Areas of Generalized Koch Snowflakes

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Abstract - In this work, we consider the areas of the regions enclosed by generalized Koch curves. We derived formulas for several classes of generalized Koch snowflakes. We also obtained a formula that can be used to compute areas of more complex snowflakes.

Keywords : Generalized Koch curves; non-self-intersecting; area

Mathematics Subject Classification (2020) **:** 37E05; 37P99

1 Introduction

The Koch snowflake, which is also known as the Koch curve, was originally constructed by Helge von Koch [8] as an example of a curve that is continuous but not differentiable at any point, though it is more well-known to public audience as a fractal. The curve is obtained by the following iterative process: begin with an equilateral triangle, replace the middle third of each side with two sides of an equilateral triangle, repeat this step for each of the resulting line segments, with equilateral triangles pointing outward. The limit of the sequence of the curves produced in this process is the classical Koch curve.

Many different generalizations of the Koch curve have been proposed and studied, even by von Koch himself [4, 8, 2, 1, 5, 6, 9]. Although it is not as interesting as the fractal dimension of the Koch curve [10], the area of the region enclosed by the classical Koch curve is a typical example of an infinite series commonly mentioned in Calculus books. The areas of the regions enclosed by generalized Koch curves, which we will simply refer to as the areas of Koch snowflakes, have also been explored [3]. In this work, we play further down along this line by finding formulas for the areas of several classes of snowflakes.

The construction of the classical Koch curve involves replacing the middle third of each side by (two sides of) an equilateral triangle. We can relax these construction elements and generalize the process to produce a large family of fractals. More specifically, generalization of the Koch curve can be achieved by allowing the replaced middle portion of a side varying in size or position at every stage, and also by allowing the middle portion to be replaced by any n-gon. Among the existing work in generalized Koch curves, Keleti and Paquetta [4] considered a class of generalization called (n, c)-Koch curves. In this generalization, instead of replacing the middle third portion of a side with an equilateral

*J.F and Y.W. were supported by NSF 1800406 and FA9550-23-1-0725. JF and CY have equal contributions to the paper

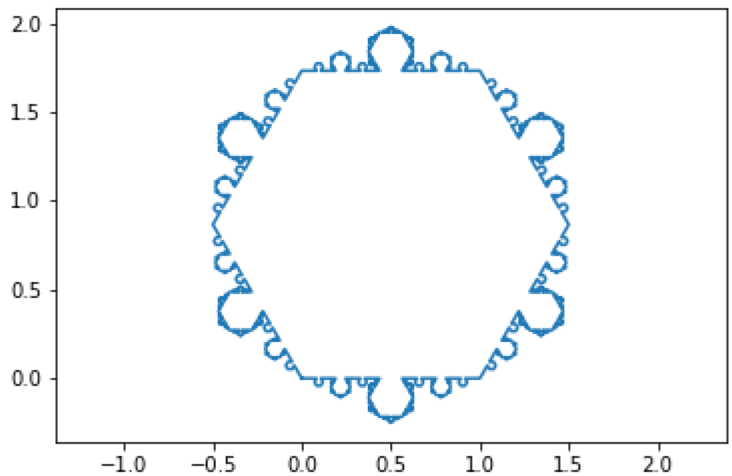

Figure 1: $(6, \frac{1}{8})$-Koch curve at 4^{th} iteration

triangle, the (n, c)-Koch curves are obtained by replacing the middle c portion [1] of each side with a regular n-gon. Fig. 1 is an example of such a curve with $n = 6$ and $c = \frac{1}{8}$. By this definition, the classical Koch curve is a $(3, \frac{1}{3})$-Koch curve. The authors of [4] showed that such a generalization could lead to self-intersecting curves. However, it was also proved in [4] that there will always be an interval of the values of c for which the (n, c)-Koch curve is not self-interacting, or say *self-avoiding*. To be more precise, we state the self-avoiding conditions from [4] in Theorem 1.1.

Theorem 1.1 ([4] Theorem 3.1) *The (n, c)-von Koch curve is self-avoiding if c satisfies*

$$c < \frac{\sin^2(\pi/n)}{\cos^2(\pi/n) + 1} \tag{1}$$

when n is even or if c satisfies

$$c < 1 - \cos(\pi/n) \tag{2}$$

when n is odd.

In this paper, we focus on self-avoiding generalized Koch curves. We found the areas for the following classes of snowflakes:

1. Self-avoiding (n, c)-Koch snowflakes: obtained by starting from a regular n-gon, iteratively replacing the middle c portion of each side with a regular n-gon. (See, Section 3)

[1] By 'middle c portion' we mean that each side is divided into three segments: two equally sized segments measuring $\frac{L(1-c)}{2}$ on each side of a middle segment measuring cL, where L is the length of the side.

2. Self-avoiding snowflakes obtained by alternating two replacing schemes every other step: replacing the middle c_i portion by a regular n_i-gon for $i = 1, 2$. (See, Section 4.)

In Section 5, we derive a formula for the area of new n-gons at any step. Our result can easily be used to calculate the areas of more general self-avoiding Koch snowflakes.

2 Area of the Classical Koch Snowflake

Even though the area of the classical Koch snowflake is well-known and can be easily computed, we include the calculation here for completeness and as well as for comparison with the approach we will take in later sections. Since the Koch snowflake is generated from an equilateral triangle with sides of unit length, the area of the snowflake is equal to the area of the initial triangle plus the areas of the additional triangles that grow from the three sides. Due to the symmetry of an equilateral triangle, the area can be represented as,

$$A = S + 3\sum_{j=1}^{\infty} A_j$$

where $S = \frac{\sqrt{3}}{4}$ is the area of the initial triangle, and A_j is the total area of new triangles that grow from each side of the initial triangle at the j^{th} iteration. Hence, we only need to compute A_j for $j \geq 1$.

The values of A_j, where $j = 0, 1, ...5$, are given in Table 1.

Step	# of lines	Length of Line	# of Triangles	Area of Each Triangle	Total Area(A_j)
0	1	1	0	0	0
1	4	$\frac{1}{3}$	1	$\frac{1}{9}S$	$A_1 = \frac{1}{9}S$
2	4^2	$\frac{1}{3^2}$	4	$\frac{1}{9^2}S$	$A_2 = \frac{4}{9^2}S$
3	4^3	$\frac{1}{3^3}$	4^2	$\frac{1}{9^3}S$	$A_3 = \frac{4^2}{9^3}S$
4	4^4	$\frac{1}{3^4}$	4^3	$\frac{1}{9^4}S$	$A_4 = \frac{4^3}{9^4}S$
5	4^5	$\frac{1}{3^5}$	4^4	$\frac{1}{9^5}S$	$A_5 = \frac{4^4}{9^5}S$

Table 1: **The values of A_j of the classical Koch Curve**

Using induction, one may quickly conclude that,

$$A_k = \frac{4^{k-1}}{9^k}S$$

Hence, the area of the snowflake is

$$\begin{aligned} A &= S + 3\sum_{k=1}^{\infty} A_k \\ &= S + 3\sum_{k=1}^{\infty} \tfrac{4^{k-1}}{9^k} S \\ &= S + \tfrac{S}{3}\sum_{k=0}^{\infty} \tfrac{4^k}{9^k} \\ &= S + \tfrac{S}{3} \cdot \tfrac{1}{1-4/9} \\ &= \tfrac{8}{5} S \end{aligned}$$

Therefore, $A = \frac{8}{5}S = \frac{2\sqrt{3}}{5}$.

3 Areas of Self-Avoiding (n,c)-Koch Snowflakes

In this section we study the areas of self-avoiding (n, c)-Koch snowflakes. Dence [3] derived the formula for the area of the $(3, c)$-Koch snowflake based on the observation that the areas of new triangles at different stages follow the pattern of Pascal's triangle.[2] Here, we give a general proof from the combinatorics point of view.

The key step to computing the area of any given self-avoiding (n, c)-Koch snowflakes is to find the number of new n-gons and the lengths of their sides at each step. To facilitate the calculation, we first introduce two terminologies and then prove a lemma.

At any step (iteration), the middle c portion of each side is replaced by $n-1$ sides of a regular n-gon with a side length cL, where L is the length of the side. Fig. 2 illustrates the process when $n = 3$. We call the two new line segments at the two ends of the resulting curve α-*sides* and the remaining $n-1$ sides c-*sides*, where $\alpha = \frac{1-c}{2}$ and c also correspond to the scaling factors. i.e, the length of an α-side is αL and the length of a c-side is cL.

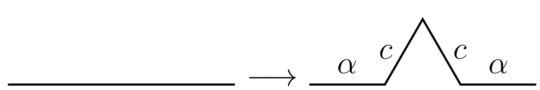

Figure 2: Replace the middle c portion of the line segment with two sides of an equilateral triangle. The two line segments at the ends of the resulting curve are called α-*sides* and the remaining two sides are denoted as c-*sides*, with $\alpha = \frac{1-c}{2}$. An α-side refers to a side with the length αL, where L is the side length of the line segment on the left. The same goes for the c-side.

For the remaining of the paper, if it is not explicitly stated, n-gon refers to a regular n-gon.

[2]triangular array of binomial coefficients.

Lemma 3.1 *Let the initial shape be an n-gon with side length L_0. Then*

1. *the side length of a new n-gon at step $k+1$ ($k \geq 1$) has the form of $a_1 a_2 \cdots a_k c L_0$ where $a_i \in \{\alpha, c\}$*
2. *there are $\binom{k}{r} 2^r (n-1)^{k-r}$ number of new n-gons with side length $\alpha^r c^{k-r} c L_0$.*
3. *given there is no self-intersection, the total area of all new n-gons at step $k+1$, generated from the middle third portion of a side on the initial n-gon, is*

$$A_{k+1} = (2\alpha^2 + (n-1)c^2)^k c^2 S \tag{3}$$

where S is the area of the initial n-gon.

Proof. First note that each line segment of the curve at the current step produces a new n-gon at the next step with the side length cL, where L is the length of the line segment. So the key to finding the total area of the new n-gons is to identify the number of line segments at the previous step and also the lengths of the respective line segments.

Note that each line segment in the curve at Step k ($k = 0, 1, 2, \cdots$) can be viewed as a result of evolving the initial line segment through k rounds of choice-makings, which we explain next. At each round, we pick one out of $(n+1)$ choices: two α-sides and $n-1$ many of c-sides. If we choose α-sides for r rounds and choose c-sides for $k-r$ rounds, then there are a total of $\binom{k}{r} 2^r (n-1)^{k-r}$ distinct such choices. For each such choice, the length of each line at Step k has the form of $a_1 \cdots a_k L_0$, where $a_i \in \{\alpha, c\}$ for $i = 1, \cdots, k$, and among them, there are r many of α's and $k-r$ many are c's. Hence there are $\binom{k}{r} 2^r (n-1)^{k-r}$ many of line segments with length $\alpha^r c^{k-r} L_0$ at Step k. Each of these line segments at Step k produce a new n-gon at Step $k+1$ of length of $\alpha^r c^{k-r} c L_0$. It follows that each of these n-gon at Step k has an area of $(\alpha^r c^{k-r} c)^2 S$.

Hence, the total area of new n-gons at Step $k+1$ is,

$$\begin{aligned} A_{k+1} &= \sum_{r=0}^{k} \binom{k}{r} 2^r (n-1)^{k-r} (\alpha^r c^{k-r} c)^2 S \\ &= \sum_{r=0}^{k} \binom{k}{r} (2\alpha^2)^r ((n-1)c^2)^{k-r} \\ &= c^2 S \sum_{r=0}^{k} \binom{k}{r} (2\alpha^2)^r ((n-1)c^2)^{k-r} \\ &= (2\alpha^2 + (n-1)c^2)^k c^2 S, \end{aligned}$$

where the last step is because of the binomial theorem.

□

Theorem 3.2 *Let $n \geq 3$ and S be the area of the initial regular n-gon. Then the area of a self-avoiding (n, c)-Koch snowflake is,*

$$A = S\left(1 + \frac{2nc^2}{1 + 2c - (2n-1)c^2}\right),$$

when $0 < c < \frac{1+\sqrt{2n}}{2n-1}$.

Proof. Since the (n, c)-Koch snowflake is generated from a regular n-gon with side length L_0, the area of the snowflake is equal to the area of the initial n-gon plus the areas of new regions growing from the n sides. Due to the symmetry of a regular n-gon, the area of the limiting Koch snowflake can be represented as.

$$A = S + n\sum_{j=1}^{\infty} A_j,$$

where A_j is the area of new n-gons growing from one side of the initial n-gon at the j^{th} iteration. By Lemma 3.1,

$$\begin{aligned} A &= S + n\sum_{k=0}^{\infty} A_{k+1} \\ &= S + n\sum_{k=0}^{\infty} (2\alpha^2 + (n-1)c^2)^k c^2 S \\ &= S + \tfrac{nc^2 S}{1-(2\alpha^2+(n-1)c^2)}. \end{aligned} \tag{4}$$

The last step is valid only when $2\alpha^2 + (n-1)c^2 < 1$. Since $\alpha = \frac{1-c}{2}$ and $0 < c < 1$, solving the inequality $2\alpha^2 + (n-1)c^2 < 1$, we have

$$0 < c < \frac{1+\sqrt{2n}}{2n-1}.$$

Replacing α with $\frac{1-c}{2}$, we arrive at $A = S\left(1 + \frac{2nc^2}{1+2c-(2n-1)c^2}\right)$. □

The result obtained by Dence [3] is a special case of Theorem 3.2 when $n = 3$.

Corollary 3.3 (Dence [3]) *Let S be the area of the initial equilateral triangle. Then the area of a self-avoiding (3, c)-Koch snowflake is*

$$A = S\left(1 + \frac{6c^2}{1+2c-5c^2}\right).$$

When $c = 1/3$ and $L_0 = 1$, then $A = \frac{2\sqrt{3}}{5}$ as we obtained in Section 1.

4 Alternating Different (n, c)-Replacing Scheme Every Other Steps

For convenience, in the remaining of the paper, we will use (n, c)*-replacing scheme* to refer to the scheme that the middle c portion of each line segment is replaced with a regular n-gon. We will also use the term (n, c)-layer referring to the step that the (n, c)-replacing scheme is applied.

In this section, we consider the areas of generalized Koch snowflakes resulting from alternating two different replacing schemes. For example, Fig. 3 shows a curve generated by alternating between a $(3, 0.3)$-layer and a $(4, 0.2)$-layer. Next we prove Lemma 4.1 which is an extension of Lemma 3.1.

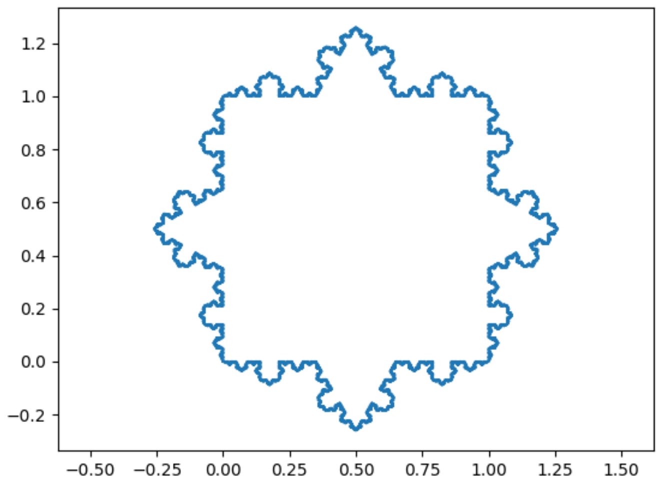

Figure 3: An example of alternating Koch snowflake obtained by by alternating between $(3, 0.3)$ layer and $(4, 0.2)$-layer

Lemma 4.1 *Suppose the first k steps involve m many of iterations following the (n_1, c_1)-replacing scheme and $k-m$ many of iterations follow the(n_2, c_2)-replacing scheme, and suppose the $(k+1)^{th}$ step follows the (n, c)-replacing scheme. Also, suppose there is no self-interaction, then the total area of all additional n-gons generated from one side of the initial n-gon at Step $k+1$ is,*

$$A_{k+1} = \left(2\alpha_1^2 + (n_1 - 1)c_1^2\right)^m \left(2\alpha_2^2 + (n_2 - 1)c_2^2\right)^{k-m} c^2 S, \tag{5}$$

where S is the area of the initial n-gon, $\alpha_1 = \frac{1-c_1}{2}$ and $\alpha_2 = \frac{1-c_2}{2}$.

Proof. Similar to Lemma 3.1, the side length of any new n-gon at step $k+1$ has the form of $a_1 \cdots a_k c L_0$, where L_0 is the side length of the initial n-gon. Since the first k steps involve m many of iterations following the (n_1, c_1)-replacing scheme and $k-m$ many of iterations following the (n_2, c_2)-replacing scheme, there are m many of $a_i \in \{\alpha_1, c_1\}$ and $k-m$ many of $a_i \in \{\alpha_2, c_2\}$. Since each iteration is independent, we can view the product $a_1 \cdots a_k$ as the product of two collections of factors: those $a_i \in \{\alpha_1, c_1\}$ form a collection and the remaining $k-m$ many of factors form another distinct collection. Suppose there are r_1 many of α_1, $m - r_1$ many of c_1, r_2 many of α_2 and $k - m - r_2$ many of c_2, by the same argument for Lemma 3.1 for each collection of the factors, there are exactly,

$$\binom{m}{r_1} 2^{r_1} (n_1 - 1)^{m-r_1} \binom{k-m}{r_2} 2^{r_2} (n_2 - 1)^{m-r_2}$$

many of new n-gons with a side length $\alpha_1^{r_1} c_1^{m-r_1} \alpha_2^{r_2} c_2^{m-r_2} c L_0$. Hence, the total area of all new n-gons at step $k+1$ is

$$\begin{aligned} A_{k+1} &= \sum_{r_1=0}^{m}\sum_{r_2=0}^{k-m}\binom{m}{r_1}2^{r_1}(n_1-1)^{m-r_1}\binom{k-m}{r_2}2^{r_2}(n_2-1)^{m-r_2}(\alpha_1^{r_1}c_1^{m-r_1}\alpha_2^{r_2}c_2^{m-r_2}c)^2S \\ &= (2\alpha_1^2+(n_1-1)c_1^2)^m(2\alpha_2^2+(n_2-1)c_2^2)^{k-m}c^2S \end{aligned} \tag{6}$$

□

Theorem 4.2 *Assume the sides of the initial regular n_1-gon have length L_0. Let S_i be the area of the n_i-gon with side length L_0 for $i = 1, 2$. The area of a self-avoiding (n, c)-Koch snowflake that is generated by alternating (n_1, c_1) layer and (n_2, c_2) layer is*

$$A = S_1 + \frac{n_1c_2^2S_2 + n_1(2\alpha_2^2 + (n_2-1)c_2^2)\cdot c_1^2S_1}{1-(2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2)}$$

where $\alpha_1 = \frac{1-c_1}{2}$ and $\alpha_2 = \frac{1-c_2}{2}$ and $0 < c_i < \frac{1+\sqrt{2n_i}}{2n_i-1}$ for $i = 1, 2$.

Proof. Since each line segment is replaced by an n_2-gon at an odd step and n_1-gon at an even step, we need to calculate the areas of new polygons at even and odd steps separately.

First we consider the total area of new n_2-gons at an odd step, say $2m + 1$ for $m \geq 0$. Since the first $2m$ steps involves m many of steps following the (n_1, c_1)-replacing scheme and m many of steps using the (n_2, c_2)-replacing scheme, by Lemma 4.1, the total area of the new n_2-gons is

$$\begin{aligned} A_{2m+1} &= (2\alpha_1^2+(n_1-1)c_1^2)^m\,(2\alpha_2^2+(n_2-1)c_2^2)^m\cdot c_2^2S_2 \\ &= \left((2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2)\right)^m\cdot c_2^2S_2 \end{aligned}$$

Next, we consider the total area of new n_1-gons at an even step, say $2m$ for $m \geq 1$. Since the first $2m-1$ steps involves $m-1$ many of steps using the (n_1, c_1)-replacing scheme and m many of steps using the (n_2, c_2)-replacing scheme, by Lemma 4.1, the total area of new n_1-gons is

$$\begin{aligned} A_{2m} &= (2\alpha_1^2+(n_1-1)c_1^2)^{m-1}\,(2\alpha_2^2+(n_2-1)c_2^2)^m\cdot c_1^2S_1 \\ &= ((2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2))^{m-1}\,(2\alpha_2^2+(n_2-1)c_2^2)\cdot c_1^2S_1 \end{aligned}$$

Note $0 < c_i < \frac{1+\sqrt{2n_i}}{2n_i-1}$ for $i = 1, 2$ implies that $(2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2) < 1$.

Hence, the total area of the resulting snowflakes is

$$\begin{aligned}
A &= S_1 + n_1\left(\sum_{m=0}^{\infty} A_{2m+1} + \sum_{m=1}^{\infty} A_{2m}\right) \\
&= S_1 + \frac{n_1 c_2^2 S_2}{1-(2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2)} \\
&+ \frac{n_1(2\alpha_2^2+(n_2-1)c_2^2)\cdot c_1^2 S_1}{1-(2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2)} \\
&= S_1 + \frac{n_1 c_2^2 S_2 + n_1(2\alpha_2^2+(n_2-1)c_2^2)\cdot c_1^2 S_1}{1-(2\alpha_1^2+(n_1-1)c_1^2)(2\alpha_2^2+(n_2-1)c_2^2)}
\end{aligned}$$

□

Remark 4.3 The conditions on c_i in Theorem 4.2 is sufficient, but may not be necessary.

5 General Mixed Steps

We can further extend Lemma 4.1 to more general scenarios.

Theorem 5.1 *Suppose the first k iterations consist of m_i steps applying (n_i, c_i)-replacing scheme for $i = 1, \cdots, N$, with $\sum_{i=1}^{N} m_i = k$, and the $(k+1)^{th}$ iteration follows the (n, c)-replacing scheme. Further suppose there is no self interaction, then the total area of all new n-gons at Step $k+1$ is*

$$A_{k+1} = \prod_{i=1}^{N}(2\alpha_i^2 + (n_i-1)c_i^2)^{m_i} c^2 S \tag{7}$$

where S is the area of the initial n-gon.

Proof. Similar to the proof of Lemma 4.1, the side length of new n-gons at Step k takes the form of $a_1 \cdots a_k L_0$, where L_0 is the side length of the initial n-gon. Since the first k iterations consists of m_i steps applying (n_i, c_i)-replacing scheme for $i = 1, \cdots, N$, there are m_i many of $a_j \in \{\alpha_i, c_i\}$. Again similar to the proof of Lemma 4.1, we can view the product $a_1 \cdots a_k$ as the product of N collections of factors: those $a_j \in \{\alpha_i, c_i\}$ form one collection for each $i \in \{1, \cdots, N\}$. Suppose there are r_i many of α_i and $m_i - r_i$ many of c_i in the sequence $a_1 \cdots a_k$, for $i = 1, \cdots, N$. Then there are exactly

$$\prod_{i=1}^{N} 2^{r_i}(n_i-1)^{m_i-r_i}$$

many of line segments associated with a given sequence $a_1 \dots a_k$. Hence, there are exactly

$$\prod_{i=1}^{N} \binom{m_i}{r_i} 2^{r_i}(n_i - 1)^{m_i - r_i}$$

many of new n-gons with a side length of $\prod_{i=1}^{N} \alpha_i^{r_i} c_i^{m-r_i} c L_0$. Thus the total area of all new n-gons at Step $k+1$ is

$$\begin{aligned} A_{k+1} &= \sum_{r_1=0}^{m_1} \cdots \sum_{r_N=0}^{m_N} \prod_{i=1}^{N} \tbinom{m_i}{r_i} 2^{r_i}(n_i-1)^{m_i-r_i}(\alpha_i^{r_i} c_i^{m_i-r_i} c)^2 S \\ &= \prod_{i=1}^{N} \sum_{r_i=0}^{m_i} \tbinom{m_i}{r_i} 2^{r_i}(n_i-1)^{m_i-r_i}(\alpha_i^{r_i} c_i^{m_i-r_i} c)^2 S \\ &= \prod_{i=1}^{N} (2\alpha_i^2 + (n_i-1)c_i^2)^{m_i} c^2 S \end{aligned} \tag{8}$$

□

6 Discussion

In this paper, we focused on computing the areas of generalized self-avoiding Koch snowflakes. We derived formulas for the areas of (n, c)-Koch snowflakes and also for the areas of mixed types of generalized Koch snowflakes. Our result in Theorem 5.1 can be further generalized to the cases that the replaced portion of a side is positioned other than the middle and the portion of side is replaced by a non-regular n-gon. In these extended cases, multinomial theorem will be needed for the calculation. However, the key combinatorics idea remains valid.

Besides the class of curves we discussed in this paper, there are other types of generalization such as those beautiful fractals constructed by Shakiban and Bergstedt [7]. As their strategy is different, our approach is likely needed to be modified. We will explore this in the future.

Acknowledgments

This work was initiated by a summer program in dynamical systems supported by grant NSF 1800406 in 2022. JF and YW were supported by NSF 1800406. YW was also supported by FA9550-23-1-0725.

References

[1] M. Cantrell, J. Palagallo, Self-intersection points of generalized Koch curves, *Fractals*, **19** (2011), 213–220. Also available at the URL:`https://worldscientific.com/doi/abs/10.1142/S0218348X11005257`

[2] R.B. Darst, J.A. Palagallo, T.E. Price, Generalizations of the Koch curve, *Fractals*, **16** (2008), 67–274. Also available at the URL:https://www.worldscientific.com/doi/abs/10.1142/S0218348X08003971

[3] T.P. Dence, Pascal's triangle and generalized Koch snowflakes, *Math. Comput. Educ.*, **34** (2000). Also available at the URL:https://www.proquest.com/openview/c80553b404a5c39708019aa17ab9dcf1/1

[4] T. Keleti, E. Paquette, The trouble with von Koch curves built from n gons, *Amer. Math. Monthly*, **117** (2010), 124–137. Also available at the URL:https://www.tandfonline.com/doi/abs/10.4169/000298910X476040

[5] A. Lakhtakia, V.K. Varadan, R. Messier, V.V. Varadan, Generalisations and randomisation of the plane Koch curve, *J. Phys. A Math. Gen.*, **20** (1987), 3537–3541. Also available at the URL:https://iopscience.iop.org/article/10.1088/0305-4470/20/11/052#

[6] M. Rani, R. Haq, D.K. Verma, Variants of Koch curve: A review, *Int. J. Comput. Appl*, **2** (2012), 20–24. Also available at the URL:https://www.researchgate.net/publication/322799091_Variants_of_Koch_curve_A_Review

[7] C. Shakiban, J.E. Bergstedt. Generalized Koch snowflakes. In Reza Sarhangi, editor, *Bridges: Mathematical Connections in Art, Music, and Science*, pages 301–308, Southwestern College, Winfield, Kansas, 2000. Bridges Conference.

[8] H. von Koch, Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire, *Arkiv för Matematik, Astronomi och Fysik*, **1** (1904), 681–702.

[9] Š. Ungar, The Koch curve: A geometric proof, *Amer. Math. Monthly*, **114** (2007), 61–66. Also available at the URL:https://www.jstor.org/stable/27642119

[10] J. Patino Ortiz, M. Patino Ortiz, M.A. Martinez-Cruz, A.S. Balankin, A brief survey of paradigmatic fractals from a topological perspective, *Fractal Fract.*, **7** (2023). Also available at the URL:https://doi.org/10.3390/fractalfract7080597

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Received: April 11, 2024 **Accepted**: October 15, 2024
Communicated by Robert Garrett Niemeyer