The Connections Between Consistent Maps and Measures

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Abstract - Let Y denote the space of places of $\overline{\mathbb{Q}}$ as defined in a 2009 article of Allcock and Vaaler. As part of an effort to classify certain dual spaces, the second author defined an object called a consistent map. Every signed Borel measure on Y can be used to construct a consistent map, however, we asserted without proof that not all consistent maps arise in this way. By constructing a counterexample, we show in the present article that not all consistent maps arise from measures, confirming claims made in the second author's earlier work.

Keywords : consistent maps; measures; Weil height

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1 Introduction

1.1 Absolute Values and Places

If F is a field then an *absolute value* on F is a function $|\cdot| : F \to [0,\infty)$ which satisfies the following properties:

(i) $|x| = 0$ if and only if $x = 0$

(ii) $|xy| = |x| \cdot |y|$ for all $x, y \in F$

(iii) $|x+y| \leq |x| + |y|$ for all $x, y \in F$.

Each absolute value on F induces a metric on F given by $(x,y) \mapsto |x-y|$, and in turn, it induces a topology on F. It is straightforward to prove that two absolute values $|\cdot|_1$ and $|\cdot|_2$ induce the same topology on F if and only if there exists a positive real number θ such that $|x|_1 = |x|_2^\theta$ for all $x \in F$. In this case, we say that $|\cdot|_1$ and $|\cdot|_2$ are *equivalent.*

We note that all fields have the *trivial absolute value* given by

$$|x|_0 = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0, \end{cases}$$

and no other absolute value may be equivalent to $|\cdot|_0$. Each equivalence class of non-trivial absolute values is called a *place of* F and we shall write M_F to denote the set of places of

F. If $|\cdot|_1$ and $|\cdot|_2$ are equivalent absolute values on F, then the Cauchy sequences in F with respect to $|\cdot|_1$ are the same as those with respect to $|\cdot|_2$. Therefore, a completion of F depends only on the choice of place and not on the particular absolute value from that place. Hence, we may write F_v to denote the completion of F with respect to a place v.

Now considering the special case where $F = \mathbb{Q}$, a theorem of Ostrowski [6] provided a complete classification of all places of $\mathbb{Q}$. Specifically, there are two categories of absolute values on $\mathbb{Q}$: the *usual absolute value* given by

$$|x|_\infty = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

and the p-adic absolute values defined in the following way. If p is prime, then each non-zero rational number x may be expressed in the form $x = p^\alpha y$, where $\alpha \in \mathbb{Z}$ and y is rational number whose numerator and denominator are not divisible by p. Then the *p-adic absolute value* is defined by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\alpha} & \text{if } x \neq 0. \end{cases}$$

It is straightforward to prove that $|\cdot|_\infty$ and $|\cdot|_p$ are absolute values on $\mathbb{Q}$, and furthermore, no pair of them are equivalent. Ostrowski [6] showed that $\{\infty, 2, 3, 5, 7, 11, \ldots\}$ is the complete collection of places of $\mathbb{Q}$. Readers interested in further detail on p-adic absolute values and the corresponding completions should consult [3].

Now suppose K is a number field (i.e., a finite extension of $\mathbb{Q}$) and L is an extension of K. If $|\cdot|$ and $|\cdot|'$ are equivalent absolute values on L, then their restrictions to K are clearly also equivalent. Therefore, each place w of L restricts to a unique place v of K, and in this case, we say that w *divides* v. We write $M_L(v)$ to denote the set of places of L that divide v. If L is a finite extension of K then we have the well-known formula

$$[L : K] = \sum_{w \in M_L(v)} [L_w : K_v] \tag{1}$$

(see [2, Sec. 4, Eq. (2)] or [5, Cor. 8.4], for example). This equality shows, in particular, that if L/K is a finite extension then $M_L(v)$ is a finite set.

As an example, consider the case where $K = \mathbb{Q}$, $v = \infty$ and L is a quadratic extension of $\mathbb{Q}$. In this scenario we have that $\mathbb{Q}_\infty = \mathbb{R}$ and (1) simplifies to

$$2 = \sum_{w \in M_L(\infty)} [L_w : \mathbb{R}]. \tag{2}$$

Each summand on the right hand side of (2) is forced to equal either 1 or 2. In the former case, L is a real quadratic extension of $\mathbb{Q}$ and $M_L(\infty)$ contains two places. In the latter case, L is an imaginary quadratic extension of $\mathbb{Q}$ and $M_L(\infty)$ contains only one place.

1.2 Places of $\overline{\mathbb{Q}}$

If L/K is an infinite extension, then $M_L(v)$ is a more complicated, typically uncountable set. Fixing an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, Allcock and Vaaler [1] began the study of the places of $\overline{\mathbb{Q}}$. Following the notation of [1], we write $Y(K,v) = M_{\overline{\mathbb{Q}}}(v)$ so that $Y(K,v)$ denotes set of places of $\overline{\mathbb{Q}}$ that divide the place v of a number field K, and furthermore, we let Y be the set of all places of $\overline{\mathbb{Q}}$. They showed that the collection of sets

$$\{Y(K,v) : [K:\mathbb{Q}] < \infty,\ v \in M_K\}$$

is a basis for a totally disconnected, locally compact Hausdorff topology on Y. Moreover, they defined a regular measure λ on the Borel sets $\mathcal{B}$ of Y satisfying

$$\lambda(Y(K,v)) = \frac{[K_v : \mathbb{Q}_p]}{[K:\mathbb{Q}]}.$$

This property means, in particular, that $\lambda(Y(\mathbb{Q},p)) = 1$ for all places p of $\mathbb{Q}$.

Allcock and Vaaler's primary purpose was to study the vector space $\mathcal{G} := \overline{\mathbb{Q}}^\times / \overline{\mathbb{Q}}^\times_{\mathrm{tors}}$ over $\mathbb{Q}$ when equipped with a certain norm arising from the Weil height. To this end, they considered the space $L^1(Y,\mathcal{B},\lambda)$ of integrable functions $f : Y \to \mathbb{R}$ for which

$$\int_Y |f(y)| d\lambda(y) < \infty. \tag{3}$$

It is well-known that the left hand side of (3) defines a norm on $L^1(Y,\mathcal{B},\lambda)$ and that $L^1(Y,\mathcal{B},\lambda)$ is a Banach space over $\mathbb{R}$ with respect to this norm, i.e., it is a complete normed vector space over $\mathbb{R}$.

For each place $y \in Y$, Allcock and Vaaler defined an absolute value $\|\cdot\|_y$ that extends one of the absolute values on $\mathbb{Q}$. Given a point $\alpha \in \mathcal{G}$, they defined the function $f_\alpha : Y \to \mathbb{R}$ by

$$f_\alpha(y) = \log \|\alpha\|_y \tag{4}$$

and showed that $\alpha \mapsto f_\alpha$ is an isometric isomorphism of $\mathcal{G}$ onto to a dense $\mathbb{Q}$-linear subspace of

$$\mathcal{X} := \left\{ f \in L^1(Y,\mathcal{B},\lambda) : \int_Y f(y) d\lambda(y) = 0 \right\}. \tag{5}$$

Any sequence of functions in $\mathcal{X}$ cannot converge to a function whose integral is different than 0, and hence, $\mathcal{X}$ is itself a Banach space over $\mathbb{R}$.

The duals of $L^1(Y,\mathcal{B},\lambda)$ and $\mathcal{X}$ have long been well understood due to various versions of the Riesz Representation Theorem (see [7, Theorem 6.19], for example). Since the image of $\mathcal{G}$ is dense in $\mathcal{X}$, one may apply the Riesz Representation Theorem to classify the space of continuous linear maps from $\mathcal{G}$ to $\mathbb{R}$. However, it may be of interest to obtain representation theorems for the following sets that are not addressed by this method:

(I) The space of all linear maps from $\mathcal{G}$ to $\mathbb{R}$

(II) The space of all linear maps from $\mathcal{G}$ to $\mathbb{Q}$

(III) The space of all continuous linear maps from $\mathcal{G}$ to $\mathbb{Q}$

While these precise problems remain unresolved, work of the second author [8, 9] provided similar results for related spaces. In [8], we established analogs of (II) and (III) for $\mathcal{V} := \overline{\mathbb{Q}}^{\times}/\overline{\mathbb{Z}}^{\times}$, where $\overline{\mathbb{Z}}^{\times}$ denotes the group of units in $\overline{\mathbb{Q}}$. Later in [9], we acquired an analog of (I) for the $\mathbb{R}$-vector space of locally constant functions from Y to $\mathbb{R}$. The crucial step in both sets of results is the construction of an object called a *consistent map* which plays a role similar to that of a measure in the Riesz Representation Theorem.

1.3 Consistent Maps

The primary purpose of the present article is to explore further the connection between consistent maps and measures on Y. To this end, we shall provide the formal definitions of both objects as given in [9] and [4, Ch. 9], respectively. From this point forward, we fix a non-empty set S of places of $\mathbb{Q}$ and let X be the set of places of $\overline{\mathbb{Q}}$ which divide a place in S. If K is a number field, we write $M_K(S)$ to denote the set of places of K that divide a place in S. Our subsequent definitions and theorems often depend on the choice of S even though we shall typically suppress that dependency in our notation.

We let

$$\mathcal{J} = \{(K,v) : [K:\mathbb{Q}] < \infty,\ v \in M_K(S)\}.$$

A map $c : \mathcal{J} \to \mathbb{R}$ is called *consistent* if it satisfies

$$c(K,v) = \sum_{w \in M_L(v)} c(L,w) \tag{6}$$

for all number fields K, all places $v \in M_K(S)$, and all finite extensions L/K. If $c, d : \mathcal{J} \to \mathbb{Q}$ are consistent maps and $r \in \mathbb{Q}$ then we let

$$(c+d)(K,v) = c(K,v) + d(K,v) \quad \text{and} \quad (rc)(K,v) = rc(K,v).$$

It is easily verified that these operations cause the set of consistent maps $c : \mathcal{J} \to \mathbb{Q}$ to be a vector space over $\mathbb{R}$, which we shall denote by $\mathcal{J}^*$. Now let $\mathcal{L}$ be the $\mathbb{R}$-vector space of locally constant functions from X to $\mathbb{R}$. The first main result of [9] provided an explicit isomorphism from $\mathcal{J}^*$ to the algebraic dual $\mathcal{L}^*$ of $\mathcal{L}$. We invite the reader to consult [9] for more detail on that particular isomorphism as well as for analogous results of other dual spaces.

We now let $\mathfrak{b} = \{Y(K,v) : (K,v) \in \mathcal{J}\}$ and note that $\mathfrak{b}$ forms a basis for the topology on X given in [1]. Now following [4, Ch. 9], a map $\mu : \mathcal{B} \to \mathbb{R} \cup \{+\infty, -\infty\}$ is called a *signed Borel measure* if it satisfies the following three properties:

(i) $\mu(\emptyset) = 0$

(ii) μ takes at most one of the values $+\infty$ and $-\infty$

(iii) We have that
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$
for all pairwise disjoint collections $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{B}$.

The final property, called *countable additivity*, includes the assumption that the right hand side either converges absolutely in $\mathbb{R}$ or diverges to $\pm\infty$.

As is noted in [8, 9], if μ is a signed Borel measure for which every element of $\mathfrak{b}$ has finite measure, then μ gives rise to a corresponding consistent map. Indeed, let $c : \mathcal{J} \to \mathbb{R}$ be given by $c(K, v) = \mu(Y(K, v))$. Assuming L/K is an extension of number fields, each place in $Y(K, v)$ divides at most one place in $M_L(v)$. Hence,
$$Y(K, v) = \bigcup_{w \in M_L(v)} Y(L, w)$$
is a disjoint union for all number fields K, all places $v \in M_K(S)$, and all finite extensions L/K. Therefore, the countable additivity property for μ implies (6), meaning that c is consistent.

We asserted in [9] that not all consistent maps arise in this way, a reasonable claim given that consistent maps need only have the finite additivity property (6) while signed Borel measures are required to have countable additivity. Nevertheless, no counterexample is provided either in [8] or [9]. As our main result, we prove the following by way of counterexample.

Theorem 1.1 *There exists a consistent map $c : \mathcal{J} \to \mathbb{R}$ for which there is no signed Borel measure μ on X satisfying $c(K, v) = \mu(Y(K, v))$ for all $(K, v) \in \mathcal{J}$.*

2 Proof of Theorem 1.1

Our proof requires a lemma showing how to construct a consistent map from a particular tower of number fields. This lemma is essentially the same as [8, Lemma 4.2], however, that article assumes that $S = \{2, 3, 5, 7, 11, \ldots\}$ and that consistent maps take only rational values. Fortunately, the proof changes very little to accommodate our more general situation, but we include it here for the sake of completeness. Although it is distinct from our introduction, we borrow the notation of [1] and [8] and write $W_v(L/K) = M_L(v)$. Additionally, we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.1 *Suppose that $\{K_i : i \in \mathbb{N}_0\}$ is a collection of number fields such that*
$$\overline{\mathbb{Q}} = \bigcup_{i=0}^{\infty} K_i \quad \text{and} \quad K_i \subseteq K_{i+1} \text{ for all } i \in \mathbb{N}_0.$$
For each $i \in \mathbb{N}_0$ and each place $v \in M_{K_i}(S)$ we let $d_i(v) \in \mathbb{R}$. Finally, assume that
$$d_i(v) = \sum_{w \in W_v(K_{i+1}/K_i)} d_{i+1}(w) \tag{7}$$

for all $i \in \mathbb{N}_0$ and all $v \in M_{K_i}(S)$. Then there exists a unique consistent map $c : \mathcal{J} \to \mathbb{R}$ such that $c(K_i, v) = d_i(v)$ for all $i \in \mathbb{N}_0$ and all $v \in M_{K_i}(S)$.

Proof. Let F be a number field and let $u \in M_F(S)$. Since F necessarily has the form $F = \mathbb{Q}(\gamma)$ for some $\gamma \in \overline{\mathbb{Q}}$, there exists $i \in \mathbb{N}_0$ such that $F \subseteq K_i$. Now let

$$c_i(F, u) = \sum_{v \in W_u(K_i/F)} d_i(v). \tag{8}$$

It is straight forward to prove by induction using (7) that

$$d_i(v) = \sum_{w \in W_v(K_j/K_i)} d_j(w)$$

for all $j \geq i$ and all $v \in M_{K_i}(S)$. Consequently, for each $j \geq i$ we have

$$c_j(F, u) = \sum_{w \in W_u(K_j/F)} d_j(w) = \sum_{v \in W_u(K_i/F)} \sum_{w \in W_v(K_j/K_i)} d_j(w) = \sum_{v \in W_u(K_i/F)} d_i(v) = c_i(F, u).$$

In other words, the definition (8) does not depend on i and we shall simply write $c(F, u) = c_i(F, u)$. To complete the proof, we must establish the following three claims:

(A) $c(K_i, v) = d_i(v)$ for all $i \in \mathbb{N}$ and all $v \in M_{K_i}(S)$.

(B) c is consistent.

(C) $c : \mathcal{J} \to \mathbb{Q}$ is the unique map satisfying (A) and (B).

We obtain (A) immediately by applying (8) with $F = K_i$ and $u = v$. To establish (B), we assume that E is a finite extension of the number field F and assume that $i \in \mathbb{N}$ is such that $E \subseteq K_i$. Then definition (8) yields

$$c(F, u) = \sum_{v \in W_u(K_i/F)} d_i(v) = \sum_{t \in W_u(E/F)} \sum_{v \in W_t(K_i/E)} d_i(v) = \sum_{t \in W_u(E/F)} c(E, t).$$

Finally, assume that $c' : \mathcal{J} \to \mathbb{Q}$ such that $c'(K_i, v) = d_i(v)$ for all $i \in \mathbb{N}_0$ and all $v \in M_{K_i}(S)$. Let F be a number field, let $u \in M_F(S)$, and let $i \in \mathbb{N}$ be such that $F \subseteq K_i$. Then we have

$$c(F, u) = \sum_{v \in W_u(K_i/F)} d_i(v) = \sum_{v \in W_u(K_i/F)} c'(K_i, v) = c'(F, u),$$

where the last equality follows from our assumption that c' is consistent. $\square$

We note that the uniqueness property established in Lemma 2.1 is not required for our proof of Theorem 1.1. However, it is sufficiently straightforward to prove so we find it worthwhile to include nevertheless. In any case, we now proceed with the proof of our main result.

Proof of Theorem 1.1. As S is a assumed to be a non-empty set of places of $\mathbb{Q}$, we shall fix an element $p \in S$. Additionally, by applying [8, Lemma 4.3], we obtain a nested sequence of number fields

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$$

such that

$$\overline{\mathbb{Q}} = \bigcup_{i=1}^{\infty} K_i \quad \text{and} \quad \#W_v(K_{i+1}/K_i) > 1 \tag{9}$$

for all $i \in \mathbb{N}_0$ and all $v \in W_p(K_i/\mathbb{Q})$.

We shall now recursively define places v_i and w_i of K_i for all $i \geq 1$. We begin by letting $v_0 = p$. Now if v_i is given, we let $v_{i+1} \in W_{v_i}(K_{i+1}/K_i)$. By our assumption (9), there exists a place $w_{i+1} \in W_{v_i}(K_{i+1}/K_i) \setminus \{v_{i+1}\}$, and for simplicity, we write

$$T_i = W_{v_i}(K_{i+1}/K_i) \setminus \{v_{i+1}\} \text{ for all } i \in \mathbb{N}_0$$

so that $w_{i+1} \in T_i$. For every $i \in \mathbb{N}_0$, we let

$$Z_i = \bigcup_{w \in T_i} Y(K_{i+1}, w)$$

so that Z_i is a given by a pairwise disjoint union and

$$Y(K_i, v_i) = Y(K_{i+1}, v_{i+1}) \cup Z_i \tag{10}$$

is also a disjoint union. Since we have chosen v_{i+1} to divide v_i we obtain that

$$Y(K_0, v_0) \supseteq Y(K_1, v_1) \supseteq Y(K_2, v_2) \supseteq \cdots . \tag{11}$$

If $i > j$ and $y \in Z_i$ then y must divide v_i so that $y \in Y(K_i, v_i) \subseteq Y(K_{j+1}, v_{j+1})$. Since (10) is a disjoint union, we conclude that $y \notin Z_j$, meaning that $Z_i \cap Z_j = \emptyset$. We now define

$$A = \bigcup_{i=0}^{\infty} Z_i = \bigcup_{i=0}^{\infty} \bigcup_{w \in T_i} Y(K_{i+1}, w) \quad \text{and} \quad B = \bigcap_{i=1}^{\infty} Y(K_i, v_i) \tag{12}$$

and observe that the collection of sets of the form $Y(K_{i+1}, w)$ in the definition of A is pairwise disjoint.

We now claim that

$$Y(\mathbb{Q}, p) = A \cup B \quad \text{and} \quad A \cap B = \emptyset. \tag{13}$$

Clearly we have that $A \cup B \subseteq Y(\mathbb{Q}, p)$. Now if $y \in Y(\mathbb{Q}, p)$, we assume that $y \notin B$. Because of (11) and the fact that $Y(\mathbb{Q}, p) = Y(K_0, v_0)$, there must exist a smallest positive integer i such that $y \notin Y(K_i, v_i)$. But then $y \in Y(K_{i-1}, v_{i-1})$ and since (10) is a disjoint union, we have that $y \in Z_{i-1}$ implying that $y \in A$. We have now shown that $Y(\mathbb{Q}, p) = A \cup B$. To see that $A \cap B = \emptyset$, we assume that $y \in A$ so that $y \in Z_i$ for some $i \in \mathbb{N}_0$. Again using the fact that (10) is a disjoint union, we obtain that $y \notin Y(K_{i+1}, v_{i+1})$ showing that $y \notin B$. We have now established both properties of (13).

We now seek to apply Lemma 2.1 to create a consistent map $c : \mathcal{J} \to \mathbb{R}$. To this end, we must select $d_i(v) \in \mathbb{R}$ for each $i \in \mathbb{N}_0$ and each place $v \in M_{K_i}(S)$ in such a way that

$$d_i(v) = \sum_{w \in W_v(K_{i+1}/K_i)} d_{i+1}(w) \tag{14}$$

is always satisfied. If $v \nmid p$, then we define $d_i(v) = 0$ so that (14) holds for all such v. For places v dividing p, we define $d_i(v)$ inductively starting with $d_0(v) = d_0(p) = 0$. Now assume that $d_i(v)$ is given for all $v \in M_{K_i}(p)$. To define $d_{i+1}(w)$ we assume that w divides a place v of K_i. As we have assumed that $w \mid p$, we know that $v \mid p$ as well. Now set

$$d_{i+1}(w) = \begin{cases} d_i(v) \cdot \frac{[(K_{i+1})_w : (K_i)_v]}{[K_{i+1} : K_i]} & \text{if } v \neq v_i \\ d_i(v) - 1 & \text{if } w = v_{i+1} \\ 1 & \text{if } w = w_{i+1} \\ 0 & \text{if } v = v_i \text{ but } w \notin \{v_{i+1}, w_{i+1}\}. \end{cases}$$

If $v \neq v_i$ then (14) follows from (1) while the remaining case $v = v_i$ is clear from the definition of $d_{i+1}(w)$. As a result, Lemma 2.1 yields a consistent map $c : \mathcal{J} \to \mathbb{R}$ such that $c(K_i, v) = d_i(v)$ for all $i \in \mathbb{N}_0$ and all places $v \in M_{K_i}(S)$.

Now assume that μ is a signed Borel measure on X such that $c(K, v) = \mu(Y(K, v))$ for all $(K, v) \in \mathcal{J}$. Using the fact that A is given by a disjoint union (12), we have that

$$\begin{aligned} \mu(A) &= \mu\left(\bigcup_{i=0}^{\infty} \bigcup_{w \in T_i} Y(K_{i+1}, w)\right) \\ &= \sum_{i=0}^{\infty} \sum_{w \in T_i} \mu(Y(K_{i+1}, w)) \\ &= \sum_{i=0}^{\infty} \sum_{w \in T_i} c(K_{i+1}, w) = \sum_{i=0}^{\infty} \sum_{w \in T_i} d_{i+1}(w). \end{aligned}$$

We recall that $T_i = W_{v_i}(K_{i+1}/K_i) \setminus \{v_{i+1}\}$ so that the interior sum equals 1. In other words, we have shown that

$$\mu(A) = \sum_{i=0}^{\infty} 1 = +\infty.$$

Because of property (ii) of signed Borel measures, we must have that $\mu(B) \in (-\infty, +\infty]$. Then using (13), we obtain that

$$\mu(Y(\mathbb{Q}, p)) = \mu(A) + \mu(B) = +\infty$$

which contradicts the fact that $\mu(Y(\mathbb{Q}, p)) = d_0(p) = 0$. □

Before closing, we note a much simpler proof of Theorem 1.1 in the case where S is infinite. Under this assumption, let $S = \{p_1, p_2, p_3, \ldots\}$ and set $\tau(p_i) = (-1)^i$. If $(K, v) \in \mathcal{J}$ then we shall write p_v to denote the unique element of S for which $v \mid p_v$. Now define $c : \mathcal{J} \to \mathbb{R}$ by

$$c(K, v) = \tau(p_v) \cdot \frac{[K_v : \mathbb{Q}_p]}{[K : \mathbb{Q}]}$$

so that (1) implies that c is consistent. We claim that c is another counterexample that would yield a proof of Theorem 1.1. To see this, assume that there exists a signed Borel measure μ such that $c(K, v) = \mu(Y(K, v))$ and let

$$A = \bigcup_{i \text{ odd}} Y(\mathbb{Q}, p_i) \quad \text{and} \quad B = \bigcup_{i \text{ even}} Y(\mathbb{Q}, p_i).$$

These are both disjoint unions, and therefore

$$\mu(A) = \sum_{i \text{ odd}} \tau(p_i) = -\infty \quad \text{and} \quad \mu(B) = \sum_{i \text{ even}} \tau(p_i) = +\infty$$

which contradicts (ii) in the definition of signed Borel measures.

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