Metrics on Permutations With the Same Descent Set

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Abstract - Let S_n be the symmetric group on the set $[n] := \{1, 2, \dots, n\}$. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, we say it has a descent at index i if $\sigma_i > \sigma_{i+1}$. Let $\mathcal{D}(\sigma)$ be the set of all descents of σ and define $\mathcal{D}(S; n) = \{\sigma \in S_n \,|\, \mathcal{D}(\sigma) = S\}$. We study the Hamming metric and ℓ_∞-metric on the sets $\mathcal{D}(S; n)$ for all possible nonempty $S \subset [n-1]$ to determine the maximum possible value that these metrics can achieve when restricted to these subsets.

Keywords : permutations; descents; Hamming metric; L-infinity metric

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1 Introduction

Modeling rankings using permutations is a classic application in statistics, and assessing the distance between rankings by studying the distance between pairs of permutations has been considered in this context since at least the 1950s [3, 11, 13]. More recently, permutations have been used to model data encoding structures, specifically error-correcting codes, including for flash memory storage [1, 8, 12]. In these data representation models, the distance between pairs of permutations is an indicator of the error correction capabilities of the code. Metrics that have been used in this context include the Hamming metric, the $\ell_\infty-$ (Chebyshev) metric, the Ulam metric, and the Kendall-tau metric (see, e.g., [6, 9, 16]). Motivated by a paper that studied metrics on sets of permutations with a given peak set [5], in this article we find the maximum values that certain permutation metrics can attain when restricted to subsets of permutations that share a given descent set.

Let $[n]$ denote the set $\{1, 2, \dots, n\}$ and S_n be the set of $n!$ symmetries of $[n]$. We write the elements of S_n in one-line notation, that is, for $\sigma \in S_n$ we write $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ to denote the permutation that sends $1 \to \sigma_1$, $2 \to \sigma_2, \dots, n \to \sigma_n$. We say σ has a **descent** at position i in $[n-1]$ if $\sigma_i > \sigma_{i+1}$. We define the **descent set** of σ, $\mathcal{D}(\sigma)$, as the set of all the indices at which σ has a descent. For example, if $\sigma = 58327164 \in S_8$ then $\mathcal{D}(\sigma) = \{2, 3, 5, 7\}$.

Given a subset $S \subseteq [n-1]$, let $\mathcal{D}(S; n)$ be the set of permutations in S_n with descent set S, that is,

$$\mathcal{D}(S; n) = \{\sigma \in S_n \,|\, \mathcal{D}(\sigma) = S\}.$$

We can partition S_n as a disjoint union of sets of the form $\mathcal{D}(S;n)$ as we range through all possible subsets S of $[n-1]$. The sets $\mathcal{D}(S;n)$ were first studied by MacMahon [14]. More recently, Diaz-Lopez et al. [4] provided some combinatorial results about them and presented some conjectures. This led to a flurry of work related to descent sets [2, 7, 10, 15].

In this article, we study the Hamming metric, which measures the number of indices at which two permutations differ, and the ℓ_∞-metric, which measures the maximum component-wise difference between two permutations. The main purpose of this article is to find the maximum Hamming and ℓ_∞-metric when restricting to permutations that share a descent set—subsets of the form $\mathcal{D}(S;n) \subset S_n$ for $S \subset [n-1]$. Theorem 3.2 provides a complete characterization of the maximum Hamming metric on all sets $\mathcal{D}(S;n)$ and Theorems 4.1 and 4.3 provide the maximum ℓ_∞-metric on $\mathcal{D}(S;n)$ for particular cases of S. Finding the maximum ℓ_∞-metric on most sets $\mathcal{D}(S;n)$ is still an open problem.

2 Preliminary Definitions

In this section, we define the main objects of study of this article. Given a set S, a **metric** d on S is a map $d : S \times S \to [0, \infty)$ such that for σ, ρ, τ in S we have

1. $d(\sigma, \rho) = 0$ if and only if $\sigma = \rho$,
2. $d(\sigma, \rho) = d(\rho, \sigma)$, and
3. $d(\sigma, \tau) \leq d(\sigma, \rho) + d(\rho, \tau)$.

In this article, we focus our attention on sets S consisting of permutations. We first define some metrics on S_n and then restrict them to subsets of S_n.

Definition 2.1 *The Hamming metric, denoted d_H, is the map $d_H : S_n \times S_n \to [0, \infty)$ such that $d_H(\sigma, \rho)$ is the number of indices where σ and ρ differ. That is, if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ and $\rho = \rho_1\rho_2 \ldots \rho_n$ then*

$$d_H(\sigma, \rho) = |\{i \mid \sigma_i \neq \rho_i\}|.$$

*Let d_ℓ, denoting the ℓ_∞-**metric**, be the map $d_\ell : S_n \times S_n \to [0, \infty)$ such that*

$$d_\ell(\sigma, \rho) = \max\{|\sigma_i - \rho_i| \mid 1 \leq i \leq n\}.$$

Example 2.2 Consider the permutations $\sigma = 14756832$ and $\rho = 13624875$ in S_8. They differ in indices $2, 3, 4, 5, 7, 8$, hence $d_H(\sigma, \rho) = 6$. To compute $d_\ell(\sigma, \rho)$ we analyze their component-wise differences to get

$$d_\ell(\sigma, \rho) = \max\{|1-1|, |3-4|, |6-7|, |2-5|, |4-6|, |8-8|, |7-3|, |5-2|\} = 4.$$

We now define descent sets.

Definition 2.3 *Given a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ in S_n, its descent set is defined as*

$$\mathcal{D}(\sigma) = \{i \in [n-1] \,|\, \sigma_i > \sigma_{i+1}\}.$$

Given a set $S \subseteq [n-1]$, we let $\mathcal{D}(S;n)$ be the set of permutations in S_n with descent set S, that is,

$$\mathcal{D}(S;n) = \{\sigma \in S_n \,|\, \mathcal{D}(\sigma) = S\}.$$

In this article, we focus on analyzing the Hamming metric and ℓ_∞-metric on subsets of the form $\mathcal{D}(S;n)$ for $S \subset [n-1]$. In particular, we study the maximum of the set

$$d(\mathcal{D}(S;n)) := \{d(\sigma,\rho) \,|\, \sigma,\rho \in \mathcal{D}(S;n) \text{ with } \sigma \neq \rho\},$$

where d is the Hamming metric or the ℓ_∞-metric. We take S to be proper and nonempty as $\mathcal{D}(\varnothing;n) = \{1\,2\,\cdots\,n\}$ and $\mathcal{D}([n-1];n) = \{n\,(n-1)\,\cdots\,1\}$, hence the sets have cardinality one and we cannot compute distances between distinct permutations.

Remark 2.4 A related question is to study the minimum of the set $d(\mathcal{D}(S;n))$, when d is the Hamming metric or the ℓ_∞-metric. Let $S \subset [n-1]$ be nonempty and proper. The minimum of the set $d(\mathcal{D}(S;n))$ is always 2 for the Hamming metric and 1 for the ℓ_∞ metric, since once you have a permutation σ that is not $1\,2\,\cdots\,n$ nor $n\,(n-1)\,\cdots\,1$, there will always be two consecutive numbers i and $i+1$ that do not appear consecutively in σ. Swapping i and $i+1$ leads to a different permutation σ' with the same descent set as σ and with $d_H(\sigma,\sigma') = 2$ and $d_\ell(\sigma,\sigma') = 1$.

3 Hamming Metric on Permutations With a Given Descent Set

Consider the Hamming metric d_H, as presented in Definition 2.1. Given a set $S \subset [n-1]$, we consider the set of permutations $\mathcal{D}(S;n)$. In this section, we find the maximum Hamming distance between pairs of distinct permutations in $\mathcal{D}(S;n)$ for all sets S. In Theorem 3.2 we show that for sets S consisting of consecutive descents starting at 1 or consecutive descents ending at $n-1$ we achieve maximum Hamming distance $n-1$, and for every other set S we achieve maximum Hamming distance n.

For example, consider S_4 and let $S \subset [3]$. Table 1 presents the maximum distance of the permutations in $\mathcal{D}(S;4)$. In this case, most of the subsets S consist of consecutive descents at the start or end of the permutation. As n increases, the proportion of sets that meet this criterion goes to 0 as the number of such sets grows linearly in n and the total number of subsets of $[n-1]$ grows exponentially in n. We now present a lemma that is useful in the proof of Theorem 3.2.

Lemma 3.1 *If $S = \{1,2,\dots,k\}$ for some $k \in [n-1]$, then any permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{D}(S;n)$ must have $\sigma_{k+1} = 1$. Similarly, if $S = \{k,k+1,\dots,n-1\}$ for some $k \in [n-1]$, then $\sigma_k = n$.*

Descent Set S	$\mathcal{D}(S;4) \subset S_4$	$d_H(\mathcal{D}(S;4))$
$\varnothing$	{1234}	—
{1}	{**2**134, **3**124, **4**123}	3
{2}	{1**3**24, 1**4**23, 2**3**14, 2**4**13, 3**4**12}	4
{3}	{12**4**3, 13**4**2, 23**4**1}	3
{1,2}	{**32**14, **42**13, **43**12}	3
{1,3}	{**2**1**4**3, **3**2**4**1, **3**1**4**2, **4**1**3**2, **4**2**3**1}	4
{2,3}	{1**43**2, 2**43**1, 3**42**1}	3
{1,2,3}	{**432**1}	—

Table 1: Table of descent sets and Hamming distances for S_4. The entries of the permutations where a descent occurs are shown in bold.

Case	$S \subset [n-1]$	$d_H(\mathcal{D}(S;n))$
1	$\{1, 2, ..., k\}$ with $k \leq (n-2)$	$n-1$
2	$\{k, (k+1), ..., n-2, n-1\}$ with $k \geq 2$	$n-1$
3	$\{1, 2, ..., k, n-1\}$ with $k \leq (n-3)$	n
4	$\{k, (k+1), ..., (n-2)\}$ with $k \geq 2$	n
5	A subset of $[n-2]$ except Cases 1 and 4	n
6	$S' \cup \{n-1\}$ where S' is as in Case 5	n

Table 2: Cases to consider in the induction step in the proof of Theorem 3.2.

Proof. In the case where $S = \{1, 2, \ldots, k\}$ for some $k \in [n-1]$, if $\sigma_i = 1$ for $i \leq k$ then i is not a descent as $\sigma_{i+1} > \sigma_i = 1$, hence this is not possible. If $\sigma_i = 1$ for $i > k+1$ then $i-1$ would be a descent as $\sigma_{i-1} > \sigma_i = 1$. This contradicts that $\sigma \in \mathcal{D}(S;n)$. Thus, we must have $\sigma_{k+1} = 1$.

In the case where $S = \{k, k+1, \ldots, n-1\}$, if $\sigma_i = n$ for $i < k$ then i is a descent as $n = \sigma_i > \sigma_{i+1}$, which contradicts that $\sigma \in \mathcal{D}(S;n)$. If $\sigma_i = n$ for $i > k$, then $i-1$ is not a descent as $\sigma_{i-1} < \sigma_i = n$, which again contradicts $\sigma \in \mathcal{D}(S;n)$. Hence, $\sigma_k = n$. $\square$

We are ready to prove our main theorem of this section, which consists of a proof by induction that breaks the set of subsets of $[n-1]$ into six cases.

Theorem 3.2 *Let $n \geq 3$ and $S \subset [n-1]$ be a non-empty proper set, then*

$$\max(d_H(\mathcal{D}(S;n))) = \begin{cases} n-1 & \text{if } S = \{i, i+1, \ldots, j\} \text{ for } i = 1 \text{ or } j = n-1 \\ n & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on n. For $n = 3$, when $S = \{1\}$ we get $\mathcal{D}(S;3) = \{312, 213\}$, thus $d(\mathcal{D}(\{1\};3)) = 2$. When $S = \{2\}$, we have $\mathcal{D}(\{2\};3) = \{132, 231\}$, so $d(\mathcal{D}(\{2\};3)) = 2$. The result is shown to be true for $n = 4$ in Table 1.

Suppose that the result is true for all descent sets $S' \subset [n-2]$ of permutations in S_{n-1} for $n \geq 4$. Let $S \subset [n-1]$ and consider $\mathcal{D}(S;n)$. We proceed by cases depending on the set S. Table 2 summarizes the cases.

Case 1: If $S = \{1, 2, \ldots, k\}$ for $1 \leq k \leq n-2$, let

$$\begin{array}{rcccccccccc} \sigma & = & n & n-1 & n-2 & \cdots & n-(k-2) & n-(k-1) & 1 & 2 & \cdots & n-k \\ \rho & = & k+1 & k & k-1 & \cdots & 3 & 2 & 1 & k+2 & \cdots & n. \end{array}$$

Then, $d_H(\sigma, \rho) = n-1$. By Lemma 3.1, this is the largest value that the Hamming metric can obtain in this set, as any permutation with descent set S will have the value of 1 at index $k+1$. Thus, $d_H(\mathcal{D}(S; n)) = n-1$.

Case 2: If $S = \{k, k+1, \ldots, n-2, n-1\}$ for $2 \leq k \leq n-1$, let

$$\begin{array}{rcccccccccc} \sigma & = & 1 & 2 & \cdots & k-2 & k-1 & n & n-1 & n-2 & \cdots & k \\ \rho & = & n-(k-1) & n-(k-2) & \cdots & n-2 & n-1 & n & n-k & n-(k+1) & \cdots & 1. \end{array}$$

Then, $d_H(\sigma, \rho) = n-1$. By Lemma 3.1, this is the largest value that the Hamming metric can obtain in this set, as any permutation with descent set S will have the value of n at index k. Thus, $d_H(\mathcal{D}(S; n)) = n-1$.

Case 3: If $S = \{1, 2, \ldots, k, n-1\}$ for $1 \leq k \leq n-3$, let

$$\begin{array}{rcccccccccccc} \sigma & = & n & n-1 & \cdots & n-(k-2) & n-(k-1) & 1 & 2 & \cdots & n-(k+2) & n-k & n-(k+1) \\ \rho & = & n-1 & n-2 & \cdots & n-(k-1) & n-k & 2 & 3 & \cdots & n-(k+1) & n & 1 \end{array}$$

Then, since σ and ρ have descent set S and $d_H(\sigma, \rho) = n$, we get $d_H(\mathcal{D}(S; n)) = n$.

Case 4: If $S = \{k, k+1, \ldots, n-2\}$ for $2 \leq k \leq n-2$, let

$$\begin{array}{rcccccccccc} \sigma & = & 1 & 2 & \cdots & k-1 & n-1 & n-2 & n-3 & \cdots & k & n \\ \rho & = & n-k & n-(k-1) & \cdots & n-2 & n & n-(k+1) & n-(k+2) & \cdots & 1 & n-1. \end{array}$$

Then, since σ and ρ have descent set S and $d_H(\sigma, \rho) = n$, we get $d_H(\mathcal{D}(S; n)) = n$.

Case 5: Let S be any nonempty, proper subset of $[n-2]$ except those in Cases 1 and 4. By the inductive assumption, $d_H(\mathcal{D}(S; n-1)) = n-1$. Hence, there is a pair of permutations $\sigma, \rho \in S_{n-1}$ with descent set S such that $d_H(\sigma, \rho) = n-1$. Since they differ at every index, assume without loss of generality that $\rho_{n-1} \neq n-1$.

Let σ' be the permutation σ with n appended at the end. Let ρ'' be the permutation ρ with n appended at the end, and let ρ' be the permutation ρ'' with the values of n and $n-1$ switched. Since $n-1$ and n do not appear consecutively in ρ'' then ρ' has descent set S. By construction, σ' and ρ' have descent set S and differ at every index; hence $d_H(\mathcal{D}(S; n)) = n$.

Case 6: Let S be any nonempty, proper subset of $[n-1]$ with $n-1 \in S$, except those in Cases 2 and 3. Then $S = S' \cup \{n-1\}$ for some subset $S' \subset [n-2]$, where S' is not of the form of the sets in Cases 1 and 4. There are two subcases.

First, suppose $n-2 \in S'$. Since S' is not one of the sets in Case 4, there must be a minimum $k \in \{3, \ldots, n-2\}$ such that $\{k, k+1, \ldots, n-2\} \subseteq S'$, and $k-1 \notin S'$. By the inductive assumption, there are permutations $\sigma, \rho \in \mathcal{D}(S'; n-1)$, with $d(\sigma, \rho) = n-1$. Since all positions of σ and ρ are distinct, at most one of σ and ρ has their k^{th} index equal to $n-1$. Without loss of generality, suppose that $\rho_k < n-1$. Since $\{k, k+1, \ldots, n-2\} \subseteq S'$, ρ is decreasing from ρ_k to ρ_{n-1}. Thus, $\rho_j = n-1$ for some $1 \leq j < k-1$. We form

$\sigma', \rho' \in S_n$ as follows. Let $\sigma'_i = \sigma_i$ for $i = 1, \ldots, k-1$. We set $\sigma'_k = n$, and $\sigma'_{k+i} = \sigma_{k+i-1}$ for $i = 1, \ldots, n-k$. Let $\rho'_i = \rho_i$ for $i = [k-1] \setminus \{j\}$. Set $\rho'_j = n$, and $\rho'_k = n-1$. Finally, let $\rho'_{k+i} = \rho_{k+i-1}$ for $i = 1, \ldots, n-k$. By construction and by the inductive assumption, we have $\sigma', \rho' \in \mathcal{D}(S; n)$ and $d(\sigma', \rho') = n$.

Next, suppose $n-2 \notin S'$. By the inductive assumption, there are permutations $\sigma, \rho \in \mathcal{D}(S'; n-1)$, with $d(\sigma, \rho) = n-1$. Since all positions of σ and ρ are distinct, at most one of σ and ρ have their $(n-1)^{th}$ position equal to $n-1$. Without loss of generality, suppose that $\rho_{n-1} \neq n-1$. Since $n-2 \notin S'$, it must be the case that $\rho_j = n-1$ for some $j < n-2$. Form $\sigma', \rho' \in S_n$ as follows. Set $\sigma'_i = \sigma_i$ for $i = 1, \ldots, n-2$, $\sigma'_{n-1} = n$, and $\sigma'_n = \sigma_{n-1}$. Similarly, set $\rho'_i = \rho_i$ for $i \in [n-2] \setminus \{j\}$. Set $\rho'_j = n$, and $\rho'_{n-1} = n-1$, and $\rho'_n = \rho_{n-1}$. By construction and by the inductive assumption, we have $\sigma', \rho' \in \mathcal{D}(S; n)$ and $d(\sigma', \rho') = n$.

Thus, by the two subcases above, $d_H(\mathcal{D}(S; n)) = n$ in this last case. □

4 ℓ_∞-Metric on Permutations With a Given Descent Set

We now shift our attention to the ℓ_∞-metric and consider the maximum distance that two permutations in $\mathcal{D}(S; n)$ can achieve under the ℓ_∞-metric. In Theorem 4.1 we consider the case of $S = [n-i]$ for some $i \in \{2, 3, \ldots, n-1\}$. In Theorem 4.3 we consider the case $S = \{i\}$. A result from Section 5 shows that analogous results hold for the complements of these sets. It is an open problem to find the maximum distance that two permutations in $\mathcal{D}(S; n)$ can achieve under the ℓ_∞-metric for all other descent sets S.

Theorem 4.1 *Fix $n \in \mathbb{N}$ with $n \geq 3$. For any $i \in \{2, \ldots, n-1\}$, let $S_i = [n-i]$, then*

$$\max(d_\ell(\mathcal{D}(S_i; n))) = \max\{i-1, n-i\} = \begin{cases} i-1 & \text{if } i \geq \lfloor \frac{n+1}{2} \rfloor \\ n-i & \text{if } i \leq \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

Proof. We first construct two permutations in $\mathcal{D}(S_i; n)$ that achieve the desired maximum distance and then show that no other pair of permutations can have a larger ℓ_∞-distance. Let σ and ρ be defined as

index		1	2	$\cdots$	$n-i$	$n-(i-1)$	$n-(i-2)$	$\cdots$	n
σ	$=$	$n-(i-1)$	$n-i$	$\cdots$	2	1	$n-(i-2)$	$\cdots$	n
ρ	$=$	n	$n-1$	$\cdots$	$i+1$	1	2	$\cdots$	$i.$

It is straightforward to verify that both $\sigma, \rho \in \mathcal{D}(S_i; n)$. To compute their ℓ_∞-distance, note that

$$|\sigma_j - \rho_j| = \begin{cases} i-1 & \text{if } j \in \{1, 2, \ldots, n-i\} \\ 0 & \text{if } j = n-(i-1) \\ n-i & \text{if } j \in \{n-(i-2), n-(i-3), \ldots, n\}. \end{cases}$$

Thus, $d_\ell(\sigma, \rho) = \max\{i-1, n-i\}$.

For any other pair of permutations $\pi, \tau \in \mathcal{D}(S_i; n)$, we will show $d_\ell(\pi, \tau) \leq \max\{i - 1, n - i\}$. Since π and τ must decrease in the first $n - i$ indices, the largest numbers that can appear in the first $n - i$ indices are $n, n-1, \ldots, i+1$, respectively. That is,

$$\pi_j, \tau_j \leq n - (j-1) = \rho_j \text{ for } j = 1, 2, \ldots, n-i.$$

Similarly, the smallest entries that can appear in the first $n-i$ indices are $n-(i-1), n-i, \ldots, 2$, respectively. Thus,

$$\pi_j, \tau_j \geq n - (i-1) - (j-1) = \sigma_j \text{ for } j = 1, 2, \ldots, n-i.$$

Since

$$n - (i-1) - (j-1) \leq \pi_j, \tau_j \leq n - (j-1) \text{ for } j = 1, 2, \ldots, n-i,$$

then

$$|\pi_j - \tau_j| \leq n - (j-1) - (n - (i-1) - (j-1)) = i - 1,$$

for $j = 1, 2, \ldots, n-i$.

We now consider the last i indices. By Lemma 3.1, $\pi_{n-(i-1)} = \tau_{n-(i-1)} = 1$. Since π and τ must increase in the last i indices, the largest numbers that can appear in the last $i-1$ indices are $n-(i-2), n-(i-3), \ldots, n$, respectively. That is,

$$\pi_j, \tau_j \leq j \text{ for } j = n-(i-2), n-(i-3), \ldots, n.$$

Similarly, the smallest entries that can appear in the last $i-1$ indices are $2, 3, \ldots, i$, respectively. Thus,

$$\pi_j, \tau_j \geq j - (n-i) \text{ for } j = n-(i-2), n-(i-3), \ldots, n.$$

Since

$$j \leq \pi_j, \tau_j \leq j - (n-i) \text{ for } j = n-(i-2), n-(i-3), \ldots, n,$$

then

$$|\pi_j - \tau_j| \leq j - (j - (n-i)) = n - i,$$

for $j = n-(i-2), n-(i-3), \ldots, n$. Therefore, for any $\pi, \tau \in \mathcal{D}(S_i; n)$, we have $d_\ell(\pi, \tau) \leq \max\{i-1, n-i\} = d_\ell(\sigma, \tau)$. □

We now consider permutations with only one descent. We start with an example.

Example 4.2 For $i \in [n-1]$, to achieve the maximum ℓ_∞-metric between two permutations with the same descent set $S = \{i\}$, we construct a permutation σ with the smallest possible entries in the first i indices and a permutation ρ with the largest possible entries in these indices. This pair of permutations always achieves the maximum possible distance ℓ_∞ between pairs of permutations in $\mathcal{D}(S; n)$, as shown in Theorem 4.3. In Tables 3 and 4 we present these permutations for all descent sets $\{i\}$ in S_n for $n = 3, 4, 5$.

S	σ	ρ	$d_\ell(\sigma,\rho)$
$\{1\}$	**2**13	**3**12	1
$\{2\}$	1**3**2	2**3**1	1

S	σ	ρ	$d_\ell(\sigma,\rho)$
$\{1\}$	**2**134	**4**123	2
$\{2\}$	1**3**24	3**4**12	2
$\{3\}$	12**4**3	23**4**1	2

Table 3: For $n = 3$ and $n = 4$, permutations σ and ρ that achieve the maximum ℓ_∞ metric.

S	σ	ρ	$d_\ell(\sigma,\rho)$
$\{1\}$	**2**1345	**5**1234	3
$\{2\}$	1**3**245	4**5**123	3
$\{3\}$	12**4**35	34**5**12	3
$\{4\}$	123**5**4	234**5**1	3

Table 4: For $n = 5$, permutations σ and ρ that achieve the maximum ℓ_∞ metric.

Theorem 4.3 *Let $n \geq 3$ and consider $\mathcal{D}(\{i\}; n)$, then*

$$\max(d_\ell(\mathcal{D}(\{i\};n))) = \begin{cases} n-2 & \text{for } i = 1, n-1 \\ n-i & \text{for } i = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor \\ i & \text{for } i = \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \dots, n-2. \end{cases}$$

Proof. We will proceed by constructing two permutations in S_n with descent set $\{i\}$ with distance given by the statement and then show that this is the maximum distance any two permutations in $\mathcal{D}(\{i\}; n)$ can achieve. Let σ, ρ be as follows:

$$\begin{array}{lcccccccccc} \sigma & = & 1 & 2 & \cdots & i-1 & i+1 & i & i+2 & i+3 & \cdots & n \\ \rho & = & n-i+1 & n-i+2 & \dots & n-1 & n & 1 & 2 & 3 & \dots & n-i. \end{array}$$

Notice that the difference between the indices of the permutations is given by

$$|\sigma_j - \rho_j| = \begin{cases} n-i & \text{if } j = 1, 2, \dots, i-1 \text{ and } i \geq 2 \\ n-(i+1) & \text{if } j = i \\ i-1 & \text{if } j = i+1 \\ i & \text{if } j = i+2, i+2, \dots, n \text{ and } i \leq n-2. \end{cases}$$

A quick check verifies that $d_\ell(\sigma, \rho)$ is given by

$$d_\ell(\sigma,\rho) = \max\{|\sigma_j - \rho_j| \mid 1 \leq j \leq n\} = \begin{cases} n-2 & \text{for } i = 1, n-1 \\ n-i & \text{for } i = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor \\ i & \text{for } i = \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \dots, n-2. \end{cases}$$

We now proceed to show that the maximum distance we can achieve among any pair of distinct permutations in $\mathcal{D}(\{i\}; n)$ is given by $d_\ell(\sigma, \rho)$.

Consider any permutations $\pi, \tau \in \mathcal{D}(\{i\}; n)$. Since they are strictly increasing until index i, then the smallest π_1 and τ_1 can be is 1, and the largest they can be is $n-(i-1)$, as you need to have $i-1$ larger entries for indices $2, 3, \ldots, i$. Thus, $1 \leq \pi_1, \tau_1 \leq n-(i-1)$. By applying a similar argument to entries $2, 3, \ldots, i-1$ we get that

$$j \leq \pi_j, \tau_j \leq n-(i-j) \text{ for } j = 1, 2, \ldots, i-1. \tag{1}$$

For index i, we have that $i+1 \leq \pi_i, \tau_i$ as the permutations have a descent at i, so they need to have indices $1, 2, \ldots, i-1, i+1$ all less than the value at index i. Hence,

$$i+1 \leq \pi_j, \tau_j \leq n \text{ for } j = i. \tag{2}$$

For index $i+1$, the smallest value π_{i+1}, τ_{i+1} can attain is 1 and the largest value they can achieve is i since there are $n-i$ indices that must have larger entries (indices $i, i+2, i+3, \ldots, n$). Hence,

$$1 \leq \pi_j, \tau_j \leq i \text{ for } j = i+1. \tag{3}$$

The final $n-i$ entries of π and τ form an increasing sequence. Thus, the smallest value that the indices $i+2, i+3, \ldots, n$ can attain are $2, 3, \ldots, n-i$, respectively. Similarly, the largest values they can attain are $i+2, i+3, \cdots, n$, respectively. Thus,

$$j-i \leq \pi_j, \tau_j \leq j \text{ for } j = i+2, i+3, \ldots, n. \tag{4}$$

Using equations (1) – (4), we get that for each $j \in \{1, 2, \ldots, n\}$,

$$|\pi_j - \tau_j| \leq |\sigma_j - \rho_j| = \begin{cases} n-i & \text{if } j = 1, 2, \ldots, i-1 \text{ and } i \geq 2 \\ n-(i+1) & \text{if } j = i \\ i-1 & \text{if } j = i+1 \\ i & \text{if } j = i+2, i+2, \ldots, n-1 \text{ and } i \leq n-2. \end{cases}.$$

Hence, $\max(d_\ell(\mathcal{D}(\{i\}; n))) = d_\ell(\sigma, \rho)$ as desired. □

5 Complements of Descent Sets

Given that the results in Section 4 cover only certain cases of descent sets S, we now present a more general result on complements of descent sets that broadens the types of descent sets for which the maximum ℓ_∞ distance is known. In this section, we use a bijection map of S_n to show that $\max(d_\ell(\mathcal{D}(S; n)))$ is equal to $\max(d_\ell(\mathcal{D}(\bar{S}; n)))$, where $\bar{S}$ is the complement of S in $[n-1]$, which leads to maximum values for the other types of sets S discussed in Corollaries 5.6 and 5.7.

Definition 5.1 *For a set $S \subseteq [n-1]$, define the* ***complement of*** *S in $[n-1]$ as $\bar{S} = [n-1] \setminus S$.*

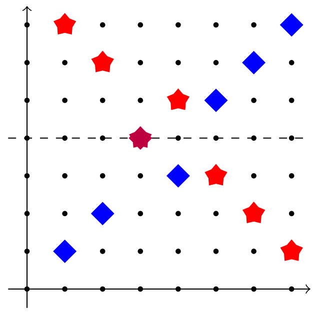

Figure 1: In (blue) squares we have the graph of the permutation $\sigma = 1243567$ and in (red) stars we have the graph of $\Phi(\sigma) = 7645321$.

Consider the map $\Phi : S_n \to S_n$ where for $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ we have

$$\Phi(\sigma) = (n + 1 - \sigma_1)\,(n + 1 - \sigma_2)\,\ldots\,(n + 1 - \sigma_n).$$

Geometrically, graphing the points (i, σ_i) in $\mathbb{R}^2$, the map Φ is flipping the graph of the permutation across the horizontal line given by $y = (n + 1)/2$ as shown in the example in Figure 1. Intuitively, Φ turns a permutation with descent set S into a permutation with descent set $\bar{S}$ (Proposition 5.3), and it preserves the ℓ_∞ distance (Lemma 5.4) and Hamming distance between permutations. This latter fact is not used in this section since all cases for the Hamming metric have been covered in Section 3.

The next two propositions show that the map Φ sends permutations in $\mathcal{D}(S; n)$ to permutations in $\mathcal{D}(\bar{S}; n)$ in a bijective manner.

Proposition 5.2 *The map $\Phi : S_n \to S_n$ is a bijection. Furthermore, it is its own inverse, that is, $\Phi \circ \Phi$ is the identity map on S_n.*

Proof. The first statement follows from the second. To prove the second, note that for $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ we have that for each $i \in [n]$,

$$(\Phi(\Phi(\sigma)))_i = n + 1 - (n + 1 - \sigma_i) = \sigma_i,$$

hence $\Phi(\Phi(\sigma)) = \sigma$. □

Proposition 5.3 *If $\sigma \in \mathcal{D}(S; n)$, then $\Phi(\sigma) \in \mathcal{D}(\bar{S}; n)$.*

Proof. If σ has a descent at index i, then $\sigma_i > \sigma_{i+1}$, which implies $-\sigma_i < -\sigma_{i+1}$. Adding $n+1$ to both sides, we get $n+1-\sigma_i < n+1-\sigma_{i+1}$. Therefore, $\Phi(\sigma_i) < \Phi(\sigma_{i+1})$, so every index that is a descent in σ will no longer be one in $\Phi(\sigma)$.

Now, let $i \notin S$, so σ does not have a descent at index i. Then $\sigma_i < \sigma_{i+1}$. This implies that $-\sigma_i > -\sigma_{i+1}$. Adding $n + 1$ to both sides, we get $n + 1 - \sigma_i > n + 1 - \sigma_{i+1}$. Therefore, $\Phi(\sigma_i) > \Phi(\sigma_{i+1})$. Every index that is not a descent in σ is now a descent in $\Phi(\sigma)$. Therefore, if $\sigma \in \mathcal{D}(S; n)$, then $\Phi(\sigma) \in \mathcal{D}(\bar{S}; n)$. □

The next result is needed to prove Theorem 5.5. It states that the ℓ_∞-metric is invariant under the map Φ.

Lemma 5.4 *For all $\sigma, \rho \in S_n$, $d_\ell(\sigma, \rho) = d_\ell(\Phi(\sigma), \Phi(\rho))$.*

Proof. Given $\sigma, \rho \in S_n$ and $i \in [n]$, note that

$$|\sigma_i - \rho_i| = |(n+1-\sigma_i) - (n+1-\rho_i)| = |\Phi(\sigma)_i - \Phi(\rho)_i|.$$

Since, when calculating d_ℓ, we are only concerned with the absolute value of the difference between indices we get that $d_\ell(\sigma, \rho) = d_\ell(\Phi(\sigma), \Phi(\rho))$. □

Theorem 5.5 *Let S be any proper, nonempty subset of $[n]$. Then*

$$\max(d_\ell(\mathcal{D}(S; n))) = \max(d_\ell(\mathcal{D}(\bar{S}; n)).$$

Proof. Let $m = \max(d_\ell(\mathcal{D}(S; n)))$ and $\sigma, \rho \in \mathcal{D}(S; n)$ such that $d_\ell(\sigma, \rho) = m$. By Proposition 5.3, $\Phi(\sigma), \Phi(\rho) \in \mathcal{D}(\bar{S}; n)$. By Lemma 5.4, $d_\ell(\Phi(\sigma), \Phi(\rho)) = m$, hence $\max(d_\ell(\mathcal{D}(\bar{S}; n)) \leq \max(d_\ell(\mathcal{D}(S; n))$. Applying the same argument to $\mathcal{D}(\bar{S}; n)$, we get $\max(d_\ell(\mathcal{D}(S; n)) \leq \max(d_\ell(\mathcal{D}(\bar{S}; n))$. □

We can apply Theorem 5.5 to the results in Theorems 4.1 and 4.3 to get the following two corollaries.

Corollary 5.6 *Let $n \geq 3$. For any $i \in \{2, \ldots, n-1\}$, let $\bar{S}_i = \{n-i+1, n-i+2, \ldots, n-1\}$ and let*

$$\mathcal{D}(\bar{S}_i; n) = \{\sigma \in S_n \,|\, \mathcal{D}(\sigma) = \bar{S}_i\},$$

then

$$\max(d_\ell(\mathcal{D}(\bar{S}_i; n))) = \max\{i-1, n-i\} = \begin{cases} i-1 & \text{if } i \geq \lfloor \frac{n+1}{2} \rfloor \\ n-i & \text{if } i \leq \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

Corollary 5.7 *Let $n \geq 6$, $\{\bar{i}\} := [n-1] \setminus \{i\}$, and consider $\mathcal{D}(\{\bar{i}\}; n)$, then*

$$\max(d_\ell(\mathcal{D}(\{\bar{i}\}; n))) = \begin{cases} n-2 & \text{for } i = 1, n-1 \\ n-i & \text{for } i = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor \\ i & \text{for } i = \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \ldots, n-2. \end{cases}$$

We end this article with two open problems for the interested reader.

Open Problem 1. Determine $\max(d_\ell(\mathcal{D}(S; n)))$ for all sets S other than those covered by Theorems 4.1 and 4.3 and Corollaries 5.6 and 5.7.

Open Problem 2. Find the maximum and minimum distance achievable by permutations in $\mathcal{D}(S; n)$ for other metrics on permutations such as the Kendall-tau metric.

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