(S, w)-Gap Shifts and Their Entropy

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Abstract - The S-gap shifts have a dynamically and combinatorially rich structure. Dynamical properties of the S-gap shift can be related to the properties of the set S. This interplay is particularly interesting when S is not syndetic such as when S is the set of prime numbers or when $S = \{2^n\}$. It is a well known result that the entropy of the S-gap shift is given by $h(X) = \log \lambda$, where $\lambda > 0$ is the unique solution to the equation $\sum_{n \in S} \lambda^{-(n+1)} = 1$. Fix a point w of the full shift $\{1, 2, \ldots, k\}^{\mathbb{Z}}$. We introduce the (S, w)-gap shift which is a generalization of the S-gap shift consisting of sequences in $\{0, 1, \ldots, k\}^{\mathbb{Z}}$ in which any two 0's are separated by a word u appearing in w such that $|u| \in S$. We extend the formula for the entropy of the S-gap shift to a formula describing the entropy of this new class of shift spaces. Additionally we investigate the dynamical properties including irreducibility and mixing of this generalization of the S-gap shift.

Keywords : S-gap shift; entropy; synchronized system; measure of maximal entropy

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1 Introduction

Given an $S \subseteq \mathbb{Z}_{\geq 0}$, an S-gap shift, $X(S)$, is defined to be the shift space consisting of all sequences in $\{0, 1\}^{\mathbb{Z}}$ such that any two nearest 1's are separated by a word 0^n for some $n \in S$. The entropy of an S-gap shift is given by $\log \lambda$, where λ is the unique positive solution to the equation $\sum_{n \in S} \lambda^{-(n+1)} = 1$. This formula for the entropy of an S-gap shift is given as an exercise in [5, Exercise 4.3.7] and proofs of the formula are described in [3, Corollary 3.21] and [1].

In [6], Matson and Sattler generalize the notion of S-gap shifts by introducing the $\mathcal{S}$-limited shifts defined on an alphabet $\{1, \ldots k\}$ by a collection of limiting sets $\mathcal{S} = \{S_1, \ldots, S_k\}$ where each $S_i \subseteq \mathbb{N}$ describes the allowed lengths of words in which the letter i may appear. They describe several dynamical properties of this class of shift spaces and prove a formula for the entropy of ordered $\mathcal{S}$-limited shifts. Dillon [2] further extends this defining the $\mathcal{S}$-graph shifts by a finite directed graph with a subset of the natural numbers assigned to each vertex. The $\mathcal{S}$-graph shifts include all shifts of finite type and both the ordered and unordered $\mathcal{S}$-limited shifts and [2] computes the entropy of these shifts. These definitions generalize the S-gap shifts by altering the set S of the allowed lengths of gaps between pairs of a symbol.

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In this paper we extend the definition of S-gap shifts by modifying the words which separate pairs of symbols rather than the set S. Fix some $S \subseteq \mathbb{Z}_{\geq 0}$ and $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$. We define an (S, w)-gap shift, denoted $X_w(S)$, to be the closure of the set of points of the form $\ldots u_{-1} 0 u_0 0 u_1 \cdots \in \{0, 1, \ldots, k\}^{\mathbb{Z}}$ where each u_i is a word appearing in w with $|u_i| \in S$. As shown in section 4, any (S, w)-gap shift is irreducible and synchronized with synchronizing word 0. Additionally we show that an (S, w)-gap shift is mixing if and only if $\gcd\{n + 1 : n \in S\} = 1$. It turns out that the entropy of an (S, w)-shift depends on the number of words of length n appearing in w, we write this number $\varphi_w(n)$. We apply Theorem 5.2 ([3, Corollary 3.12]) to prove the following formula for the entropy of (S, w)-gap shifts.

Theorem 1.1 *Let $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ and $S \subseteq \mathbb{Z}_{\geq 0}$. Then $h(X_w(S)) = \log \lambda$ where $\lambda > 0$ is the unique positive solution to*

$$1 = \sum_{n \in S} \varphi_w(n) \lambda^{-(n+1)}.$$

2 Background

Let $\mathcal{A}$ be a finite set of symbols (or *letters*), we refer to $\mathcal{A}$ as an *alphabet*. The collection of all bi-infinite sequences of symbols in $\mathcal{A}$ is the *full shift* on $\mathcal{A}$ denoted

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} \ : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$$

which is equivalent to the standard notation of $\mathcal{A}^{\mathbb{Z}}$ for the set of all maps $\mathbb{Z} \to \mathcal{A}$.

Each sequence $x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is a *point* of the full shift. A *block* or *word* over $\mathcal{A}$ is a finite sequence of symbols from the alphabet $\mathcal{A}$. The empty word containing no letters is denoted ε. The *length* of a word $u = u_1 u_2 \cdots u_n \in \mathcal{A}^n$ is $|u| = n$ and the length of the empty word is $|\varepsilon| = 0$. The restriction of $x \in \mathcal{A}^{\mathbb{Z}}$ to the set of integers in the interval $[i, j]$ with $i \leq j$ is

$$x_{[i,j]} = x_i x_{i+1} \ldots x_j,$$

the block of coordinates in x from position i to position j. Similar for intervals (i, j), $[i, j)$, $(i, j]$. By extension we define

$$x_{[i,\infty)} = x_i x_{i+1} \ldots,$$

$$x_{(-\infty,i]} = \ldots x_{i-1} x_i.$$

A word u *appears* in the point $x = (x_i)_{i \in \mathbb{Z}}$ if $u = x_{[i,j]}$ for some integers $i \leq j$.

Let $u \in \mathcal{A}^m$ and $v \in \mathcal{A}^n$ with $m \leq n$. If $v_{[0,m)} = u$ we say u is a *prefix* of v. If $v_{[n-m,n)} = u$ we say u is a *suffix* of v.

The index i of a bi-infinite sequence $x = (x_i)_{i \in \mathbb{Z}}$ may be thought of as the time. Then shifting the sequence one position to the left corresponds to moving forward in time.

Definition 2.1 *The* shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ *maps a point x to $y = \sigma(x)$ defined by $y_i = x_{i+1}$.*

Definition 2.2 *A point $x \in \mathcal{A}^{\mathbb{Z}}$ is said to be* periodic for σ *if $\sigma^p(x) = x$ for some integer $p \geq 1$, in this case we say x has* period p under σ. *Equivalently, x is a periodic point of period p under the shift map σ if $x = u^\infty = \ldots uuu \ldots$ for some word u of length p.*

If x is periodic, the smallest positive integer p such that $\sigma^p(x) = x$ is the minimal period *of x. The point x is a* fixed point *for σ if $\sigma(x) = x$.*

There are no restriction on what sequences can occur in the full shift $\mathcal{A}^{\mathbb{Z}}$. Interesting dynamical properties arise when we impose constraints on which bi-infinite sequences occur in our space. Let $\mathcal{F}$ denote some set of words over the alphabet $\mathcal{A}$, the elements of $\mathcal{F}$ are known as *forbidden words*. We define

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{ no word of } \mathcal{F} \text{ appears in } x\}.$$

The notion of forbidden words leads us to the definition of a shift space.

Definition 2.3 *A* shift space *(or* shift*) is a subset X of the full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words over $\mathcal{A}$.*

If there is some finite set of forbidden words $\mathcal{F}$ such that $X = X_{\mathcal{F}}$, we say X is a *shift of finite type*. In many cases it's easier to describe what words are allowed to appear in a shift space rather than what words are forbidden.

Definition 2.4 *Let X be a subset of a full shift $\mathcal{A}^{\mathbb{Z}}$. The set of all words of length n that appear in some point of X is denoted $\mathcal{B}_n(X)$. The* language *of X is the collection*

$$\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X).$$

If w is any point of X, we denote the set of all words of length n appearing in w as $\mathcal{B}_n(w)$.

The language of X consists of all words which appear in some point of X. By [5, Proposition 1.3.4], the language of a shift space determines the shift space. For any subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the condition that $x \in \mathcal{A}^{\mathbb{Z}}$ has each $x_{[i,j]} \in \mathcal{B}(X)$ for integers $i \leq j$ is equivalent to $X = X_{\mathcal{B}(X)^c}$. So $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if whenever $x \in \mathcal{A}^{\mathbb{Z}}$ and each $x_{[i,j]} \in \mathcal{B}(X)$ then $x \in X$. Therefore, a shift space may be equivalently defined by its language.

For any shift space X, the *cylinder set* of $u \in \mathcal{B}_n(X)$ is defined as

$$[u] = \{x \in X : x_{[0,n)} = u\}.$$

Take the discrete product topology on the full shift $\mathcal{A}^{\mathbb{Z}}$. The collection of cylinder sets forms a basis for the topology on $X \subseteq \mathcal{A}^{\mathbb{Z}}$. The *extender set* of $u \in \mathcal{B}_n(X)$ is

$$E_X(u) = \{x_{(-\infty,0)} x_{[n,\infty)} : x \in [u]\}.$$

Example 2.5 The extender set of any word u in the language of $X = \{0,1\}^{\mathbb{Z}}$ is $E_X(u) = \{0,1\}^{\mathbb{Z}}$.

Now we discuss several important dynamical properties that a shift space may have.

Definition 2.6 *Let X be a shift space. A word $v \in \mathcal{B}(X)$ is* synchronizing *if $uv, vw \in \mathcal{B}(X)$ implies $uvw \in \mathcal{B}(X)$. If $\mathcal{B}(X)$ contains a synchronizing word, we say X is* synchronized.

Definition 2.7 *A shift space X is said to be* irreducible *if for all ordered pairs (u, w) with $u, w \in \mathcal{B}(X)$ there is some $v \in \mathcal{B}(X)$ such that $uvw \in \mathcal{B}(X)$.*

Definition 2.8 *A shift space X is* mixing *if for all $u, w \in \mathcal{B}(X)$, there is some positive integer N such that for each $n \geq N$ there exists a word $v \in \mathcal{B}_n(X)$ such that $uvw \in \mathcal{B}(X)$.*

Definition 2.9 *The* (topological) entropy *of a shift space X is defined as*

$$h(X) = h_{top}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

For the remainder of this paper any measure μ will be assumed to be a Borel probability measure on a shift space X. Additionally, we assume that μ is invariant under the shift map σ, that is, $\mu(\sigma^{-1}(B)) = \mu(B)$ for any measurable set B. We denote the set of all such measures on X as $\mathcal{M}(X)$.

Definition 2.10 *The* metric entropy *or* measure-theoretic entropy *of X with a given measure μ is defined as*

$$h_\mu(X) = \lim_{n\to\infty} \frac{1}{n} \sum_{\alpha\in\mathcal{A}^n} -\mu[\alpha]\log(\mu[\alpha]).$$

Since any word $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{A}^{n+m}$ can be broken into two words $\alpha_{[1,n]} \in \mathcal{A}^n$, $\alpha_{[n+1,n+m]} \in \mathcal{A}^m$, the sequence $\sum_{\alpha\in\mathcal{A}^n} -\mu[\alpha]\log(\mu[\alpha])$ is subadditive in n. By Fekete's Lemma, the limit defining the measure-theoretic entropy exists and we have

$$h_\mu(X) = \lim_{n\to\infty} \frac{1}{n} \sum_{\alpha\in\mathcal{A}^n} -\mu[\alpha]\log(\mu[\alpha]) = \inf_{n\geq 1} \frac{1}{n} \sum_{\alpha\in\mathcal{A}^n} -\mu[\alpha]\log(\mu[\alpha]).$$

The topological entropy and metric entropy of a shift X are related by the *variational principle* (see, e.g. [8, Theorem 8.6]) which states:

$$h(X) = \sup_{\mu\in\mathcal{M}(X)} h_\mu(X)$$

A measure μ is called a *measure of maximal entropy*, or MME, if $h_\mu(X) = h(X)$.

Remark 2.11 Any shift space X is a compact metric space with the metric

$$d_\theta(x, y) = \theta^{\max\{k: x_i = y_i, |i|\leq k\}}$$

for $0 < \theta < 1$ and the shift map $\sigma : X \to X$ is an expansive homeomorphism. By [8, Theorem 8.2], the entropy map $\mu \mapsto h_\mu(X)$ is upper semi-continuous. Then the entropy map attains a maximum $h_\mu(X)$ on $\mathcal{M}(X)$ and by the variational principle, $h(X) = h_\mu(X)$. Hence, a measure of maximal entropy exists on any shift space X.

3 S-Gap Shifts

Definition 3.1 *Let $S \subseteq \mathbb{Z}_{\geq 0}$. The S-*gap shift *is*

$$X_S = \overline{\{\dots 10^{n_{-1}}10^{n_0}10^{n_1}\cdots : n_i \in S\}} \subseteq \{0,1\}^{\mathbb{Z}}.$$

Equivalently, an S-gap shift is the shift space $X_w(S)$ on the alphabet $\{0,1\}$ consisting of points of the form

$$\dots 10^{n_{-1}}10^{n_0}10^{n_1}\dots$$

with $n_i \in S$. In the case that S is infinite, we also allow sequences which begin and/or end with an infinite string of 0's.

Theorem 3.2 ([4, p. 1407]) *An S-gap shift is mixing if and only if*

$$\gcd\{n+1 : n \in S\} = 1.$$

A formula for the entropy of the S-gap shift is given as an exercise in [5, Exercise 4.3.7] and is computed in [3, Corollary 3.21]. Several other proofs of this formula are described in [1] one of which is generalized to the ordered $\mathcal{S}$-limited shifts in [6]. Additionally, we present a proof of the S-gap entropy formula, Theorem 3.3. This proof has the advantage of taking a combinatorial approach, but does not generalize well for the (S,w)-gap shifts. So we adopt an approach similar to that of [3, Corollary 3.21] for our proof of Theorem 1.1.

Theorem 3.3 *The* entropy *of any S-gap shift is $h(X(S)) = \log\lambda$ where $\lambda > 0$ is the unique positive solution to*

$$1 = \sum_{n\in S} \lambda^{-(n+1)}.$$

Proof. Let $a_n = |\mathcal{B}_n(X(S))|$ and consider the function

$$H(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We compute that the radius of convergence of $H(z)$ is $\xi = e^{-h(X(S))}$ as follows

$$\begin{aligned}\log(\xi) &= \log(\lim_{n\to\infty} \sqrt[n]{a_n}^{-1}) \\ &= \lim_{n\to\infty} -\log(\sqrt[n]{a_n}) \\ &= -\lim_{n\to\infty} \frac{1}{n}\log(a_n) \\ &= -h(X(S)).\end{aligned}$$

Put $G = \{0^n 1 : n \in S\}$ and let A_n^k be the set of all words of length n that may be written as the concatenation of exactly $k \geq 1$ words in G, $A_n^0 := \{\varepsilon\}$. Define

$$F_k(z) = \sum_{n=1}^{\infty} |A_n^k| z^n.$$

Note that in the case of $k = 1$,

$$F_1(z) = \sum_{n=1}^{\infty} |A_n^1| z^n = \sum_{n\in S} z^{n+1}$$

because $|A_n^1| = \mathbf{1}_S(n-1)$.

We claim that $F_k(z)F_\ell(z) = F_{k+\ell}(z)$. It is sufficient to show that

$$\sum_{m=1}^{n} |A_{n-m}^k||A_m^\ell| = |A_n^{k+\ell}|.$$

First consider words $u \in A_m^\ell, v \in A_{n-m}^k$ for any $1 \le m \le n$. Since uv has length n and can be written as the concatenation of ℓ words in G followed by k words in G, we have $uv \in A_n^{k+\ell}$ and thus

$$\sum_{m=1}^{n} |A_{n-m}||A_m^\ell| \le |A_n^{k+\ell}|.$$

Now consider any $w \in A_n^{k+\ell}$ and break the word after the ℓ-th 2 denoting the first part as u and the last part as v. Then $uv = w$, $u \in A_m^\ell, v \in A_{n-m}^k$ where $m = |u|$. So

$$|A_n^{k+\ell}| \le \sum_{m=1}^{n} |A_{n-m}||A_m^\ell|.$$

By induction, it follows that $F_k(z) = (F_1(z))^k$.

Since A_n^k, $k \ge 1$ are disjoint subsets of $\mathcal{B}_n(X(S))$, we have

$$\sum_{k\ge 1} |A_n^k| \le |\mathcal{B}_n(X(S))|.$$

Any word of $\mathcal{B}_n(X(S))$ is of the form 0^n or $0^i 1 w 0^j$ where $w \in A_{n-i-j-1}^k$ for some $k \ge 0$ so

$$|\mathcal{B}_n(X(S))| \le 1 + \sum_{k\ge 0}\sum_{i\ge 0}\sum_{j\ge 0} |A_{n-i-j-1}^k|.$$

If $z \ge 0$, then we may apply Tonelli's theorem and compute

$$\sum_{k\ge 1} |A_n^k| \le |\mathcal{B}_n(X(S))| \le 1 + \sum_{k\ge 0}\sum_{i\ge 0}\sum_{j\ge 0} |A_{n-i-j-1}^k|,$$

$$\sum_{n\ge 1}\sum_{k\ge 1} |A_n^k| z^n \le H(z) \le \sum_{n\ge 1} z^n + \sum_{n\ge 1}\sum_{k\ge 0}\sum_{i\ge 0}\sum_{j\ge 0} |A_{n-i-j-1}^k| z^n,$$

$$\sum_{k\ge 1} F_k(z) \le H(z) \le \sum_{n\ge 1} z^n + \sum_{k\ge 0} F_k(z) \left(\sum_{i\ge 0} z^i\right)\left(\sum_{j\ge 0} z^{j+1}\right)$$

where the last inequality is because

$$\begin{aligned}\sum_{n\geq 1}\sum_{k\geq 0}\sum_{i\geq 0}\sum_{j\geq 0}|A^k_{n-i-j-1}|z^n &= \sum_{n\geq 1}\sum_{k\geq 0}\sum_{i\geq 0}\sum_{j\geq 0}|A^k_n|z^{n+i+j+1}\\ &= \sum_{n\geq 1}\sum_{k\geq 0}|A^k_n|z^n\left(\sum_{i\geq 0}\sum_{j\geq 0}z^{i+j+1}\right)\\ &= \sum_{k\geq 0}F_k(z)\left(\sum_{i\geq 0}z^i\right)\left(\sum_{j\geq 0}z^{j+1}\right).\end{aligned}$$

If $\hat{H}(z) = \sum_{k\geq 1} F_k(z) = \sum_{k\geq 1}(F_1(z))^k$, then

$$\hat{H}(z) \leq H(z) \leq \sum_{n\geq 1} z^n + (F_0(z) + \hat{H}(z))\left(\sum_{i\geq 0}z^i\right)\left(\sum_{j\geq 0}z^{j+1}\right).$$

Since $\sum_{n\geq 1} z^n$ converges for all $z \in [0, 1)$ and $F_0(z) = \sum_{n\geq 1} z^n$, $\hat{H}(z)$ converges if and only if $H(z)$ converges. Then the radius of convergence of $H(z)$ is the unique $z > 0$ such that $F_1(z) = 1$ and by the first paragraph $h(X(S)) = \log(1/z)$. □

4 (S, w)-Gap Shifts

Definition 4.1 *Let $S \subseteq \mathbb{Z}_{\geq 0}$ and let $w \in \{1, 2, \dots, k\}^{\mathbb{Z}}$ where $k > 0$ is an integer. We define the* (S, w)-gap shift *to be*

$$X_w(S) = \overline{\{\dots 0u_{-1}0u_00u_1\dots : u_i \textit{ appears in } w, |u_i| \in S\}} \subseteq \{0, 1, \dots, k\}^{\mathbb{Z}}.$$

Recall that $\mathcal{B}_n(w)$ is the set of all length n words that appear in w. Let $B = \mathcal{B}(\{1, 2, \dots, k\}^{\mathbb{Z}}) \setminus (\bigcup_n \mathcal{B}_n(w))$. If S is infinite, then the forbidden word set of $X_w(S)$ is

$$\mathcal{F} = B \cup \{0u : u \in B\} \cup \{u0 : u \in B\} \cup \{0u0 : |u| \in \mathbb{Z}_{\geq 0} \setminus S\}. \tag{1}$$

If S is finite, then the forbidden word set of $X_w(S)$ is

$$\mathcal{F} \cup \{1, 2, \dots, k\}^{1+\max S}.$$

So the (S, w)-gap shift is in fact a shift space.

Example 4.2 Let $S \subseteq \mathbb{Z}_{\geq 0}$. By relabeling symbols of the alphabet $\{0, 1\}$ in Definition 3.1, we note that an alternative definition of the S-gap shift $X(S)$ is the set of points of $\{0, 1\}^{\mathbb{Z}}$ such that any pair of nearest 0's is separated by a block 1^n with $n \in S$. Therefore, any S-gap shift is the (S, w)-gap shift with $w = 1^\infty$.

Example 4.3 If $S \subseteq \mathbb{Z}_{\geq 0}$ is cofinite and words in the language of $\{1, \ldots, k\}^{\mathbb{Z}}$ appear in w, then the set described in (1) is finite and thus $X_w(S)$ is a shift of finite type. For example we may take a point $w \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $w_i = 1$ for $i < 0$ and $w_{[0,\infty)} = \alpha_0\alpha_1 \ldots \alpha_n \ldots$ where α_n denotes the concatenation of all elements of $\{1, \ldots, k\}^n$.

Proposition 4.4 *Any (S, w)-gap shift is synchronized with synchronizing word* 0.

Proof. If $u0 \in \mathcal{B}(X_w(S))$ and $0v \in \mathcal{B}(X_w(S))$, then $u0v \in \mathcal{B}(X_w(S))$. So 0 is a synchronizing word for $X_w(S)$. □

Proposition 4.5 *Any (S, w)-gap shift is irreducible.*

Proof. Let $u, v \in \mathcal{B}(X_w(S))$. To show $X_w(S)$ is irreducible, we will construct a word $t0s$ such that $ut0sv \in \mathcal{B}(X_w(S))$.

If u ends with 0, put $t = w_{[0,n)}$ for some $n \in S$. Otherwise, u has a suffix of the form $w_{[i,i+k)}$ with k less than or equal to some element $n \in S$. If $k \in S$, put $t = \varepsilon$. If $k \notin S$, put $t = w_{[i+k,i+n)}$. Then $ut0 \in \mathcal{B}(X_w(S))$. Similarly, if v begins with 0, put $t = w_{[0,n)}$ for some $n \in S$. Otherwise, has a prefix of the form $w_{[i-k,i)}$ with k less than or equal to some element $n \in S$. If $k \in S$, put $s = \varepsilon$. If $k \notin S$, put $s = w_{[i-n,i-k)}$. Then $0sv \in \mathcal{B}(X_w(S))$.

Since 0 is a synchronizing word for $X_w(S)$, $ut0sv \in L(X_w(S))$. Hence $X_w(S)$ is irreducible. □

Remark 4.6 Note that the construction in the proof of Proposition 4.5 shows that for all $u, v \in \mathcal{B}(X_w(S))$ there are words $t, s \in \mathcal{B}(X_w(S))$ such that $ut0sv \in \mathcal{B}(X_w(S))$.

The following proposition uses this observation and is a slight variant of the proof for S-gap shifts in [4, p. 1407] and for the $\mathcal{S}$-graph shifts in [2, Proposition 3.4.].

Proposition 4.7 *An (S, w)-gap shift is mixing if and only if*

$$\gcd\{n + 1 : n \in S\} = 1.$$

Proof. If $X_w(S)$ is mixing, then there is some $N > 0$ such that for all $n \geq N$ there is some word $\gamma \in \mathcal{B}_n(X_w(S))$ such that $0\gamma0 \in \mathcal{B}(X_w(S))$. In particular, we may choose $\gamma_1 \in \mathcal{B}_N(X_w(S))$,$\gamma_2 \in \mathcal{B}_{N+1}(X_w(S))$ such that $0\gamma_10, 0\gamma_20 \in \mathcal{B}(X_w(S))$ and γ_1, γ_2 are of the form

$$u_1 0 u_2 \cdots 0 u_m$$

for $u_i \in \bigcup_{k \in S} \mathcal{B}_k(w)$. Then $0\gamma_1 \in \mathcal{B}_{N+1}(X_w(S))$ and $0\gamma_2 \in \mathcal{B}_{N+2}(X_w(S))$ are concatenations of words of the form $0u_i$. Then $N+1$ and $N+2$ are equal to sums of elements of $\{n+1 : n \in S\}$ because $|0u_i| = |u_i| + 1 \in \{n + 1 : n \in S\}$. Since $N + 1$ and $N + 2$ are relatively prime, $\gcd\{n + 1 : n \in S\} = 1$.

Now suppose that $\gcd\{n + 1 : n \in S\} = 1$ and let $\alpha, \beta \in \mathcal{B}(X_w(S))$. Then there is some $M > 0$ such that any integer $n \geq M + 1$ may be written as a sum of elements of $\{n + 1 : n \in S\}$ and thus there is a word

$$0v_1 0v_2 \cdots 0v_{m'} 0$$

of length n with $v_i \in \bigcup_{k\in S} \mathcal{B}_k(w)$. As noted in remark 4.6, there is a word $\gamma_1 0 \gamma_2 \in \mathcal{B}(X_w(S))$ such that $\alpha\gamma_1 0\gamma_2\beta \in \mathcal{B}(X_w(S))$. Let $N = M + |\gamma_1\gamma_2| + 1$. For any $n \geq N$ there is a word

$$0u_1 0u_2 \cdots 0u_m 0$$

of length $n - |\gamma_1\gamma_2| \geq M + 1$ with $u_i \in \bigcup_{k\in S} \mathcal{B}_k(w)$. Since 0 is a synchronizing word for $X_w(S)$,

$$\gamma := \gamma_1 0u_1 0u_2 \cdots 0u_m 0\gamma_2 \in \mathcal{B}(X_w(S))$$

is a word of length n such that $\alpha\gamma\beta \in \mathcal{B}(X_w(S))$. This shows $X_w(S)$ is mixing. □

Definition 4.8 *A shift space X is a* coded system *if it is the closure of the set of points obtained by freely concatenating elements of some set G of words over an alphabet $\mathcal{A}$. The elements of G are called* generators *of the coded system.*

The (S, w)-gap shifts are an example of a coded system with the generating set

$$G = \bigcup_{n\in S} \bigcup_{u\in B_n(w)} u0 = \{u0 : u \text{ appears in } w, |u| \in S\}.$$

5 Entropy

Lemma 5.1 ([3, Lemma 3.14]) *If v is a synchronizing word for a shift space X, then for any word of the form $vuv \in \mathcal{B}(X)$, we have*

$$E_X(vuv) = E_X(v).$$

Theorem 5.2 ([3, Corollary 3.12]) *Let X be a subshift, μ a measure of maximal entropy and $u, v \in \mathcal{B}(X)$. If $E_X(u) = E_X(v)$, then*

$$\mu[v] = \mu[u]e^{h(X)(|u|-|v|)}.$$

Lemma 5.3 *For any $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ and $S \subseteq \mathbb{Z}_{\geq 0}$ there is some measure of maximal entropy μ on $X_w(S)$ such that $\mu[0] > 0$.*

Proof. By Remark 2.11, there exists a measure of maximal entropy on $X_w(S)$. If S is finite, then any pair of nearest 0's appearing in a point of $X_w(S)$ are separated by a word u appearing in w with $|u| \leq \max S$. Then the sets $[0], \sigma[0], \ldots, \sigma^{\max S}[0]$ cover $X_w(S)$, so any measure of maximal entropy μ on $X_w(S)$ satisfies $\mu[0] \geq (1 + \max S)^{-1} > 0$.

Now consider the case when S is infinite. Suppose that μ is measure of maximal entropy on $X_w(S)$ such that $\mu[0] = 0$. Then μ is supported on the orbit closure of w, that is, the shift space $\mathcal{O}$ defined by the language $\mathcal{B}(w) = \bigcup_n \mathcal{B}_n(w)$. By the variational principle, $h(\mathcal{O}) \geq h_\mu(\mathcal{O}) = h(X_w(S))$. Since $\log(t!)/t \to \infty$ as $t \to \infty$, we may choose some positive integer t such that $\log(t!)/t > h(\mathcal{O})$. Since S is infinite, we there are distinct $n_1, \ldots, n_t \in \mathbb{Z}_{\geq 0}$ such that $n_1 - 1, \ldots, n_t - 1 \in S$. For any $k > 0$ the set $\mathcal{B}_{k(n_1+n_2+\cdots+n_t)}(X_w(S))$ contains all word of the form

$$(0u_{1,1}0u_{1,2}\ldots 0u_{1,t})(0u_{2,1}0u_{2,2}\ldots 0u_{2,t})\ldots(0u_{k,1}0u_{k,2}\ldots 0u_{k,t})$$

where the lengths $|u_{i,1}|, |u_{i,2}|, \ldots, |u_{i,t}|$ are distinct elements of $\{n_1 - 1, n_2 - 1, \ldots, n_t - 1\}$ for each $i = 1, \ldots, k$. So

$$|\mathcal{B}_{k(n_1+n_2+\cdots+n_t)}(X_w(S))| \geq (t!)^k |\mathcal{B}_{n_1-1}(w)|^k |\mathcal{B}_{n_2-1}(w)|^k \cdots |\mathcal{B}_{n_t-1}(w)|^k.$$

Since $|\mathcal{B}_n(w)| \geq e^{nh(\mathcal{O})}$ for all values of n, we have

$$\begin{aligned}\frac{\log |\mathcal{B}_{k(n_1+n_2+\cdots+n_t)}(X_w(S))|}{k(n_1+n_2+\cdots+n_t)} &\geq \frac{\log((t!)^k |\mathcal{B}_{n_1-1}(w)|^k \cdots |\mathcal{B}_{n_t-1}(w)|^k)}{k(n_1+n_2+\cdots+n_t)} \\ &\geq \frac{k\log(t!) + k(n_1+n_2+\cdots+n_t - t)h(\mathcal{O})}{k(n_1+n_2+\cdots+n_t)} \\ &= \frac{\log(t!) - t\cdot h(\mathcal{O}) + (n_1+n_2+\cdots n_t)h(\mathcal{O})}{n_1+n_2+\cdots+n_t}.\end{aligned}$$

By taking the limit $k \to \infty$,

$$h(X_w(S)) = \frac{\log(t!) - t\cdot h(\mathcal{O}) + (n_1+n_2+\cdots n_t)h(\mathcal{O})}{n_1+n_2+\cdots+n_t} > h(\mathcal{O})$$

since $\log(t!) > t \cdot h(\mathcal{O})$. This is a contradiction to $h(\mathcal{O}) \geq h(X_w(S))$. □

For any point $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ define $\varphi_w(n)$ for $n \geq 0$ to be the number of distinct words of length n appearing in w. We say φ_w is the *complexity function of* w. Observe that $\varphi_w(n) = |\mathcal{B}_n(w)|$.

Theorem 1.1 *Let $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ and $S \subseteq \mathbb{Z}_{\geq 0}$. Then $h(X_w(S)) = \log \lambda$ where $\lambda > 0$ is the unique positive solution to*

$$1 = \sum_{n\in S} \varphi_w(n)\lambda^{-(n+1)}.$$

Proof. In any (S, w)-gap shift $X_w(S)$, we have

$$[0] = \left(\bigcup_{n\in S}\bigcup_{u\in\mathcal{B}_n(w)} [0u0]\right) \cup \left(\bigcup_{i\in\mathbb{Z}} [0w_{[i,\infty)}]\right) \tag{2}$$

where $[0w_{[i,\infty)}]$ denotes the set of all $x \in X_w(S)$ such that $x_{[0,\infty)} = 0w_{[i,\infty)}$. By Lemma 5.3, there is some measure of maximal entropy μ on $X_w(S)$ such that $\mu[0] > 0$. Suppose that for some $i \in \mathbb{Z}$, $\mu[0w_{[i,\infty)}] \neq 0$. Since $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$, all $\sigma^{-n}[0w_{[i,\infty)}]$ for $n \geq 0$ are pairwise disjoint. By the property that σ is measure preserving,

$$\mu\left(\bigcup_{n\geq 0} \sigma^{-n}[0w_{[i,\infty)}]\right) = \sum_{n\geq 0} \mu(\sigma^{-n}[0w_{[i,\infty)}]) = \sum_{n\geq 0} \mu[0w_{[i,\infty)}],$$

a contradiction to the fact that $\mu(X_w(S)) = 1$. Hence, $\mu[0w_{[i,\infty)}] = 0$ for all $i \in \mathbb{Z}$ and we have $\mu\left(\bigcup_{i\in\mathbb{Z}} [0w_{[i,\infty)}]\right) = 0$.

Since 0 is a synchronizing word for $X_w(S)$, Lemma 5.1 shows that

$$E_{X_w(S)}(0u0) = E_{X_w(S)}(0)$$

for all $u \in \mathcal{B}_n(w)$. By Theorem 5.2,

$$\mu[0u0] = \mu[0]e^{h(X_w(S))(-n-1)}.$$

For any pair of words $u \neq v$ appearing in w, the cylinder sets $[0u0]$ and $[0v0]$ are disjoint. By taking the measure of the set in (2),

$$\begin{aligned}\mu[0] &= \mu\left(\bigcup_{n\in S}\bigcup_{u\in\mathcal{B}_n(w)}[0u0]\right)\\ &= \sum_{n\in S}\sum_{u\in\mathcal{B}_n(w)}\mu[0u0]\\ &= \sum_{n\in S}\varphi_w(n)\mu[0]e^{h(X_w(S))(-n-1)}.\end{aligned}$$

Dividing by $\mu[0] > 0$ shows

$$1 = \sum_{n\in S}\varphi_w(n)e^{h(X_w(S))(-n-1)}.$$

Since the complexity function of w is positive, the positive solution to

$$1 = \sum_{n\in S}\varphi_w(n)\lambda^{-(n+1)}$$

is unique. □

Corollary 5.4 *If $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ has minimal period p and $S \subseteq \mathbb{Z}_{\geq 0}$, then $h(X_w(S)) = \log\lambda$ where $\lambda > 0$ is the unique solution to*

$$1 = \sum_{n\in S, n<p-1}\varphi_w(n)\lambda^{-(n+1)} + p\sum_{n\in S, n\geq p-1}\lambda^{-(n+1)}.$$

Proof. It suffices to show that $\varphi_w(n) = p$ for $n > p-1$. Let $w \in \{1, 2, \ldots, k\}^{\mathbb{Z}}$ have minimal period p. Consider the case when $n = p-1$. Showing that the elements in the set

$$\mathcal{T} = \{\sigma^m_{[0,n-1]}(w) \mid 0 \leq m \leq p-1\}$$

are distinct is sufficient enough to complete the proof. Here we denote $\sigma^m_{[0,n-1]}(w)$ as the shift map on the cylinder set w from the coordinates 0 to $n-1$. Now suppose, for contradiction, that the elements in the set $\mathcal{T}$ were not all distinct. Since w has period p

and if we know the first $p-1$ symbols, then we know what the last symbol must be to satisfy that w has period p. This allows us to write

$$\sigma^i(w) = \sigma^j(w)$$

for some $0 \leq i < j \leq p-1$. Taking σ^{-i} on both sides yields

$$w = \sigma^{-i}(\sigma^j(w)) = \sigma^{j-i}(w)$$

which contradicts the fact that w has period p. Now consider the case for when $n > p-1$. Once again, we assume the elements in $\mathcal{T}$ are not distinct. We have the following equality

$$\sigma^i_{[0,n-1]}(w) = \sigma^j_{[0,n-1]}(w)$$

for some $0 \leq i < j \leq p-1$. The above equality states that the two words agree with each other on the the first n symbols, so they must agree everywhere with their periodicity infinitely in both directions. This allows us to write the equality as $\sigma^i(w) = \sigma^j(w)$. We achieve the same contradiction with a similar argument. So, $\varphi_w(n) = p$ for $n > p-1$. □

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