A Recursive Approach to a Multi-State Cylindrical Lights Out Game

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Abstract - Lights Out is a game featuring a grid of light-up buttons that begins with some lights on and some off. The goal is to turn off all lights but pressing a button changes its state and the states of the cardinal neighboring buttons. In this paper, we explore a Lights Out game in which the board is placed on a cylinder and the lights have k states with a specific starting configuration. We try to turn off all lights using a light chasing strategy in which we methodically turn off the lights row by row. We model this process using recursive equations. A connection to the Fibonacci sequence then allows us to determine the number of rows of buttons the board should have in order for us to turn off all lights using our light chasing strategy.

Keywords : Lights Out; recursion; Fibonacci sequence

Mathematics Subject Classification (2020) **:** 91A99; 11B39

1 Introduction

Lights Out is an electronic handheld game from the 1990s that features a 5×5 grid of light-up buttons. In the classic version, the game begins with a random collection of the lights turned on and others turned off. The goal is to turn all lights off. Pressing a button changes its state (from on to off or vice versa) *and* changes the states of the cardinal neighboring buttons (above, below, left, and right). See Figure 1.

This classic Lights Out game has been studied extensively using linear algebra. For example, in [1] Anderson and Feil show that a strategy for winning can be obtained using matrices, Gauss-Jordan elimination and facts about the column and null space of a matrix, and in [10] Martín-Sánchez and Parejo-Flores provide a dual analysis of the game in which they demonstrate how the methodology for winning directly aligns with solving a specific problem in matrix algebra. Other analyses of Lights Out or its generalizations can be found in [3], [5], [6], [7], and [8].

A common variation of Lights Out allows for the lights to have more states that just "on" and "off." We can think of the states as different levels of brightness, and pressing a button brings the state of its light and the states of all adjacent lights to the next brightest setting. If a light is at the brightest setting, it turns off. Many of the sources listed above use linear algebra explore this multi-state Lights Out game. In this paper we consider a multi-state game in which the lights have k states for $k \geq 2$, and we play

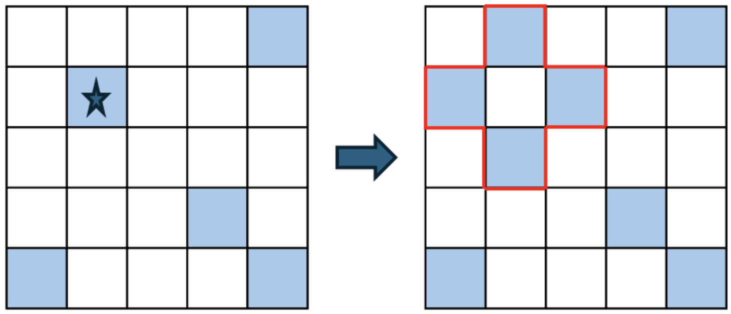

Figure 1: The grids above represent the board of a classic Lights Out game in which blue indicates the light is on, and white indicates the light is off. In the classic Lights Out game pressing the starred button on the board on the left results in the board on the right.

the game using a variation on a strategy that is popularly known as *light chasing*. Our mathematical approach differs from the standard approach in that we do not use linear algebra.

The light chasing strategy focuses on the fact that we can turn a light off by pressing the button directly below it. So, the light chasing strategy starts by pressing the buttons in the second row to turn off the lights in the first row. Then we press the lights in the third row to turn off the lights in the second row, etc. In many cases, when we get to the bottom of the board, i.e., when we press the lights in the last row to turn off the lights in the second-to-last row, not all lights will be turned off. Hence, the game will not yet have been won. However, depending on which lights remain on after pressing the buttons in the last row, it may be possible to press specific buttons in the top row and proceed down the board using the light chasing process again in order to turn off all lights. Knowing what buttons to press at the top of the board requires access to a "look up" table that can be found online at sites such as [13]. In [9], Leach shows how to use linear algebra to generate the look up tables.

But what if we do not have access to a look up table (and we do not remember the linear algebra necessary to generate one)? In this case, we want to know when does the light chasing process turn off all lights on the first pass. We think of this strategy in which we chase the lights to the bottom of the board and then stop instead of pressing buttons at the top again as *naive light chasing*. This paper explores naive light chasing for a Lights Out game in which the lights have k states and the board is wrapped around the cylinder so that the left side connects to the right. (See Figure 2). Our approach to studying this question depends on the starting configuration of the lights. In this paper we focus on starting with the lights in each row alternating between the brightest state and the off state. Hence, our goal is to explore the question below. (For an analysis of a similar game in which all lights start at the same setting see [2].)

Main Question *Suppose we use a naive light chasing strategy in a Lights Out game with* *k states for the lights and in which the board wraps around a cylinder. Further, suppose the game begins with the lights in each row alternating between the brightest state and the off state. How many rows of buttons must the board have in order for naive light chasing to turn off all lights?*

We analyze this game using recursion and modular arithmetic instead of linear algebra, and a connection to the Fibonacci numbers allows us to generate answers to the Main Question using known results about the Fibonacci numbers under various moduli. In the next section we formally define the Lights Out game with k states for the lights and whose board wraps around a cylinder, provide details on the naive light chasing process, and introduce terminology used throughout the paper. In Section 3 we develop the recursion that models the naive light chasing process, and in Section 4 we develop the relationship between the recursion and the Fibonacci numbers that allows us to provide answers to the Main Question.

2 The k-State Alternating-Start Cyl-Lights Out Game

We consider a *cylindrical* Lights Out game in which the board is an $m \times n$ grid of light-up buttons on a cylinder. Hence, the board "wraps around" in the sense that pressing the leftmost button in a row changes the state of the rightmost button (as well as the buttons above, below and to the right of the leftmost button, as in the classic game). Similarly, pressing the rightmost button in a row changes the state of the leftmost button.

Since the lights in our game have k states for some natural number $k \geq 2$, we think of the states as different levels of brightness and denote them by $0, 1, 2, \ldots, k-1$ where 0 is the off state and $k-1$ is the brightest state. Pressing a button one time increases the state of its light by 1 (mod k). In particular, if we press a button whose light is at the $k-1$ state (i.e., brightest state) one time, the light changes to the 0 state (i.e., it turns off). Going forward, we write -1 for the $k-1$ state. See Figure 2 and Figure 3. The goal of the game is still to turn off all lights, which means the goal is to get all lights to a state of 0.

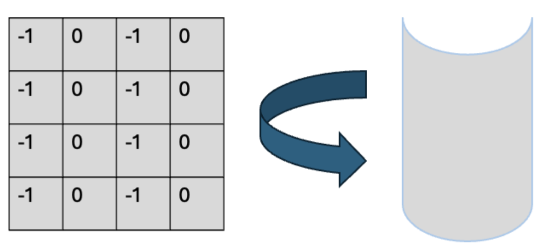

Figure 2: Wrapping an original Lights Out board around a cylinder turns it into a Cyl-Lights Out board.

Definition 2.1 (k-State Cyl-Lights Out Game) We refer to a Lights Out game in which the board is an $m \times n$ grid of light up buttons on a cylinder and in which the lights have k states as a *k-state Cyl-Lights Out game*.

Example 2.2 In Figure 3 we show an example of a 4-state Cyl-Lights Out game. The lights have states 0, 1, 2, and 3, and since the grid of buttons is on a cylinder, pressing the button marked with a star on the left results in the board on the right.

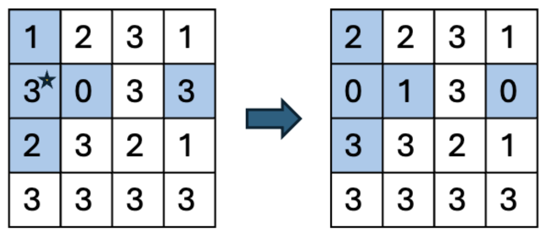

Figure 3: An example of a 4-State Cyl-Lights Out Game. Pressing the starred button on the left affects the five shaded buttons and results in the board on the right.

In this paper, we further assume that the board has an even number of columns and at the start of the game the lights in each row alternate between the brightest state and the off state. This inspires the following definition.

Definition 2.3 (k-state alternating-start Cyl-Lights Out game) A *k-state alternating-start Cyl-Lights Out game* is a k-state Cyl-Lights Out game with $2n$ columns for $n \geq 2$ with the following starting configuration: the lights in each row alternate between the -1 state and the 0 state such that the lights in each column have the same starting state. See Figure 4.

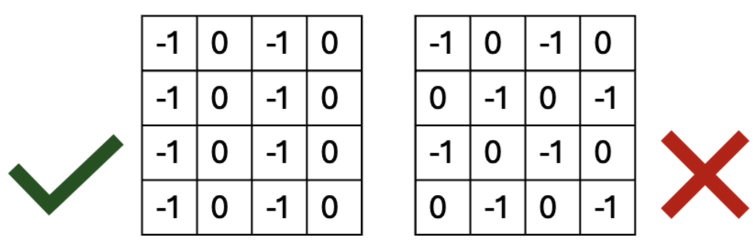

Figure 4: The board on the left is the starting configuration of a k-state alternating-start Cyl-Lights Out game. The board on the right is not. (It is possible to consider games with the starting configuration as show on the right, however the mathematical analysis is not as nice.)

We approach the k-state alternating-start Cyl-Lights out game using a strategy that we call *naive light chasing*. Naive light chasing begins in the same way as classic light chasing;

we work down the board, turning off the lights in a row by pressing the appropriate buttons in the row below. However, we stop the process after pressing the last row, i.e., we do not generate a look up table, and we do not start pressing buttons at the top again. It is not always possible to turn off all lights of a k-state Cyl-Lights Out game using naive light chasing. (i.e, see Figure 5.) This inspires the following definition.

Definition 2.4 (Naively Solvable) If naive light chasing turns off all lights of a k-state Cyl-Lights Out game, then we say the game is *naively solvable*.

The goal of this paper is to determine what size board a k-state alternating-start Cyl-Lights Out game should have in order to ensure that it is naively solvable. If a board has $2n$ columns for $n \geq 2$ (i.e., the number of columns is an even number greater than 2), then no two neighboring buttons have the same starting state. This means that lights in a given row with the same starting state are all affected in the same way by the naive light chasing process. (If the board has an odd number of columns, then there will be a pair of adjacent buttons in the same row that have the same starting state, and the naive light chasing process would affect these buttons differently than the rest of the buttons in their row.) Thus, since we are assuming that the board has $2n$ columns for $n \geq 2$, the number of columns will not affect whether a game is naively solvable. Hence, in this paper we investigate how many *rows* of buttons a k-state alternating-start Cyl-Lights Out game must have in order to be naively solvable.

Example 2.5 Consider a 3-state alternating-start Cyl-Lights Out game. If the board consists of four rows of buttons, then the game is not naively solvable. We show the naive light chasing strategy for this game in Figure 5; notice that the strategy does not turn off all lights. (We will see in Section 3 that such a 3-state game must have at least 11 rows in order to be naively solvable.)

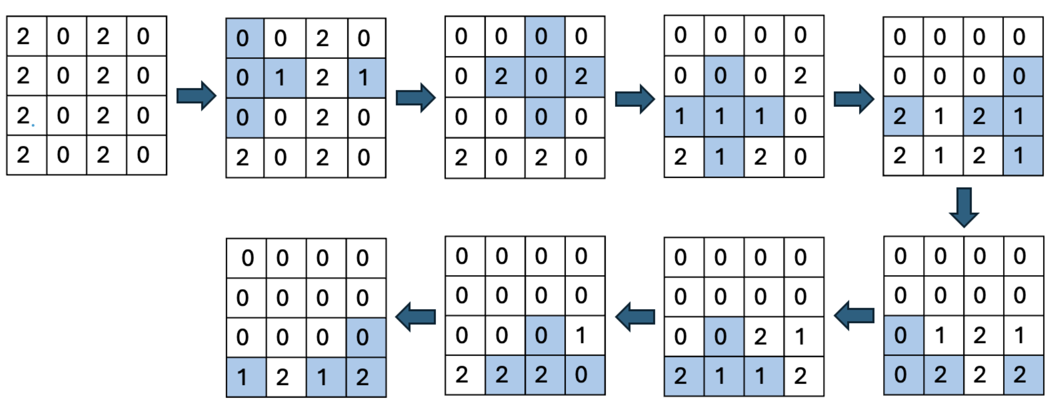

Figure 5: The naive light chasing strategy applied to a 3-state alternating-start Cyl-Lights Out game with four rows of buttons. This game is not naively solvable.

To determine if a k-state alternating-start Cyl-Lights Out game is naively solvable we rely on the fact that naive light chasing is recursive. This means that the state of the lights in the i^{th} row after turning off the lights in the $(i-1)^{st}$ row (i.e., after pressing the lights in the i^{th} row) is determined by the states of the lights above it. The remainder of this paper is dedicated to developing and analyzing a recursion for the k-state alternating-start Cyl-Lights Out game. We then use the recursion to determine how many rows are needed to ensure that the game is naively solvable.

3 A Recursive Approach

To build a recursion for the k-state alternating-start Cyl-Lights Out game we need to keep track of the starting state of each light in a row and how many times we press each button during the naive light chasing process. However, since we are assuming the game has an even number of columns, every light with the same starting state in a row will be pressed the same amount of times. This means that the naive light chasing process is only affected by the number of rows of buttons and not by the number of columns. Further, since there are two starting states (-1 and 0) for the lights in each row, the i^{th} term of the recursion consists of two equations; one gives the state of the lights in the i^{th} row that start at -1, and another one gives the state of the lights in the i^{th} row that start at 0.

More precisely, we let B_i be the state of the lights that start at the -1 state in the i^{th} row after turning off all lights in the $(i-1)^{st}$ row, and we let C_i be the state of the lights that start at the 0 state in the i^{th} row after turning off all lights in the $(i-1)^{st}$ row. We think of the lights in the 0^{th} row as being already turned off, making $B_0 = C_0 = 0$. Then B_1 and C_1 give the states of the lights in the first row at the start of the naive light chasing process, and so $B_1 = -1$ and $C_1 = 0$. We develop formulas for B_i and C_i for $i \geq 2$ using the fact that the states of the lights in the i^{th} row are affected when we press the buttons in the i^{th} and $(i-1)^{st}$ rows.

To develop a formula for B_i we note that that state of the lights in the i^{th} row that start at -1 is affected when we press the buttons in the $(i-1)^{st}$ row that are directly above the buttons whose state is B_i in order to turn off the corresponding lights in the $(i-2)^{nd}$ row. We need to press the buttons enough times so that the lights in the $(i-2)^{nd}$ row turn to the 0 state. Since pressing a button adds one (mod k) to the state of its light, we can think of the number of times we need to press the buttons as $-B_{i-2} \pmod{k}$.

Next, the state of the lights in the i^{th} row that start at -1 is also affected when we press the buttons in the i^{th} row to turn the lights in the $(i-1)^{st}$ row to the 0 state. First, we press the buttons whose state is given by B_i to turn off the light directly above them. The number of times we need to press these buttons is $-B_{i-1} \pmod{k}$. Then the state of the lights in the i^{th} row that start at -1 is affected when we press the two buttons on either side to turn off the lights above those buttons. These lights start at the 0 state. Since their state is given by C_{i-1}, we turn them off by pressing $-C_{i-1} \pmod{k}$ times. Thus, for $i \geq 2$, $B_i = -1 - B_{i-2} - B_{i-1} - 2C_{i-1}$. The equation for C_i is analogous; the only difference being that these lights start at the 0 state.

Recursion 3.1 In the k-state alternating-start Cyl-Lights Out game, let B_i be the state of the lights that start at the -1 state in the i^{th} row after turning off the lights in the $(i-1)^{st}$ row, and let C_i be the state of the lights that start at the 0 state in the i^{th} row after turning off the lights in the $(i-1)^{st}$ row. Then $B_0 = C_0 = 0$, $B_1 = -1$, $C_1 = 0$, and B_i and C_i are given by the following equations.

$$\begin{aligned} B_i &= -1 - B_{i-2} - B_{i-1} - 2C_{i-1} \\ C_i &= -C_{i-2} - C_{i-1} - 2B_{i-1}. \end{aligned}$$

The first 13 terms of Recursion 3.1 are shown in Table 1.

i	0	1	2	3	4	5	6	7	8	9	10	11	12
B_i	0	-1	0	-4	7	-20	52	-137	356	-936	2447	-6408	16776
C_i	0	0	2	-2	8	-20	52	-136	358	-934	2448	-6408	16776

Table 1: The first 13 terms of Recursion 3.1

The following fact will allow us to use Recursion 3.1 to determine the number of rows needed for a k-state alternating-start Cyl-Lights Out game to be naively solvable.

Fact 3.2 The k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if $B_i \equiv C_i \equiv 0 \pmod{k}$.

For example, from Table 1 we see that both the 4-state and 5-state alternating-start Cyl-Lights Out games are naively solvable when the board had 5 rows because -20 is divisible by both 4 and 5. On the other hand, the 8-state game is not naively solvable when the board has 4 rows because while C_4 is divisible by 8, B_4 is not.

Our goal is then to determine when $B_i \equiv C_i \equiv 0 \pmod{k}$. To help us in this endeavor, from Table 1 we notice that B_i and C_i are closely related.

Lemma 3.3 *Let B_i and C_i be the i^{th} terms of Recursion 3.1. Then*

$$B_i = \begin{cases} C_i - 1 & \text{if } i \equiv 1 \text{ or } i \equiv 4 \pmod{6} \\ C_i - 2 & \text{if } i \equiv 2 \text{ or } i \equiv 3 \pmod{6} \\ C_i & \text{if } i \equiv 0 \text{ or } i \equiv 5 \pmod{6}. \end{cases}$$

Proof. Writing i as $i = 6x + r$ for some non-negative integers x and r with $0 \leq r \leq 5$, we proceed by induction on x. For the base case, let $x = 0$. If $i = 1$ or $i = 4$, from Table 1 we see that $B_1 = -1$ and $C_1 = 0$ and, $B_4 = 7$ and $C_4 = 8$, as desired. Similarly, if $i = 2$ or $i = 3$ then $B_2 = 0$, $C_2 = 2$, $B_3 = -4$, and $C_3 = -2$. Finally, if $i = 0$ or $i = 5$, we see that $B_0 = C_0 = 0$ and $B_5 = C_5 = -20$.

For the induction step we assume that the statement holds for an arbitrary non-negative integer x and show that it holds for $x + 1$. Thus, we assume the following:

$$B_{6x} = C_{6x} \text{ and } B_{6x+5} = C_{6x+5},$$

$$B_{6x+1} = C_{6x+1} - 1 \text{ and } B_{6x+4} = C_{6x+4} - 1,$$
$$B_{6x+2} = C_{6x+2} - 2 \text{ and } B_{6x+3} = C_{6x+3} - 2.$$

We will show

$$B_{6(x+1)} = C_{6(x+1)},\ B_{6(x+1)+1} = C_{6(x+1)+1} - 1,\ \text{and } B_{6(x+1)+2} = C_{6(x+1)+2} - 2.$$

The proofs of that $B_{6(x+1)+5} = C_{6(x+1)+5}$, $B_{6(x+1)+4} = C_{6(x+1)+4} - 1$, and $B_{6(x+1)+3} = C_{6(x+1)+3} - 2$ are analogous.

We first show $B_{6(x+1)} = C_{6(x+1)}$. By definition,

$$B_{6(x+1)} = B_{6x+6} = -1 - B_{6x+4} - B_{6x+5} - 2C_{6x+5}.$$

Then from the inductive hypothesis it follows that $B_{6x+4} = (C_{6x+4} - 1)$ and $B_{6x+5} = C_{6x+5}$. Thus, we have

$$\begin{aligned} B_{6(x+1)} &= -1 - B_{6x+4} - B_{6x+5} - 2C_{6x+5} \\ &= -1 - (C_{6x+4} - 1) - C_{6x+5} - 2B_{6x+5} \\ &= C_{6(x+1)}. \end{aligned}$$

Similarly, to show that $B_{6(x+1)+1} = C_{6(x+1)+1} - 1$ we start with

$$B_{6(x+1)+1} = B_{6x+7} = -1 - B_{6x+5} - B_{6x+6} - 2C_{6x+6}.$$

Then by the inductive hypothesis and by what we showed above, it follows that $B_{6x+5} = C_{6x+5}$ and $B_{6x+6} = C_{6x+6}$. Hence,

$$\begin{aligned} B_{6(x+1)+1} &= -1 - C_{6x+5} - C_{6x+6} - 2B_{6x+6} \\ &= C_{6(x+1)+1} - 1. \end{aligned}$$

Finally,

$$\begin{aligned} B_{6(x+1)+2} = B_{6x+8} &= -1 - B_{6x+6} - B_{6x+7} - 2C_{6x+7} \\ &= -1 - C_{6x+6} - (C_{6x+7} - 1) - 2(B_{6x+7} + 1) \\ &= -C_{6x+6} - C_{6x+7} - 2B_{6x+7} - 2 \\ &= C_{6(x+1)+2} - 2. \end{aligned}$$

□

Since a k-state alternating-start Cyl-Lights Out game with i rows is only naively solvable when both B_i and C_i are congruent to 0 (mod k), from Lemma 3.3 we immediately see that there are infinitely many board sizes for which the game is not naively solvable for any $k \geq 2$.

Theorem 3.4 *Let $k \geq 2$ be a natural number and let $i \equiv 1$ (mod 3). Then the k-state alternating-start Cyl-Lights Out game with i rows is not naively solvable.*

Proof. First, $i \equiv 1 \pmod 3$ means that $i \equiv 1 \pmod 6$ or $i \equiv 4 \pmod 6$. By Lemma 3.3, if $i \equiv 1 \pmod 6$ or $i \equiv 4 \pmod 6$, then $C_i = B_i + 1$. Hence, B_i and C_i are relatively prime. It follows that B_i and C_i are not both congruent to $0 \pmod k$ for any $k \geq 2$. $\square$

On the other hand, in Table 1 we see that B_i and C_i are both even except when $i \equiv 1 \pmod 3$. It turns out that this is true for all $i \not\equiv 1 \pmod 3$.

Lemma 3.5 *Let B_i and C_i be the i^{th} terms of Recursion 3.1. If $i \equiv 0$ or $i \equiv 2$ (mod 3) then both B_i and C_i are even.*

Proof. Letting $i = 3x$ or $i = 3x + 2$, respectively, for some non-negative integer x we proceed by induction on x. For the base case, let $x = 0$ so that $i = 0$ or $i = 2$. From Table 1, we see that $B_0 = C_0 = 0$, $B_2 = 0$, and $C_2 = 2$.

For the inductive step let x be an arbitrary non-negative integer and suppose that B_{3x}, B_{3x+2}, C_{3x}, and C_{3x+2} are even. We show that $B_{3(x+1)}$ and $C_{3(x+1)}$ are even. The proofs that $B_{3(x+1)+2}$ and $C_{3(x+1)+2}$ are even are analogous. First, by definition,

$$\begin{aligned} B_{3(x+1)} = B_{3x+3} &= -1 - B_{3x+1} - B_{3x+2} - 2C_{3x+2} \\ &= -1 - B_{3x+1} - (-1 - B_{3x} - B_{3x+1} - 2C_{3x+1}) - 2C_{3x+2} \\ &= B_{3x} + 2C_{3x+1} - 2C_{3x+2} \end{aligned}$$

By the inductive hypothesis, B_{3x} is even. Thus, since $B_{3(x+1)}$ is a sum of even numbers, it is even as well.

Similarly,

$$\begin{aligned} C_{3(x+1)} = C_{3x+3} &= -C_{3x+1} - C_{3x+2} - 2B_{3x+2} \\ &= -C_{3x+1} - (-C_{3x} - C_{3x+1} - 2B_{3x+1}) - 2B_{3x+2} \\ &= C_{3x} + 2B_{3x+1} - 2B_{3x+2}, \end{aligned}$$

which is even as well. $\square$

We now know the number of rows needed for the 2-state alternating-start game to be naively solvable.

Theorem 3.6 *The 2-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if $i \equiv 0$ (mod 3) or $i \equiv 2$ (mod 3).*

Proof. If $i \equiv 1 \pmod 3$, then the 2-state alternating-start game is not naively solvable by Theorem 3.4. If $i \equiv 0$ or $i \equiv 2 \pmod 3$ then the 2-state alternating-start game is naively solvable by Fact 3.2 and Lemma 3.5. $\square$

In the next lemma we show that when $k > 2$, the k-state alternating-start game only has a chance of being naively solvable when $i \equiv 0$ or $i \equiv 5 \pmod 6$.

Lemma 3.7 *Let $k > 2$. If the k-state alternating-start Cyl-Lights Out game is naively solvable then $i \equiv 0$ or $i \equiv 5$ (mod 6).*

Proof. Let $k > 2$. First, by Theorem 3.4, if $i \equiv 1$ or $i \equiv 4 \pmod 6$ then the k-state alternating-start game is not naively solvable. It remains to show that if $i \equiv 2$ or $i \equiv 3 \pmod 6$ then the k-state alternating-start game is also not naively solvable.

If $i \equiv 2$ or $i \equiv 3 \pmod 6$, then by Lemma 3.3 it follows that $B_i = C_i - 2$, and by Lemma 3.5, both B_i and C_i are even. Thus, there is an integer z such that $B_i = 2z$ and $C_i = 2z + 2 = 2(z + 1)$. Since z and $z + 1$ are relatively prime, it follows that $\gcd(B_i, C_i) = 2$. Hence, there is no $k > 2$ for which B_i and C_i are congruent to 0 (mod k), and so by Fact 3.2 there is no $k > 2$ for which the k-state alternating-start game with i rows is naively solvable. □

The converse of Lemma 3.7 is not true; it is not the case that the k-state alternating-start game will always be naively solvable when $i \equiv 0 \pmod 6$ or $i \equiv 5 \pmod 6$. For example, in Table 1 we see that $B_5 = C_5 = -20$ and $B_6 = C_6 = 52$. Since neither of these are divisible by 3, it follows that the 3-state alternating-start Cyl-Lights Out game is not naively solvable when the board has five or six rows.

4 A Connection to the Fibonacci Numbers

In order to determine the number of rows needed for a k-state alternating-start Cyl-Lights Out game to be naively solvable we connect Recursion 3.1 to the Fibonacci numbers. We then use known results about the Fibonacci numbers (mod k) to draw conclusions on when the game is naively solvable.

Recall that the Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$. We show the first 16 terms of this sequence in Table 2. In Theorem 4.2 we show that the sum $B_i + C_i$ can be recognized as a product of Fibonacci numbers. We need the following identity for the proof.

Fact 4.1 (Cassini's Identity) Let F_i be the i^{th} Fibonacci number. Then

$$F_{i-1}F_{i+1} - F_i^2 = (-1)^i.$$

For multiple proofs of the identity see [4].

Theorem 4.2 *Let B_i and C_i be the i^{th} terms of Recursion 3.1 for $i \geq 0$. Then*

$$B_i + C_i = (-1)^i F_i F_{i+1}.$$

Proof. We proceed it by induction on i. For the base cases let $i = 0$ and $i = 1$. Then

$$B_0 + C_0 = 0 = (-1)^0 F_0 F_1 \text{ and } B_1 + C_1 = -1 = (-1)^1 F_1 F_2.$$

For the inductive step, let n be an arbitrary natural number and assume $B_i + C_i = (-1)^i F_i F_{i+1}$ for all $i \leq n$. We show $B_{n+1} + C_{n+1} = (-1)^{n+1} F_{n+1} F_{n+2}$. In other words, using Recursion 3.1, we show

$$B_{n+1} + C_{n+1} = -1 - B_{n-1} - B_n - 2C_n - C_{n-1} - C_n - 2B_n = (-1)^{n+1} F_{n+1} F_{n+2}. \quad (1)$$

Looking at the left side of Equation 1, by adjusting some terms and using the inductive hypothesis, we get

$$\begin{aligned}
-1 - B_{n-1} - B_n - 2C_n - C_{n-1} - C_n - 2B_n &= -1 - (B_{n-1} + C_{n-1}) - 3(B_n + C_n) \\
&= -1 - (-1)^{n-1}F_{n-1}F_n - 3(-1)^n F_n F_{n+1} \\
&= -1 + (-1)^n F_{n-1}F_n - 3(-1)^n F_n F_{n+1} \\
&= -1 + (-1)^n F_n(F_{n-1} - 3F_{n+1}).
\end{aligned}$$

Now looking at the right side of Equation 1, by applying the definition of the Fibonacci numbers, adjusting some terms, and using Fact 4.1, we get

$$\begin{aligned}
(-1)^{n+1}F_{n+1}F_{n+2} &= -(-1)^n(F_{n-1} + F_n)(F_n + F_{n+1}) \\
&= -(-1)^n(F_{n-1}F_n + F_{n-1}F_{n+1} + F_n^2 + F_nF_{n+1}) \\
&= -(-1)^n(F_{n-1}F_n + ((-1)^n + F_n^2) + F_n^2 + F_nF_{n+1}) \\
&= -(-1)^{2n} - (-1)^n(F_{n-1}F_n + 2F_n^2 + F_nF_{n+1}) \\
&= -1 - (-1)^n(F_{n-1}F_n + 2F_n(F_{n+1} - F_{n-1}) + F_nF_{n+1}) \\
&= -1 - (-1)^n(3F_nF_{n+1} - F_{n-1}F_n) \\
&= -1 + (-1)^n F_n(F_{n-1} - 3F_{n+1}).
\end{aligned}$$

Therefore, Equation 1 holds and $B_i + C_i = (-1)^i F_i F_{i+1}$ for all i. □

We use Theorem 4.2 to determine the amount of rows needed for a k-state alternating-start game to be naively solvable. Since we know that the game is not naively solvable when the number of rows i is not congruent to 0 or 5 (mod 6), we focus on $i \equiv 0$ and $i \equiv 5 \pmod 6$. The following corollary follows directly from Theorem 4.2 and Lemma 3.7.

Corollary 4.3 *Let B_i and C_i be the i^{th} terms of Recursion 3.1. If $i \equiv 0$ (mod 6) or $i \equiv 5$ (mod 6) then*

$$B_i = C_i = \frac{1}{2}(-1)^i F_i F_{i+1}.$$

From Lemma 3.7 and Corollary 4.3, it follows that when $k > 2$, the k-state alternating-start game with i rows will be naively solvable when $i \equiv 0 \pmod 6$ or $i \equiv 5 \pmod 6$ and either $F_i \equiv 0 \pmod k$ or $F_{i+1} \equiv 0 \pmod k$. The Fibonacci numbers have been extensively studied under different moduli. See for example [11] and [12]. The following definition and lemma first appeared in [12] and are well-known to Fibonacci number enthusiasts.

Definition 4.4 [Restricted Period of the Fibonacci Numbers, $\alpha(k)$] The *restricted period* of the Fibonacci sequence (mod k), denoted $\alpha(k)$, is the least positive integer i such that $F_i \equiv 0 \pmod k$.

Lemma 4.5 (Theorem 3, [12]) *Let F_i be the i^{th} term of the Fibonacci sequence and let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). Then $F_i \equiv 0$ (mod k) if and only if $i \equiv 0$ (mod $\alpha(k)$).*

For example, Table 2 shows the first 16 terms of the original Fibonacci sequence along with the sequence taken mod 3, mod 4, and mod 5. Notice that $F_4 \equiv 0 \pmod 3$. Thus, $\alpha(3) = 4$, and by Lemma 4.5, $F_i \equiv 0 \pmod 3$ if and only if $i \equiv 0 \pmod 4$. Taking the Fibonacci sequence mod 4, we see that the first 0 occurs at F_6. Hence, $\alpha(4) = 6$ and $F_i \equiv 0 \pmod 4$ if and only if $i \equiv 0 \pmod 6$. Similarly, $\alpha(5) = 5$ so $F_i \equiv 0 \pmod 5$ if and only if $i \equiv 0 \pmod 5$. More values for $\alpha(k)$ can be found on [14].

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_i	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_i \pmod 3$	0	1	1	2	**0**	2	2	1	**0**	1	1	2	**0**	2	2	1
$F_i \pmod 4$	0	1	1	2	3	1	**0**	1	1	2	3	1	**0**	1	1	2
$F_i \pmod 5$	0	1	1	2	3	**0**	3	3	1	4	**0**	4	4	3	2	**0**

Table 2: The first 16 terms of the Fibonacci sequence, the Fibonacci sequence (mod 3), and the Fibonacci sequence (mod 4). Notice that $\alpha(3) = 4$, $\alpha(4) = 6$, and $\alpha(5) = 5$.

If we know the restricted period of the Fibonacci numbers (mod k), the following theorem gives us a quick way to find infinitely many board sizes for which the k-state alternating-start game is naively solvable.

Theorem 4.6 *Let $k > 2$, and let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). If $i \equiv 0$ (mod lcm$(\alpha(k), 6)$) or $i \equiv -1$ (mod lcm$(\alpha(k), 6)$), then the k-state alternating-start game with i rows is naively solvable.*

Proof. If $i \equiv 0 \pmod{\operatorname{lcm}(\alpha(k), 6)}$ then $i \equiv 0 \pmod 6$ and $i \equiv 0 \pmod{\alpha(k)}$. In this case, by Theorem 4.2, $B_i = C_i = \frac{1}{2}F_iF_{i+1}$, and by Lemma 4.5, $F_i \equiv 0 \pmod k$. Thus, $B_i = C_i \equiv 0 \pmod k$, making the game naively solvable. Similarly, if $i \equiv -1 \pmod{\operatorname{lcm}(\alpha(k), 6)}$, then $i \equiv 5 \pmod 6$ and $F_{i+1} \equiv 0 \pmod k$. Again, it follows that $B_i = C_i = \frac{1}{2}F_iF_{i+1} \equiv 0 \pmod k$. □

While the above theorem gives infinitely many board sizes for which the k-state alternating-start game is naively solvable, it does not give all possible board sizes. For example, when $k = 5$, $\alpha(5) = 5$ and $\operatorname{lcm}(\alpha(5), 6) = 30$. Thus, using Theorem 4.6 we see that the 5-state alternating-start game is naively solvable when the number of rows is congruent to 0 or $-1 \pmod{30}$. However, from Table 1 we see that $B_5 = C_5 = -20$, and thus the 5-state game is also naively solvable when the board has five rows. In the remainder of this section we seek to identify *all* possible board sizes for which a k-state alternating-start Cyl-Lights Out game is naively solvable. Specifically, we focus on all values of $k > 2$ for which we do not run into a product of zero divisors (mod k) when $B_i = C_i$. (Recall from Lemma 3.7 that when $k > 2$ the game with i rows is not naively solvable when $B_i \neq C_i$.) This inspires the following definition.

Definition 4.7 (No Consecutive Zero Divisors) We say the Fibonacci sequence has *no consecutive zero divisors (mod k)* if for all $i \equiv 0 \pmod 6$ and all $i \equiv 5 \pmod 6$, $F_iF_{i+1} \equiv 0 \pmod k$ implies either $F_i \equiv 0 \pmod k$ or $F_{i+1} \equiv 0 \pmod k$.

When we assume that the Fibonacci sequence has no consecutive zero divisors (mod k) we are able to fully characterize the conditions on i under which the k-state alternating-start game with i rows is naively solvable.

Theorem 4.8 *Let $k > 2$. Assume that the Fibonacci sequence has no consecutive zero divisors (mod k), and let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). Then the k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if one of the following properties hold:*

Property 1: $\alpha(k)$ *and 6 both divide* i,

Property 2: $\alpha(k)$ *and* 6 *both divide* $i + 1$,

Property 3: $\alpha(k)$ *divides* i *and* 6 *divides* $i + 1$, *or*

Property 4: $\alpha(k)$ *divides* $i + 1$ *and* 6 *divides* i.

Proof. The k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if $B_i = C_i \equiv 0 \pmod{k}$. Using Lemma 3.7 and Corollary 4.3, since we assume that the Fibonacci sequence has no zero divisors (mod k), it follows that $B_i = C_i \equiv 0 \pmod{k}$ if and only if $i \equiv 0 \pmod{6}$ or $i \equiv 5 \pmod{6}$ and either $F_i \equiv 0 \pmod{k}$ or $F_{i+1} \equiv 0 \pmod{k}$. By the definition of $\alpha(k)$, $i \equiv 0 \pmod{6}$ and $F_i \equiv 0 \pmod{k}$ if and only if Property 1 holds, and $i \equiv 0 \pmod{6}$ and $F_{i+1} \equiv 0 \pmod{k}$ if and only if Property 3 holds. Properties 2 and 4 hold if and only if $i \equiv 5 \pmod{6}$ and $F_{i+1} \equiv 0 \pmod{k}$ and $i \equiv 5 \pmod{6}$ and $F_i \equiv 0 \pmod{k}$, respectively. □

While Theorem 4.8 characterizes the values for i for which the k-state alternating-start game with i rows is naively solvable, it does not provide us with a nice way of computing such i values. In the remaining results we provide insight into how to determine which values of i satisfy the four conditions of Theorem 4.8. For example, $\alpha(k)$ and 6 both divide i if and only if $i \equiv 0 \pmod{\operatorname{lcm}(\alpha(k), 6)}$, and $\alpha(k)$ and 6 both divide $i + 1$ if and only if $i \equiv -1 \pmod{\operatorname{lcm}(\alpha(k), 6)}$. Hence, Theorem 4.6 is reflected in Properties (1) and (2) of Theorem 4.8. Going one step further, in Theorem 4.9 we prove that if the Fibonacci sequence has no consecutive zero divisors (mod k) and $\gcd(\alpha(k), 6) \neq 1$ then Properties (3) and (4) cannot occur, which turns Theorem 4.6 into an if and only if statement.

Theorem 4.9 *Let $k > 2$. Assume that the Fibonacci sequence has no consecutive zero divisors (mod k), and let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). Further assume* $\gcd(\alpha(k), 6) \neq 1$. *Then the k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if $i \equiv 0$ (mod lcm$(\alpha(k), 6)$) or $i \equiv -1$ (mod lcm$(\alpha(k), 6)$).*

Proof. First, $\alpha(k)$ and 6 both divide i if and only if $i \equiv 0 \pmod{\operatorname{lcm}(\alpha(k), 6)}$, and $\alpha(k)$ and 6 both divide $i + 1$ if and only if $i \equiv -1 \pmod{\operatorname{lcm}(\alpha(k), 6)}$. It remains to show that if $\gcd(\alpha(k), 6) \neq 1$ then Conditions (3) and (4) of Theorem 4.8 do not occur.

We show that Condition (3) does not occur. The proof that Condition (4) does not occur is analogous. Assume for contradiction that $\alpha(k)$ divides i and 6 divides $i+1$. Then the divisors of $\alpha(k)$ also divide i, and the divisors of 6 also divide $i+1$. Thus, $\gcd(\alpha(k), 6)$ divides both i and $i+1$. However, i and $i+1$ are relatively prime. This contradicts our assumptions that $\gcd(\alpha(k), 6) \neq 1$. □

Example 4.10 When $k = 3$, we see from Table 2 that the restricted period, $\alpha(3) = 4$, so $\gcd(\alpha(3), 6) = 2$ and $\operatorname{lcm}(\alpha(3), 6) = 12$. (Since 3 is prime we do not need to worry about zero divisors.) Thus, by Theorem 4.9, the 3-state alternating-start game with i rows is naively solvable if and only if $i \equiv 0 \pmod{12}$ or $i \equiv 11 \pmod{12}$.

Example 4.11 When $k = 4$, Table 2 also shows that $\alpha(k) = 6$ and that the Fibonacci sequence has no consecutive zero divisors (mod 4). (There are no consecutive zero divisors in the first six terms, and the Fibonacci sequence (mod 4) is an infinite loop of those six terms.) By Theorem 4.9, the 4-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if $i \equiv 0$ or $i \equiv 5 \pmod 6$. Since $\alpha(4) = 6$, it follows that the 4-state alternating-start game is naively solvable for the most board sizes for $k > 2$.

As we saw after Theorem 4.6, the 5-state alternating-start game with i rows is naively solvable for more values of i than just $i \equiv 0 \pmod{\operatorname{lcm}(\alpha(5), 6)}$ or $i \equiv -1 \pmod{\operatorname{lcm}(\alpha(5), 6)}$. This is because when $k = 5$, there exists values for i for which Conditions (3) and (4) of Theorem 4.8 hold. In the following lemma we show that such i always exist when $\gcd(\alpha(k), 6) = 1$.

Lemma 4.12 *Let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). Further assume* $\gcd(\alpha(k), 6) = 1$. *Then*

- *there exists i such that $\alpha(k)$ divides i and 6 divides $i+1$, and*
- *there exists j such that $\alpha(k)$ divides $j+1$ and 6 divides j.*

Proof. Since $\gcd(\alpha(k), 6) = 1$, there exists integers r and s such that $\alpha(k)r + 6s = 1$. Then either $r < 0$ or $s < 0$. We consider each case separately. First, if $r < 0$ let $i = |\alpha(k)r|$ and $j = 5|\alpha(k)r| - 1$. Then $\alpha(k)$ divides i and $i + 1 = |\alpha(k)r| + 1 = |1 - 6s| + 1 = (6s - 1) + 1 = 6s$. Further, $\alpha(k)$ divides $j + 1$ and $j = 5(6s - 1) - 1 = 6(5s - 1)$ so 6 divides j.

On the other hand, if $s < 0$, let $j = |6s|$ and $i = 5|6s| + 5$. Then 6 divides j and $j + 1 = |6s| + 1 = |1 - \alpha(k)r| + 1 = \alpha(k)r$. So, $\alpha(k)$ divides $j + 1$. Since $i = 5|6s| + 5$ then 6 divides $i + 1$ and $i = 5(\alpha(k)r - 1) + 5 = 5\alpha(k)r$. So, $\alpha(k)$ divides i. □

When $\gcd(\alpha(k), 6) = 1$ we can characterize the values of i for which Properties (3) and (4) of Theorem 4.8 occur in terms of the smallest of such i's. More specifically, letting i_1 be the smallest natural number such that $\alpha(k)$ divides i_1 and 6 divides $i_1 + 1$ and i_2 be the smallest natural number such that $\alpha(k)$ divides $i_2 + 1$ and 6 divides i_2, we first show that both i_1 and i_2 are less than $\alpha(k)6$, i.e., less than $\operatorname{lcm}(\alpha(k), 6)$. This allows us to view i_1 and i_2 as congruence classes (mod $\alpha(k)6$). Then in Theorem 4.14 we show that all other i's that satisfy Properties (3) and (4) of Theorem 4.8 are congruent to i_1 and i_2 (mod $\alpha(k)6$), respectively.

Lemma 4.13 *Let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k) and assume $\gcd(\alpha(k), 6) = 1$. If i_1 is the smallest natural number such that $\alpha(k)$ divides i_1 and 6 divides $i_1 + 1$ and i_2 is the smallest natural number such that $\alpha(k)$ divides i_2 and 6 divides $i_2 + 1$ then*

$$i_1 < 6\alpha(k) \text{ and } i_2 < 6\alpha(k).$$

Proof. Let i_1 be the smallest natural number such that $\alpha(k)$ divides i_1 and 6 divides i_1+1. We will show that $i_1 < 6\alpha(k)$. The proof that $i_2 < 6\alpha(k)$ is analogous. Since 6 divides $i_1 + 1$ and thus doesn't divide i_1, it follows that $i_1 \neq \operatorname{lcm}(\alpha(k), 6) = 6\alpha(k)$. Assume for contradiction that $i_1 > 6\alpha(k)$, so there exists an $r > 0$ such that $i_1 = 6\alpha(k) + r$. Because $6\alpha(k) \neq 0$, $i_1 > r$. Further, since $i_1 = 6\alpha(k)+r$, it follows that $\alpha(k)$ divides $\alpha(k)6+r$ and 6 divides $\alpha(k)6 + r + 1$. So $\alpha(k)$ divides r and 6 divides $r + 1$. But this is a contradiction because $r < i_1$ and we assumed i_1 is the smallest natural number such that $\alpha(k)$ divides i_1 and 6 divides $i_1 + 1$. Therefore, $i_1 < 6\alpha(k)$. □

Theorem 4.14 *Let $k > 2$. Assume that the Fibonacci sequence has no consecutive zero divisors (mod k), and let $\alpha(k)$ be the restricted period of the Fibonacci sequence (mod k). Further assume $\gcd(\alpha(k), 6) = 1$. Then the k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if one of the following conditions holds.*

Condition 1: $i \equiv 0 \pmod{6\alpha(k)}$,

Condition 2: $i \equiv -1 \pmod{6\alpha(k)}$,

Condition 3: $i \equiv i_1 \pmod{6\alpha(k)}$ *where i_1 is the smallest natural number such that $\alpha(k)$ divides i_1 and 6 divides $i_1 + 1$, or*

Condition 4: $i \equiv i_2 \pmod{6\alpha(k)}$ *where i_2 is the smallest natural number such that $\alpha(k)$ divides $i_2 + 1$ and 6 divides i_2.*

Proof. By Theorem 4.8, the k-state alternating-start Cyl-Lights Out game with i rows is naively solvable if and only if (1) $\alpha(k)$ and 6 both divide i, (2) $\alpha(k)$ and 6 both divide $i + 1$, (3) $\alpha(k)$ divides i and 6 divides $i + 1$, or (4) $\alpha(k)$ divides $i + 1$ and 6 divides i.

Since $\gcd(\alpha(k), 6) = 1$, it follows that $\operatorname{lcm}(\alpha(k), 6) = 6\alpha(k)$. Hence, $\alpha(k)$ and 6 both divide i if and only if $i \equiv 0 \pmod{6\alpha(k)}$, and $\alpha(k)$ and 6 both divide $i + 1$ if and only if $i \equiv -1 \pmod{6\alpha(k)}$.

We now need to show that $\alpha(k)$ divides i and 6 divides $i+1$ if and only if $i \equiv i_1 \pmod{6\alpha(k)}$, and $\alpha(k)$ divides $i+1$ and 6 divides i if and only if $i \equiv i_2 \pmod{6\alpha(k)}$. We prove the first statement. The proof of the second statement is analogous.

If $i \equiv i_1 \pmod{6\alpha(k)}$ then $i = i_1 + 6\alpha(k)x$ for some integer x. (By Lemma 4.13, $i_1 < 6\alpha(k)$.) Since $\alpha(k)$ divides i_1, it follows that $\alpha(k)$ divides $i_1 + 6\alpha(k)$, so $\alpha(k)$ divides i. Similarly, since 6 divides $i_1 + 1$ and $i = i_1 + 6\alpha(k)x$, it follows that 6 divides $i + 1$.

Next we show that if $\alpha(k)$ divides i and 6 divides $i+1$ then $i \equiv i_1 \pmod{6\alpha(k)}$. Since $\alpha(k)$ divides both i and i_1, it follows that $\alpha(k)$ divides $i - i_1$. Hence $i \equiv i_1 \pmod{\alpha(k)}$. Since 6 divides $i+1$ and 6 divides i_1+1, it follows that 6 divides $i+1-(i_1+1) = i-i_1$, and $i \equiv i_1 \pmod 6$. Therefore, $i \equiv i_1 \pmod{6\alpha(k)}$. □

While there is no formula for calculating i_1 and i_2 of Theorem 4.14, they are often easy to determine for specific k. We show an example below.

Example 4.15 (The 5-state alternating-start Cyl-Lights Out game) When $k = 5$, the restricted period $\alpha(k) = 5$, so $\gcd(\alpha(k), 6) = 1$, and the 5-state alternating-start game with i rows is naively solvable if and only if one of the conditions of Theorem 4.14 holds. To determine i_1 of Condition (3) we see that $i_1 = 5$ is the smallest integer such that $5|i_1$ and $6|i_1 + 1$. Similarly, for i_2 of Condition (4) we see that $i_2 = 24$ is the smallest integer such that $5|i_2 + 1$ and $6|i_2$. Therefore, since $\alpha(5) \cdot 6 = 30$, we see that the 5-state alternating-start game with i rows is naively solvable if and only if one of the following holds;

1. $i \equiv 0 \pmod{30}$,
2. $i \equiv -1 \pmod{30}$,
3. $i \equiv 5 \pmod{30}$, or
4. $i \equiv 24 \pmod{30}$.

So, the 5-state alternating-start game with $i \leq 100$ rows is naively solvable for

$$i = 5, 24, 29, 30, 35, 54, 59, 60, 65, 84, 89, 90, 95.$$

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