: $(-2, 3)$, $(-1, 3)$, $(3,-1)$, and $(3, 3)$; and $(-2, 3)$, $(3,-1)$, $(3, 0)$, and $(3, 3)$, both of which have $(3, 3)$ too distant from the other firefighters and the PPF set. The five placements are not distinguished by CP2, but four have a good PPF. The one that does not is $(-2, 3)$, $(-1, 3)$, $(3,-1)$, and $(3, 0)$. The remaining four are not distinguished by CP4. Two of them minimize the total distance as five. The two that do not are $(-2, 3)$, $(2, 3)$, $(3,-1)$, and $(3, 3)$; and $(-2, 3)$, $(3,-1)$, $(3, 2)$, and $(3, 3)$. As usual, although the Containment Protocol requires us to choose one of the remaining two placements at random, we will instead continue with both to see how the protocol performs. For Solution 1: $(-2,3)$, $(-1,3)$, $(0,3)$, and $(3,-1)$; and Solution 2: $(-2,3)$, $(3,-1)$, $(3,0)$, and $(3,1)$.

Solution 1: Time step 3. Again there are $\binom{10}{4} = 210$ placements of the four firefighters and five realize the minimum $|\mathrm{BV2}| = 10$. Two others would give a low count but are not allowed as there is a firefighter too far from the others: $(0,4)$, $(1,4)$, $(4,-1)$, and $(4,4)$ where $(4,4)$ is too far; and $(0,4)$, $(4,-1)$, $(4,0)$, and $(4,4)$, again with $(4,4)$ too far. The remaining five placements are not distinguished by CP2. Three have a good PPF and the other two, which are eliminated, are $(0,4)$, $(1,4)$, $(4,-1)$, and $(4,0)$; and $(0,4)$, $(4,-1)$, $(4,0)$, and $(4,1)$. Of the remaining three, two have a bad PPF and are eliminated by CP4: $(0,4)$, $(3,4)$, $(4,-1)$, and $(4,4)$; and $(0,4)$, $(4,-1)$, $(4,3)$, and $(4,4)$, both having a good PPF at $(3,-1)$ and a bad one at $(0,3)$. The remaining placement is the one we will continue with; $(0,4)$, $(1,4)$, $(2,4)$, and $(4,-1)$ with a good PPF at $(3,-1)$.

Solution 1: Time step 4. Once again, there are $\binom{10}{4} = 210$ placements of the four firefighters and seven realize the minimum $|\mathrm{BV2}| = 10$. Another, $(2,5)$, $(5,-1)$, $(5,0)$ and $(5,5)$ would also give a low count but the firefighter at $(5,5)$ is more than distance two from the others. We reject one of the seven using CP2: $(2,5)$, $(3,5)$, $(4,5)$, and $(5,5)$ due to the BV2 vertex at $(4,-2)$ which is adjacent to the PPF $(4,-1)$. CP3 allows us to eliminate two placements that have no good PPF: $(2,5)$, $(3,5)$, $(5,-1)$, and $(5,0)$; and $(2,5)$, $(5,-1)$, $(5,0)$, and $(5,1)$. Two more are removed due to a bad PPF: $(2,5)$, $(4,5)$, $(5,5)$, and $(5,-1)$; and $(2,5)$, $(5,-1)$, $(5,4)$, and $(5,5)$, both having a good PPF at $(4,-1)$ and a bad one at $(2,4)$. The remaining two placements are distinguished by CP5. The minimum total distance is five for $(2,5)$, $(3,5)$, $(4,5)$, and $(5,-1)$ and we will continue with this placement. We reject the placement with total distance six: $(2,5)$, $(3,5)$, $(5,-1)$, and $(5,5)$.

Solution 1: Time step 5. We continue to have $\binom{10}{4} = 210$ placements. In this case, there is a unique placement that realizes the minimum $|\mathrm{BV2}| = 8$: $(4,6)$, $(5,6)$, $(6,6)$, and $(6,-1)$.

Solution 1: Time step 6. Now there are $\binom{8}{4} = 70$ placements. Of these, three realize the minimum $|\mathrm{BV2}| = 6$ and none of those have a PPF adjacent to a BV2. Two have a good PPF. By CP3, we eliminate the third: $(7,-1)$, $(7,0)$, $(7,5)$, and $(7,6)$. The remaining two have no bad PPF and both have a total distance of five. Ordinarily, the Containment Protocol would have us choose one of them at random. Instead we will continue with both. For Solution 1a: $(7,-1)$, $(7,4)$, $(7,5)$, and $(7,6)$ with a good PPF at $(6,-1)$; for Solution 1b: $(7,-1)$, $(7,0)$, $(7,1)$, and $(7,6)$ with a good PPF at $(6,6)$.

Solution 1a: Time step 7. We are down to $\binom{6}{4} = 15$ placements. There are three that realize the minimum $|\mathrm{BV2}| = 4$. This means all three are complete in one more time step with the same amount of vertices burned. There is one that we eliminate by CP3, as it's the only one lacking a good PPF: $(8,-1)$, $(8,0)$, $(8,3)$, and $(8,4)$. From the remaining two, we randomly chose the first: $(8,-1)$, $(8,2)$, $(8,3)$, $(8,4)$ with a good PPF at $(7,-1)$, rather than the second: $(8,-1)$, $(8,0)$, $(8,1)$, and $(8,4)$ with a good PPF at $(7,4)$.

This leads to the solution shown at left in Figure 13; the fire is contained in nine time steps with 41 vertices burned.

Solution 1b: Time step 7. We are down to $\binom{6}{4} = 15$ placements. Of these, three

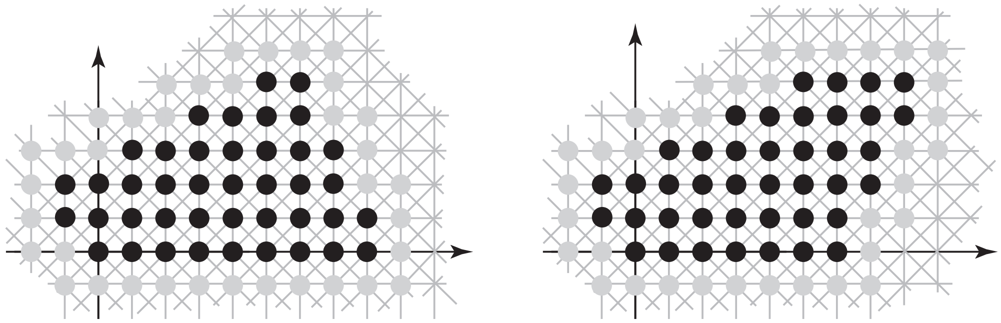

Figure 13: Solutions 1a (left) and 1b (right) contain the fire in nine time steps.

realize the minimum $|\mathrm{BV2}| = 4$. One lacks a good PPF and is rejected by CP3: $(8,1)$, $(8,2)$, $(8,5)$, and $(8,6)$. The other two are equal with respect to CP4 and CP5 and also in terms of the number of vertices burned and time steps for the final solution. So, we choose one at random: $(8,1)$, $(8,2)$, $(8,3)$, and $(8,6)$ with a good PPF at $(7,6)$; and reject the other $(8,1)$, $(8,4)$, $(8,5)$, and $(8,6)$ with a good PPF at $(7,1)$.

This leads to the solution shown at right in Figure 13; again, the fire is contained in nine time steps with 41 vertices burned.

Solution 2: Time step 3. There are $\binom{10}{4} = 210$ placements of which eight would yield $|\mathrm{BV2}| = 10$. One of these, $(-2,4)$, $(-1,4)$, $(4,1)$, and $(4,4)$, has $(4,4)$ more than two away from the other firefighters and is not allowed for that reason. Another is rejected due to CP2: $(4,1)$, $(4,2)$, $(4,3)$, and $(4,4)$, since the BV2 $(-3,3)$ is adjacent to the PPF $(-2,3)$. Next, there are placements that have a good PPF, so we reject the two that do not: $(-2,4)$, $(-1,4)$, $(0,4)$, and $(4,1)$; and $(-2,4)$, $(-1,4)$, $(4,1)$, and $(4,2)$. Of the remaining four placements, we reject two that have a bad PPF: $(-2,4)$, $(3,4)$, $(4,1)$, and $(4,4)$; and $(-2,4)$, $(4,1)$, $(4,3)$, and $(4,4)$, both of which have $(3,1)$ as a bad PPF.

The remaining two placements are distinguished by total distance (see CP5). The minimal distance is five, which is achieved by $(-2,4)$, $(4,1)$, $(4,2)$, and $(4,3)$, with which we continue. The final placement $(-2,4)$, $(4,1)$, $(4,2)$, and $(4,4)$ is rejected due to a total distance of six.

Solution 2: Time step 4. There are $\binom{10}{4} = 210$ placements only one of which realizes the minimum $|\mathrm{BV2}| = 8$: $(-2,5)$, $(5,3)$, $(5,4)$, and $(5,5)$.

Solution 2: Time step 5. There are $\binom{8}{4} = 70$ placements and three realize the minimum $|\mathrm{BV2}| = 6$. Two have a good PPF, so, by CP3, we reject the third: $(-2,6)$, $(-1,6)$, $(4,6)$, and $(5,6)$. The remaining two placements are not distinguished by CP4 or CP5. Ordinarily, the Containment Protocol would have us choose between them at random, but we will carry both forward to see how the protocol performs in all cases. For solution 2a, we place firefighters at $(-2,6)$, $(-1,6)$, $(0,6)$, and $(5,6)$. In this case, $(5,5)$ is a good PPF. For solution 2b, we continue with $(-2,6)$, $(3,6)$, $(4,6)$, and $(5,6)$ which has $(-2,5)$ as a good PPF.

Solution 2a: Time step 6. There are $\binom{6}{4} = 15$ placements and two realize the minimum $|\mathrm{BV2}| = 4$. One has a good PPF, so, by CP3, we reject the other: $(0,7)$, $(3,7)$,

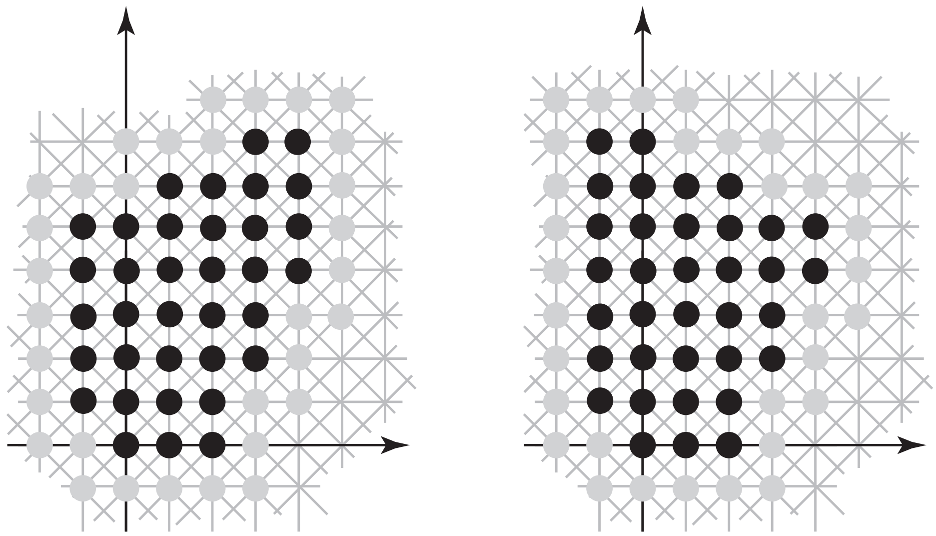

Figure 14: Solutions 2a (left) and 2b (right) contain the fire in eight time steps.

$(4, 7)$, and $(5, 7)$ and continue with $(0, 7)$, $(1, 7)$, $(2, 7)$, and $(5, 7)$. In the next time step, we place firefighters at the four BV2 vertices and contain the fire in eight time steps with 35 vertices burned, see Figure 14.

Solution 2b: Time step 6. There are $\binom{6}{4} = 15$ placements and three realize the minimum $|\text{BV2}| = 4$. Of these, only one has a good PPF, so by CP3, we reject the other two: $(-2, 7)$, $(-1, 7)$, $(0, 7)$, and $(3, 7)$; and $(-2, 7)$, $(-1, 7)$, $(2, 7)$, and $(3, 7)$. We continue with the placement $(-2, 7)$, $(1, 7)$, $(2, 7)$, and $(3, 7)$. We next place firefighters at the four BV2 vertices to terminate the fire in eight time steps, with 35 vertices burned, see Figure 14.

In summary, the Containment Protocol produces one of four solutions, all with equal likelihood. Two of the four match the best known solution with containment in eight time steps and 35 vertices burned. The other two are close to optimal with 41 vertices burned after nine time steps.

4 The Pentagon Graph

We have seen that the Containment Protocol performs quite well, often meeting the best known outcome for some well studied regular infinite graphs. One virtue of the protocol is that it can be applied to firefighting on any graph. As an example, we investigate a graph resulting from a pentagonal tiling of the plane as in Figure 15. This results in an infinite graph that is not regular as it has vertices of degree three and four.

Conjecture 4.1 One firefighter per time step is not sufficient to control a fire on the Pentagon Graph.

It is easy to see that, much like the hexagonal grid, a fire that begins on a degree three

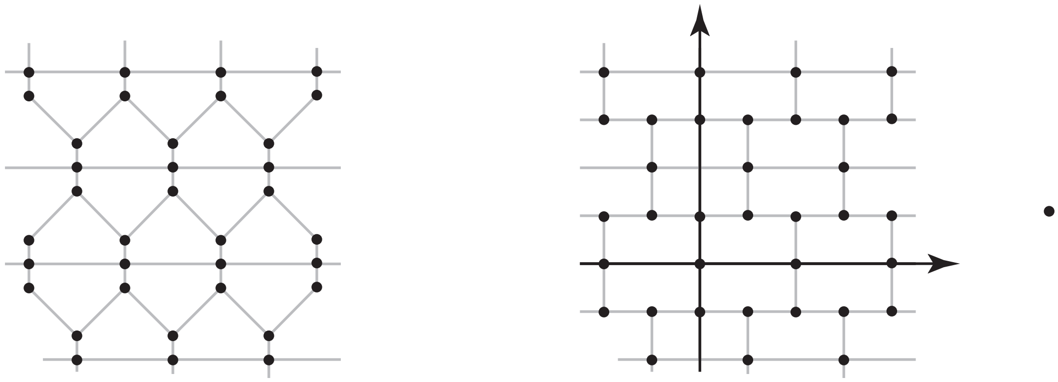

Figure 15: A pentagonal tiling (left). The resulting graph on the lattice (right).

vertex can be contained in two time steps if there are two firefighters placed per time step. Let's see how the Containment Protocol handles this situation. As in Figure 15, we imagine the fire beginning at $(0, 1)$. Up to symmetry there are two ways to place the two firefighters: $(-1, 1)$ and $(0, 0)$ with $|\mathrm{BV2}| = 2$; and $(-1, 1)$ and $(1, 1)$ with $|\mathrm{BV2}| = 3$. Both placements fail to have the firefighters adjacent (see CP0), and CP1 says we should continue with the first placement. We next place our two firefighters at the two BV2 vertices to contain the fire in two steps with two vertices burned, which is the optimal solution.

We next consider a fire that starts at the degree four vertex $(0, 0)$ and allow ourselves two firefighters per time step.

Time step 0. Up to symmetry, there are three ways to place two firefighters adjacent to the initial degree four vertex at $(0, 0)$. For none of those three are the two firefighters adjacent, so CP0 does not apply. However, only one of them realizes the minimum $|\mathrm{BV2}| = 4$: $(-2, 0)$ and $(2, 0)$.

Time step 1. Up to symmetry, there are three ways to place the next two firefighters, none of which can be distinguished by CP1 to CP5. Ordinarily, the Containment Protocol would have us choose one at random, but we will continue our practice of following through with all three: Solution 1: $(-1, 1)$ and $(1, 1)$; Solution 2: $(-1, -1)$ and $(-1, 1)$; and Solution 3: $(-1, 1)$ and $(1, -1)$.

Solution 1: Time step 2. Among the $\binom{4}{2} = 6$ ways to place two firefighters, the only one that realizes the minimum $|\mathrm{BV2}| = 2$ is $(-1, -2)$ and $(1, -2)$. Placing two firefighters at the BV2 positions contains the fire in four time steps with seven vertices burned.

Solutions 2 and 3 are similar, also stopping the fire in four time steps with seven vertices burned, see Figure 16.

As the analysis of a fire starting at a single vertex was straightforward, we also investigated what would happen if a fire began with a pair of adjacent vertices burned. There are three cases as the two vertices may have the same degree three or four, or different degrees.

Two degree three vertices burned: Suppose a fire begins with two adjacent degree

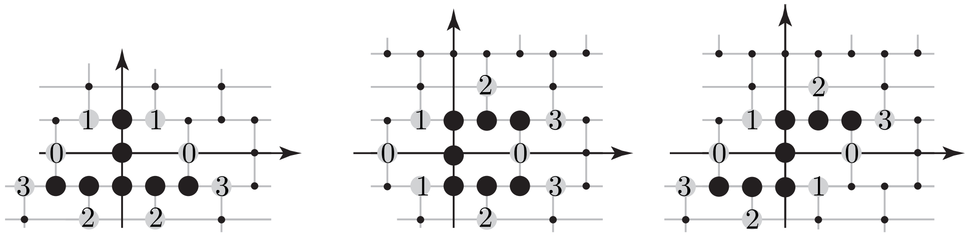

Figure 16: Starting from a single degree four vertex, Solution 1 (left), Solution 2 (center), and Solution 3 (right).

three vertices at $(0, 1)$ and $(1, 1)$. Given Conjecture 4.1, it is likely not possible to contain the fire with a single firefighter per time step, so we will allow ourselves two.

Time step 0. Up to symmetry, there are four ways to place two firefighters adjacent to the burned vertices. None have the firefighters adjacent to one another, so CP0 does not help. Two of the four realize the minimum $|BV2| = 4$, and we will continue with both, even though the Containment Protocol says we should pick one at random. Solution 1: $(0, 0)$ and $(1, 2)$ or Solution 2: $(-1, 1)$ and $(1, 2)$.

Solution 1: Time step 1. Up to symmetry, there are four ways to place the two firefighters and all four result in $|BV2| = 4$. Using CP2, we eliminate all but one placement: $(-2, 1)$ and $(3, 1)$.

Solution 1: Time step 2. Up to symmetry, there are four ways to place the two firefighters and only one realizes the minimum $|BV2| = 4$: $(-3, 2)$ and $(4, 0)$.

Solution 1: Time step 3. Up to symmetry, there are four ways to place the two firefighters. These four are all equivalent with respect to CP1 through 5. The Containment Protocol would have us choose one at random. Instead we will continue with all four so that we can compare the results: Solution 1a: $(-2, 3)$ and $(0, 3)$; Solution 1b: $(-2, 3)$ and $(1, -1)$; Solution 1c: $(-2, 3)$ and $(3, -1)$; and Solution 1d: $(0, 3)$ and $(1, -1)$.

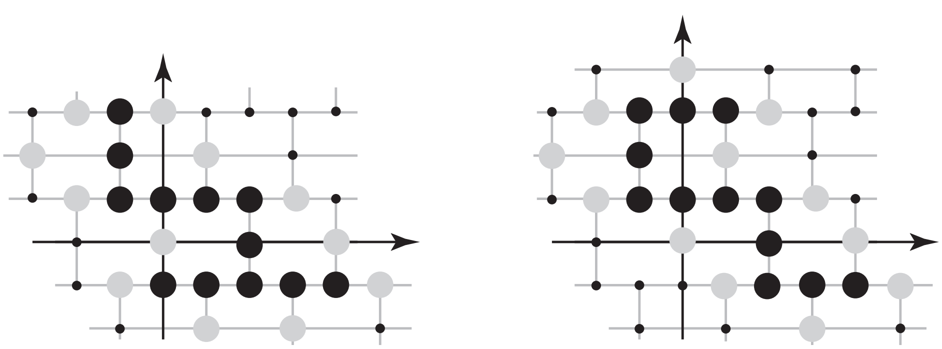

Figure 17: Solution 1a (left) and 1b (right).

Solution 1a: Time step 4. There are $\binom{4}{2} = 6$ ways to place firefighters and only one realizes the minimum $|BV2| = 2$: $(1, -2)$ and $(3, -2)$. Covering those two vertices in the next step contains the fire in six time steps with 12 vertices burned, see Figure 17.

Solution 1b: Time step 4. Again, there are $\binom{4}{2} = 6$ ways to place firefighters and only one realizes the minimum $|\mathrm{BV2}| = 2$: $(0, 4)$ and $(3, -2)$. Placing firefighters at those two vertices contains the fire in six time steps with 12 vertices burned, see Figure 17.

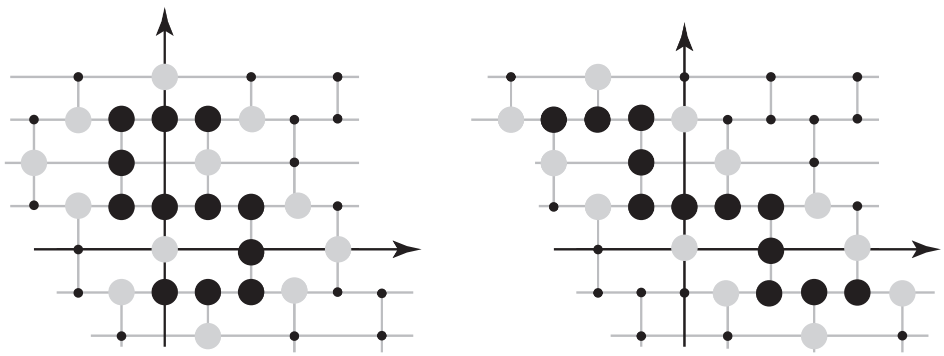

Figure 18: Solution 1c (left) and 1d (right).

Solution 1c: Time step 4. There are $\binom{4}{2} = 6$ ways to place firefighters and only one realizes the minimum $|\mathrm{BV2}| = 2$: $(0, 4)$ and $(1, -2)$. Covering those two vertices in the next step contains the fire in six time steps with 12 vertices burned, see Figure 18.

Solution 1d: Time step 4. Again, there are $\binom{4}{2} = 6$ ways to place firefighters and only one realizes the minimum $|\mathrm{BV2}| = 2$: $(-2, 4)$ and $(3, -2)$. Covering those two vertices in the next step contains the fire in six time steps with 12 vertices burned, see Figure 18.

Solution 2: Time step 1. There are $\binom{4}{2} = 6$ ways to place firefighters and three realize the minimum $|\mathrm{BV2}| = 4$. However, only one has no PPF adjacent to an BV2, so we continue with that one: $(-2, 0)$ and $(3, 1)$.

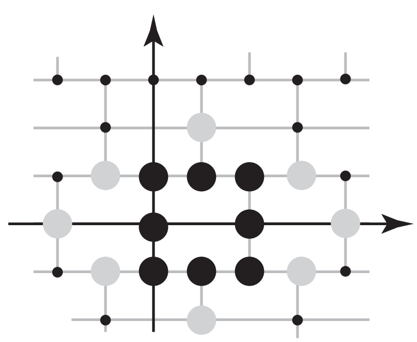

Figure 19: Solution 2.

Solution 2: Time step 2. There are $\binom{4}{2} = 6$ ways to place firefighters but only one realizes the minimum $|\mathrm{BV2}| = 2$: $(-1, -1)$ and $(4, 0)$. Then placing firefighters at the two BV2 vertices completes the fire in four time steps with eight vertices burned, see Figure 19

In summary, for a fire that begins with two adjacent degree three vertices burned, using the Containment Protocol, we found two types of solutions. Solutions 1a, 1b, 1c, and 1d contain the fire in six time steps with 12 vertices burned. The best outcome is solution 2 with eight vertices burned after four time steps.

Two vertices including one of degree four burned: If a fire begins with two adjacent vertices burned, one of degree four, the analysis of the Containment Protocol becomes quite involved, branching off into many possible solutions with a wide range of time steps and number of vertices burned. Rather than chasing through all the details, we will present the best solutions we found using the Containment Protocol. We conjecture that these are the best possible solutions.

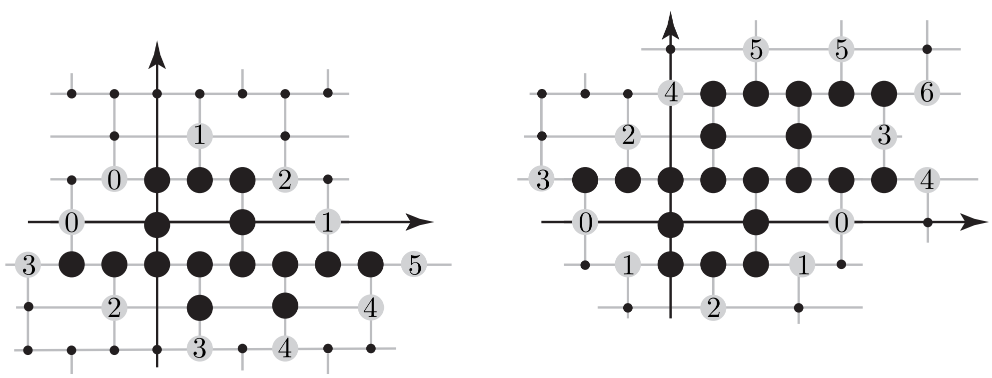

Figure 20: Conjectural best solutions for a fire that starts with two adjacent vertices, one of degree 4.

There are two cases. Suppose first that a fire begins with the adjacent vertices $(0, 0)$, of degree four, and $(0, 1)$, of degree three. Using the Containment Protocol, the best solution that we found is at left in Figure 20 and entails containing the fire in six time steps with 15 vertices burned. Next, imagine a fire that starts with vertices $(0, 0)$ and $(2, 0)$, both of degree four. To the right in Figure 20, we present the best solution we found with the Containment Protocol, which sees the fire terminated in seven times steps with 20 vertices burned.

Acknowledgments

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