

The Straightening Identity in the Universal Enveloping Algebra of the Heisenberg Algebra

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Abstract - This work aims to formulate and prove the nontrivial straightening identity in the universal enveloping algebra of the Heisenberg algebra. This identity is a special case of the Baker-Campbell-Hausdorff formula, but was proven using a new proof method of double induction. This general formula is applicable for straightening identities in many different Lie algebras, one application being the Onsager algebra of $\mathfrak{sl}_4$.

Keywords : Lie algebras; straightening identities; Baker-Campbell-Hausdorff formula; Heisenberg algebra

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1 Introduction

Straightening identities are reordering formulas within the universal enveloping algebra of a Lie algebra. These identities can be used to turn products of the basis elements into linear combinations of products of basis elements where the factors are in a preferred order. The Baker-Campbell-Hausdorff Theorem gives straightening identities in the universal enveloping algebra of general Lie algebras. See a standard reference like [8, Theorem 5.3] for details on the Baker-Campbell-Hausdorff Theorem.

For context on straightening identities, in [9, Lemma 1] B. Kostant proved some straightening identities in the universal enveloping algebra of any simple complex finite dimensional Lie algebra, $\mathfrak{g}$. H. Garland in [7, Lemma 7.5] proved some straightening identities in the universal enveloping algebras of the untwisted loop algebras, $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Garland's work was later expanded to untwisted map algebras in [3, Lemma 5.4] by S. Chamberlin proving some straightening identities in the universal enveloping algebras of the map algebras $\mathfrak{g} \otimes A$, where A is any commutative associative algebra with unity.

Further straightening identities were developed by D. Mitzman in [10], who was the first to do this for twisted Lie algebras. E. Date and S. Roan in [6] introduced the structure of the Onsager algebra as we see it now. They noted the relationship between the Onsager algebra and the loop algebra of $\mathfrak{sl}_2$. In [5, Theorem 3.3] S. Chamberlin, A. Rafizadeh, M. Samsel, and D. Strong formulated and proved some straightening identities in the universal enveloping algebra of the Onsager algebra of $\mathfrak{sl}_2$. In [1, Proposition 2.5] A. Bianchi and S. Chamberlin formulated and proved complete straightening identities in this algebra. Building off of these publications is S. Chamberlin and J. Niraula's work



in [4, Theorems 3.1 and 3.2] formulating some straightening identities in the universal enveloping algebras of the twisted multiloop algebras of $\mathfrak{sl}_4$ with Chevalley involution twist. This work also formulated a nice basis in $\mathfrak{sl}_4$ which we used in the example in this paper.

This paper is predicated on two main theorems, the Poincare-Birkhoff-Witt (PBW) Theorem and the Baker-Campbell-Hausdorff Theorem. The Poincare-Birkhoff-Witt Theorem gives a basis for the universal enveloping algebra of a Lie algebra L where the basis elements of L are in a preferred order. See a standard reference like [2, Theorem 9.4] for details on the PBW Theorem. The Baker-Campbell-Hausdorff theorem gives straightening identities in the universal enveloping algebras of general Lie algebras. In Theorem 3.3, we state and prove the nontrivial straightening identity in the universal enveloping algebra of the Heisenberg algebra (defined in [8, Proposition 3.26]) which is a special case of the Baker-Campbell-Hausdorff Theorem. Our proof uses double induction which is a new and more accessible proof technique. As an application, in Corollary 4.8, we formulate some straightening identities in the universal enveloping algebra of the Onsager algebra of $\mathfrak{sl}_4$ defined in [11, Example 3.9]. To prove these identities we make use of the Lie algebra homomorphisms defined in Proposition 4.7.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{Z}$ be the set of integers, and $\mathbb{N}$ be the set of positive integers.

Recall the definition of a Lie algebra over $\mathbb{C}$.

2.1 Lie Algebras

Definition 2.1 *A vector space, L , over the complex numbers, with a bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ (called the bracket), is called a Lie Algebra if the following axioms are satisfied for all x, y, z in L .*

$$\begin{aligned} [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned}$$

The second axiom is called the Jacobi Identity. For further reading on Lie Algebras, see the standard text [2].

2.2 Universal Enveloping Algebra

Definition 2.2 *For any given complex Lie algebra L , the Universal Enveloping Algebra, $U(L)$, is the unique unital associative algebra over $\mathbb{C}$ such that there is a linear function $\tau : L \rightarrow U(L)$ such that, for all $x, y \in L$,*

$$\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x) \tag{1}$$



and the following holds: for any unital associative algebra, A , over $\mathbb{C}$ and any linear function $\sigma : L \rightarrow A$ satisfying (1), there exists a unique algebra homomorphism $\phi : U(L) \rightarrow A$ such that $\phi(1) = 1$ and $\phi \circ \tau = \sigma$.

3 Main Theorem

In this section we prove a special case of the Baker-Campbell-Hausdorff Formula for the Heisenberg algebra, we use double induction on r and s , which is a new proof technique.

Definition 3.1 Let L be a complex Lie algebra, u in $U(L)$, and k in $\mathbb{Z}$. Define the divided powers in $U(L)$ as follows: $u^{(k)} = 0$ if $k < 0$, $u^{(0)} = 1$, and

$$u^{(k)} = \frac{u^k}{k!}$$

if $k > 0$.

Definition 3.2 Define the Heisenberg algebra, H , to be the complex Lie algebra with basis a, b, c and Lie bracket given by $[a, b] = c$, $[a, c] = [b, c] = 0$.

The proof of the following theorem will use double induction. This technique is used to prove a property $P(r, s)$ depending on two natural numbers r and s . It involves first proving the base case $P(1, 1)$, then proving the implication $P(r, 1) \Rightarrow P(r+1, 1)$ for some $r \in \mathbb{N}$ and finally proving the implication $P(r, s) \Rightarrow P(r, s+1)$ for some $s \in \mathbb{N}$ where r is general in $\mathbb{N}$.

Theorem 3.3 Given $r, s \in \mathbb{N}$,

$$a^{(r)}b^{(s)} = \sum_{j=0}^{\min\{r,s\}} b^{(s-j)}c^{(j)}a^{(r-j)}$$

in the universal enveloping algebra of the Heisenberg algebra.

Proof. We will prove this by double induction on r and s .

Base case $r = s = 1$:

$$\begin{aligned} ab &= ba + [a, b] \\ &= ba + c. \end{aligned}$$

If $s = 1$, the formula becomes:

$$a^{(r)}b = ba^{(r)} + ca^{(r-1)}. \quad (2)$$

We will prove this equation by induction on r . The base case was done above. Now assume (2) for some $r \in \mathbb{N}$, then



$$\begin{aligned}
(r+1)a^{(r+1)}b &= aa^{(r)}b \\
&= a(ba^{(r)} + ca^{(r-1)}) \text{ by the induction hypothesis} \\
&= aba^r + aca^{(r-1)} \\
&= (ba + c)a^{(r)} + caa^{(r-1)} \\
&= baa^{(r)} + ca^{(r)} + rca^{(r)} \\
&= (r+1)ba^{(r+1)} + (r+1)ca^{(r)} \\
&= (r+1)(ba^{(r+1)} + ca^{(r)}) .
\end{aligned}$$

Induction Hypothesis: Assume for $r, s \in \mathbb{N}$ that r, s holds and show that $s+1, r$ holds. We will use double induction.

$$\begin{aligned}
(s+1)a^{(r)}b^{(s+1)} &= a^{(r)}bb^{(s)} \\
&= (ba^{(r)} + ca^{(r-1)})b^{(s)} \text{ by (2) } s = 1 \\
&= ba^{(r)}b^{(s)} + ca^{(r-1)}b^{(s)} \\
&= b \left(\sum_{j=0}^{\min\{r,s\}} b^{(s-j)}c^{(j)}a^{(r-j)} \right) + c \left(\sum_{j=0}^{\min\{r-1,s\}} b^{s-j}c^{(j)}a^{(r-j-1)} \right) \\
&\quad \text{by the induction hypothesis} \\
&= \sum_{j=0}^{\min\{r,s\}} bb^{(s-j)}c^{(j)}a^{(r-j)} + \sum_{j=0}^{\min\{r-1,s\}} b^{(s-j)}cc^{(j)}a^{(r-j-1)} \\
&= \sum_{j=0}^{\min\{r,s\}} (s+1-j)b^{(s-j+1)}c^{(j)}a^{(r-j)} \\
&\quad + \sum_{j=0}^{\min\{r-1,s\}} (j+1)b^{(s-j)}c^{(j+1)}a^{(r-j-1)} \\
&= \sum_{j=0}^{\min\{r,s\}} (s+1-j)b^{(s-j+1)}c^{(j)}a^{(r-j)} + \sum_{j=1}^{\min\{r,s+1\}} jb^{(s-j+1)}c^{(j)}a^{(r-j)} \\
&= (s+1)b^{(s+1)}a^{(r)} + \sum_{j=1}^{\min\{r,s+1\}} (s+1-j)b^{(s-j+1)}c^{(j)}a^{(r-j)} \\
&\quad + \sum_{j=1}^{\min\{r,s+1\}} jb^{(s-j+1)}c^{(j)}a^{(r-j)} \\
&= \sum_{j=0}^{\min\{r,s+1\}} (s+1)b^{(s-j+1)}c^{(j)}a^{(r-j)}
\end{aligned}$$

□



4 Application

In this section, we will apply our theorem to give some straightening identities in the Onsager algebra of $\mathfrak{sl}_4$. We will begin with some definitions.

Definition 4.1 Define $\mathfrak{sl}_4$ to be the vector space of all 4×4 matrices with complex entries and trace zero. $\mathfrak{sl}_4$ is a Lie algebra under the commutator bracket $[A, B] = AB - BA$.

Definition 4.2 Given $j, k \in \{1, 2, 3, 4\}$ define $e_{j,k}$ to be the 4×4 matrix with 1 in the j th row and k th column and 0 in every other entry.

Then the matrices $e_{j,k}$ and $e_{l,l} - e_{l+1,l+1}$ for distinct $j, k \in \{1, 2, 3, 4\}$ and $l \in \{1, 2, 3\}$ form the standard basis for $\mathfrak{sl}_4$.

Definition 4.3 Let σ be the Chevalley involution on $\mathfrak{sl}_4$. Then σ is a Lie algebra involution and acts on the standard basis for $\mathfrak{sl}_4$ in the following way:

$$\begin{aligned}\sigma(e_{j,k}) &= -e_{k,j}, \\ \sigma(e_{l,l} - e_{l+1,l+1}) &= -(e_{l,l} - e_{l+1,l+1}).\end{aligned}$$

Definition 4.4 Given a Lie algebra we can build a new Lie algebra out of it called the loop algebra as follows: if $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and $\mathbb{C}[t, t^{-1}]$ is the space of Laurent polynomials, then the loop algebra of $\mathfrak{g}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with Lie bracket given by linearly extending the bracket

$$[x \otimes t^k, y \otimes t^l] = [x, y] \otimes t^{k+l}.$$

where $x, y \in \mathfrak{g}$ and $k, l \in \mathbb{Z}$.

Define the action of σ on the Laurent polynomials by extending the following action $\sigma(t) = t^{-1}$ and $\sigma(t^{-1}) = t$.

The diagonal action of σ on the loop algebra of $\mathfrak{sl}_4$, $\mathfrak{sl}_4 \otimes \mathbb{C}[t, t^{-1}]$, is defined by

$$\sigma(g \otimes f) = \sigma(g) \otimes \sigma(f),$$

for all $g \in \mathfrak{sl}_4$ and all $f \in \mathbb{C}[t, t^{-1}]$. This action is an automorphism of the loop algebra and therefore the fixed points form a subalgebra.

Definition 4.5 Let $\mathcal{O}$ be the subalgebra of the loop algebra $\mathfrak{sl}_4 \otimes \mathbb{C}[t, t^{-1}]$ fixed by σ . We call $\mathcal{O}$ the Onsager algebra of $\mathfrak{sl}_4$.

We will work with the following basis for $\mathcal{O}$, which is built from the basis for $\mathfrak{sl}_4$ originally given in [4, Page 336]. As stated there, the fact that it is a basis for $\mathfrak{sl}_4$ follows from its linear independence and the finite dimension of the Lie algebra.



Definition 4.6 For $j \in \{1, 2\}$, $k \in \{1, 2, 3\}$, $l \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$ define $x_{j,l}^\pm, h_{j,l}, y_{k,m}^\pm, Y_{j,m}^\pm, z_m^\pm \in \mathcal{O}$ where $i = \sqrt{-1} \in \mathbb{C}$ as follows:

$$\begin{aligned}
x_{1,l}^+ &= \frac{1}{2} ((e_{1,2} - e_{2,1}) - (e_{3,4} - e_{4,3}) - i(e_{1,3} - e_{3,1}) - i(e_{2,4} - e_{4,2})) \otimes (t^l + t^{-l}) \\
x_{1,l}^- &= \frac{1}{2} (-(e_{1,2} - e_{2,1}) + (e_{3,4} - e_{4,3}) - i(e_{1,3} - e_{3,1}) - i(e_{2,4} - e_{4,2})) \otimes (t^l + t^{-l}) \\
x_{2,l}^+ &= \frac{1}{2} ((e_{1,2} - e_{2,1}) + (e_{3,4} - e_{4,3}) - (e_{1,3} - e_{3,1}) + i(e_{2,4} - e_{4,2})) \otimes (t^l + t^{-l}) \\
x_{2,l}^- &= \frac{1}{2} (-(e_{1,2} - e_{2,1}) - (e_{3,4} - e_{4,3}) + i(e_{1,3} - e_{3,1}) - i(e_{2,4} - e_{4,2})) \otimes (t^l + t^{-l}) \\
h_{1,l} &= i((e_{2,3} - e_{3,2}) - (e_{1,4} - e_{4,1})) \otimes (t^l + t^{-l}) \\
h_{2,l} &= -i((e_{2,3} - e_{3,2}) + (e_{1,4} - e_{4,1})) \otimes (t^l + t^{-l}) \\
y_{1,m}^+ &= \frac{1}{2} ((e_{1,2} + e_{2,1}) + (e_{3,4} + e_{4,3}) - i(e_{1,3} + e_{3,1}) + i(e_{2,4} + e_{4,2})) \otimes (t^m - t^{-m}) \\
y_{1,m}^- &= \frac{1}{2} ((e_{1,2} + e_{2,1}) + (e_{3,4} + e_{4,3}) + i(e_{1,3} + e_{3,1}) - i(e_{2,4} + e_{4,2})) \otimes (t^m - t^{-m}) \\
y_{2,m}^+ &= \frac{1}{2} ((e_{1,2} + e_{2,1}) - (e_{3,4} + e_{4,3}) + i(e_{1,3} + e_{3,1}) + i(e_{2,4} + e_{4,2})) \otimes (t^m - t^{-m}) \\
y_{2,m}^- &= \frac{1}{2} ((e_{1,2} + e_{2,1}) - (e_{3,4} + e_{4,3}) - i(e_{1,3} + e_{3,1}) - i(e_{2,4} + e_{4,2})) \otimes (t^m - t^{-m}) \\
y_{3,m}^+ &= \frac{1}{2} ((e_{1,2} + e_{2,1}) + (e_{3,4} + e_{4,3}) - i(e_{1,3} + e_{3,1}) + i(e_{2,4} + e_{4,2})) \otimes (t^m - t^{-m}) \\
y_{3,m}^- &= \frac{1}{2} (i(e_{1,4} + e_{4,1}) + (e_{1,1} - e_{2,2}) + (e_{2,2} - e_{3,3}) + (e_{3,3} + e_{4,4})) \otimes (t^m - t^{-m}) \\
y_{3,m}^- &= \frac{1}{2} (-i(e_{1,4} + e_{4,1}) + (e_{1,1} - e_{2,2}) + (e_{2,2} - e_{3,3}) + (e_{3,3} - e_{4,4})) \otimes (t^m - t^{-m}) \\
Y_{1,m}^+ &= (i(e_{2,3} + e_{3,2}) - (e_{2,2} - e_{3,3})) \otimes (t^m - t^{-m}) \\
Y_{2,m}^+ &= (i(e_{2,3} + e_{3,2}) + (e_{2,2} - e_{3,3})) \otimes (t^m - t^{-m}) \\
z_m^+ &= ((e_{1,1} - e_{2,2}) - (e_{3,3} - e_{4,4})) \otimes (t^m - t^{-m})
\end{aligned}$$

4.1 Some Straightening Identities in the Onsager Algebra of $\mathfrak{sl}_4$.

We wish to use Theorem 3.3 to prove some straightening identities in the Onsager algebra of $\mathfrak{sl}_4$. To do this, we make use of the following Lie algebra homomorphisms.

Proposition 4.7 Given $1 \leq j \leq 8$, the linear functions $\psi_j : H \rightarrow \mathcal{O}$ defined by the following equations are Lie algebra homomorphisms.

$$\begin{aligned}
\psi_1(a) &= x_{1,k}^+, \quad \psi_1(b) = y_{2,l}^+, \quad \psi_1(c) = y_{3,k+l}^+ - y_{3,k-l}^+ \\
\psi_2(a) &= x_{2,k}^+, \quad \psi_2(b) = y_{1,l}^+, \quad \psi_2(c) = y_{3,k+l}^+ - y_{3,k-l}^+ \\
\psi_3(a) &= x_{1,k}^-, \quad \psi_3(b) = y_{2,l}^-, \quad \psi_3(c) = y_{3,k-l}^- - y_{3,k+l}^-
\end{aligned}$$



$$\begin{aligned}
\psi_4(a) &= x_{2,k}^-, & \psi_4(b) &= y_{1,l}^-, & \psi_4(c) &= y_{3,k-l}^- - y_{3,k+l}^- \\
\psi_5(a) &= x_{1,k}^+, & \psi_5(b) &= y_{2,l}^-, & \psi_5(c) &= Y_{1,k+l}^+ - Y_{1,k-l}^+ \\
\psi_6(a) &= x_{2,k}^+, & \psi_6(b) &= y_{1,l}^-, & \psi_6(c) &= Y_{2,k-l}^+ - Y_{2,k+l}^+ \\
\psi_7(a) &= y_{2,l}^+, & \psi_7(b) &= x_{1,k}^-, & \psi_7(c) &= Y_{2,k-l}^+ - Y_{2,k+l}^+ \\
\psi_8(a) &= y_{1,l}^+, & \psi_8(b) &= x_{2,k}^-, & \psi_8(c) &= Y_{1,k+l}^+ - Y_{1,k-l}^+
\end{aligned}$$

Proof. The following bracket calculations are sufficient to prove that these linear functions are Lie algebra homomorphisms:

$$\begin{aligned}
[\psi_1(a), \psi_1(b)] &= [x_{1,k}^+, y_{2,l}^+] = y_{3,k+l}^+ - y_{3,k-l}^+ = \psi_1(c) = \psi_1([a, b]) \\
[\psi_2(a), \psi_2(b)] &= [x_{2,k}^+, y_{1,l}^+] = y_{3,k+l}^+ - y_{3,k-l}^+ = \psi_2(c) = \psi_2([a, b]) \\
[\psi_3(a), \psi_3(b)] &= [x_{1,k}^-, y_{2,l}^-] = y_{3,k-l}^- - y_{3,k+l}^- = \psi_3(c) = \psi_3([a, b]) \\
[\psi_4(a), \psi_4(b)] &= [x_{2,k}^-, y_{1,l}^-] = y_{3,k-l}^- - y_{3,k+l}^- = \psi_4(c) = \psi_4([a, b]) \\
[\psi_5(a), \psi_5(b)] &= [x_{1,k}^+, y_{2,l}^-] = Y_{1,k+l}^+ - Y_{1,k-l}^+ = \psi_5(c) = \psi_5([a, b]) \\
[\psi_6(a), \psi_6(b)] &= [x_{2,k}^+, y_{1,l}^-] = Y_{2,k-l}^+ - Y_{2,k+l}^+ = \psi_6(c) = \psi_6([a, b]) \\
[\psi_7(a), \psi_7(b)] &= [y_{2,l}^+, x_{1,k}^-] = Y_{2,k-l}^+ - Y_{2,k+l}^+ = \psi_7(c) = \psi_7([a, b]) \\
[\psi_8(a), \psi_8(b)] &= [y_{1,l}^+, x_{2,k}^-] = Y_{1,k+l}^+ - Y_{1,k-l}^+ = \psi_8(c) = \psi_8([a, b])
\end{aligned}$$

□

Applying each of the homomorphisms $\psi_1, \psi_2, \dots, \psi_8$ to Theorem 3.3 gives us the following straightening identities in the universal enveloping algebra of the Onsager algebra of $\mathfrak{sl}_4$.

Corollary 4.8 *Let $k \in \mathbb{N} \cup \{0\}$ and $l, r, s \in \mathbb{N}$. Then in $U(\mathcal{O})$:*

$$\begin{aligned}
(x_{1,k}^+)^{(r)} (y_{2,l}^+)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{2,l}^-)^{(s-j)} (y_{3,k+l}^+ - y_{3,k-l}^+)^{(j)} (x_{1,k}^+)^{(r-j)} \\
(x_{2,k}^+)^{(r)} (y_{1,l}^+)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{1,l}^+)^{(s-j)} (y_{3,k+l}^+ - y_{3,k-l}^+)^{(j)} (x_{2,k}^+)^{(r-j)} \\
(x_{1,k}^-)^{(r)} (y_{2,l}^-)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{2,l}^-)^{(s-j)} (y_{3,k-l}^- - y_{3,k+l}^-)^{(j)} (x_{1,k}^-)^{(r-j)} \\
(x_{2,k}^-)^{(r)} (y_{1,l}^-)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{1,l}^-)^{(s-j)} (y_{3,k-l}^- - y_{3,k+l}^-)^{(j)} (x_{2,k}^-)^{(r-j)}
\end{aligned}$$



$$\begin{aligned}
(x_{1,k}^+)^{(r)} (y_{2,l}^-)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{2,l}^-)^{(s-j)} (Y_{1,k+l}^+ - Y_{1,k-l}^+)^{(j)} (x_{1,k}^+)^{(r-j)} \\
(x_{2,k}^+)^{(r)} (y_{1,l}^-)^{(s)} &= \sum_{j=0}^{\min\{r,s\}} (y_{1,l}^-)^{(s-j)} (Y_{2,k-l}^+ - Y_{2,k+l}^+)^{(j)} (x_{2,k}^+)^{(r-j)} \\
(y_{2,l}^+)^{(s)} (x_{1,k}^-)^{(r)} &= \sum_{j=0}^{\min\{r,s\}} (x_{1,k}^-)^{(r-j)} (Y_{2,k-l}^+ - Y_{2,k+l}^+)^{(j)} (y_{2,l}^+)^{(s-j)} \\
(y_{1,l}^+)^{(s)} (x_{2,k}^-)^{(r)} &= \sum_{j=0}^{\min\{r,s\}} (x_{2,k}^-)^{(r-j)} (Y_{1,k+l}^+ - Y_{1,k-l}^+)^{(j)} (y_{1,l}^+)^{(s-j)}
\end{aligned}$$

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