The Smallest Bipartite Intrinsically Knotted Graph

B. Johnson and S. Li*

Abstract - We use the restoring method of Kim, Mattman, and Oh to prove that the Heawood graph is the only bipartite graph with 21 edges that is intrinsically knotted.

Keywords : intrinsically knotted graph; bipartite graph; graph embedding

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1 Introduction

A planar graph is a graph that can be embedded in a plane, i.e., that can be drawn on a plane without having any edges intersect one another. Kazimierz Kuratowski provided a characterization of planar graphs by proving that a graph is planar if and only if it does not contain a subgraph that is a subdivision of one of two graphs, K_5 or $K_{3,3}$ [13]. Klaus Wagner proved, equivalently, that a graph is planar if and only if it does not contain one of these two graphs as a minor [21].

The problem of intersecting edges disappears when one considers spatial embeddings of finite graphs – all finite graphs can be embedded in 3-space without edge intersections – but new questions arise. Horst Sachs and, separately, John Conway and Cameron Gordon replaced the edge intersection condition with conditions about the existence of nontrivial knots and links [3],[20]. They proved that every spatial embedding of the complete graph on 6 vertices, K_6, contains a non-splittable link, and using a similar strategy, Conway and Gordon proved that every spatial embedding of the complete graph on 7 vertices, K_7, contains a nontrivial knot. Neil Robertson, Paul Seymour, and Robin Thomas proved that a graph contains a non-splittable link in every spatial embedding if and only if it does not contain any member of the Petersen family of graphs as a minor [19]. A corresponding characterization of *intrinsically knotted graphs*, that is, of graphs that have nontrivial knots in every spatial embedding, remains an open problem. The work of Robertson and Thomas, and in particular their Forbidden Minor Theorem, guarantees that such a characterization exists.

To formulate such a characterization of intrinsically knotted graphs, one seeks to identify *minor-minimal intrinsically knotted graphs*, that is, intrinsically knotted graphs all of whose minors are not intrinsically knotted. Conway and Gordon's work provided the first identification of such a graph, K_7. Joel Foisy and collaborators made important

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contributions in identifying a linking condition that led to the identification of several new examples [4], [5], [2]. Breakthrough work of Noam Goldberg, Thomas Mattman, Jonathan Miller, and Ramin Naimi developed and utilized a computer algorithm based on Foisy's linking condition that identified over 200 new examples of minor minimal intrinsically knotted graphs [16], [6].

A different approach to this problem seeks to systematically identify "small" intrinsically knotted graphs. For example, Thomas Mattman, and independently the first author together with Mark Kidwell and T.S. Michael proved that any intrinsically knotted graph must have at least 21 edges. Mattman and Jamison Barsotti, showed that there are exactly 14 graphs with 21 edges that are intrinsically knotted [1]. Minjung Lee et al. obtained the same result in [14]. Using the techniques of [14], the authors, together with Mattman, identified all intrinsically knotted triangle-free graphs with at least two degree five vertices, and classified all intrinsically knotted bipartite graphs with 22 edges [8], [9]. Kim, Mattman, and Oh added the *restoring method* to this collection of techniques in [10] and used it to establish that there are exactly five triangle-free intrinsically knotted graphs that have 22 edges and one vertex of degree 5. The restoring method is also used in [11] to identify the bipartite intrinsically knotted graphs that have 23 edges and no vertex with degree less than 3.

In this paper, we take a second look, through the lens of the techniques used by Mattman, Kim, Lee, Lee, and Oh, at the case of graphs with 21 edges. We use their methods to prove that there is exactly one bipartite graph, known as the Heawood graph, that has 21 edges and is intrinsically knotted. Of course, one can also obtain this result by showing directly that among the 14 intrinsically knotted graphs with 21 edges the only graph that is bipartite is the Heawood graph. By applying the techniques of Mattman et al. to a simpler case, our goals are also to provide a guide to the methods of [14], [9], [11], [8], and [10], and to highlight their effectiveness.

1.1 Organization

In section 1, we review results about graphs with 21 edges that are needed in the rest of the paper. Section 2 reviews the techniques of Mattman et al. that we use to prove our main result, as well as the results and notation needed for these techniques. In section 3, we show that the vertices in an intrinsically knotted bipartite graph with 21 edges must have degree greater than 2 and less than 6, and use this to identify possible combinations of degrees for the vertices in such a graph. We eliminate all but one of these possibilities in section 5, and complete the proof that the Heawood graph is the only possibility in section 6.

1.2 Conventions and Notation

For a graph G, we write $G = (V(G), E(G))$ where $V(G)$ is the set of vertices in the graph, and $E(G)$ is the set of edges. When the context is clear, we use V and E in place of $V(G)$ and $E(G)$. For vertices v and w in a graph G, we use (v, w) to denote an edge between them. The degree of a vertex is the number of edges that are adjacent to that vertex. For a vertex v, we use $\deg(v)$ to denote its degree.

For an integer $n \geq 1$, the *complete graph on n vertices* is the graph K_n that has n vertices and all possible edges between those vertices.

A *bipartite* graph is a graph G whose vertex set is the disjoint union of sets A and B where $(u, v) \in E(G)$ implies that exactly one of u and v is in A and exactly one is in B. For such a bipartite graph, we write $V(G) = A \sqcup B$ to indicate that the vertex set of G is the disjoint union of sets A and B.

For integers $m, n \geq 1$, the *complete bipartite graph* $K_{m,n}$ has $V(K_{m,n}) = A \sqcup B$ and $E(K_{m,n}) = \{(u, v) \mid u \in A, v \in B\}$ where A has m elements and B has n elements. In other words, $K_{m,n}$ has $m + n$ vertices divided into subsets A and B of size m and n, respectively, and all possible edges between vertices in A and vertices in B.

2 Graphs With 21 Edges

We use this section to review some results about graphs with 21 edges. The first theorem establishes a lower bound on the number of edges in an intrinsically knotted graph. Conway and Gordon's proof that K_7 is intrinsically knotted implies that this lower bound is sharp.

Theorem 2.1 *[7, 15] An intrinsically knotted graph has at least* 21 *edges.*

A $\nabla\mathbf{Y}$-move on a graph is the operation that replaces three edges that form a triangle in the graph with a new vertex with edges adjacent to the three vertices from the original triangle. See Figure 1. A graph G' is said to be a descendant of G if G' is obtained

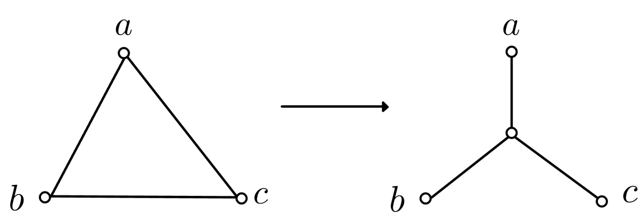

Figure 1: $\nabla\mathbf{Y}$ moves

by one or more $\nabla\mathbf{Y}$-moves on G. Motwani et. al [17] showed that $\nabla\mathbf{Y}$-moves preserve intrinsic knotting. Applying $\nabla\mathbf{Y}$-moves repeatedly to K_7, Kohara and Suzuki established the following.

Theorem 2.2 *[12] The thirteen graphs obtained from K_7 by one or a series of $\nabla\mathbf{Y}$-moves are intrinsically knotted.*

The *Heawood graph*, as shown in Figure 2, is a bipartite graph with 14 vertices and 21 edges. Kohara and Suzuki identified the Heawood graph (C_{14} in their notation) as a descendant of K_7, giving us the following corollary.

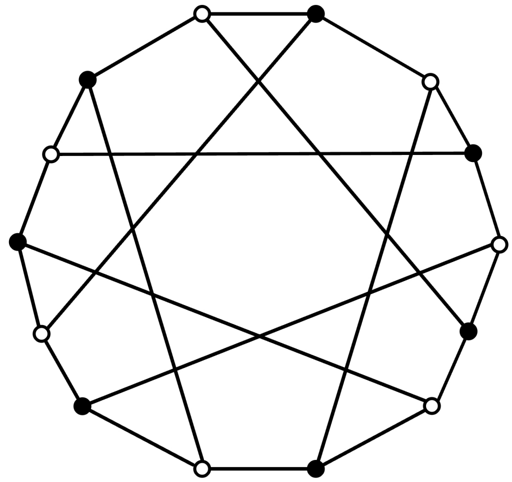

Figure 2: The Heawood graph

Corollary 2.3 *[12] The Heawood graph is intrinsically knotted.*

3 Tools, Notation, and Strategy

The *restoring method* of Kim, Mattman, and Oh [11] is part of a strategy for using initial partial information about the edges and vertices for a graph, what we call a *starting configuration*, to determine if it's possible to build an intrinsically knotted graph from that starting configuration. In the cases where it may be possible to do so, the restoring method provides information about what must be added to a starting configuration to build such a graph. We use this section to review the results that underlie this strategy, outline the strategy, and include an example of how the strategy can be used to determine that a particular starting configuration cannot be used to construct an intrinsically knotted graph.

3.1 Planarity and Intrinsically Knotted Graphs

The strategy for identifying potential intrinsically knotted graphs stems from the next result. Before stating it, we remind the reader of the following construction for creating a new graph from graphs G_1 and G_2.

Definition 3.1 *For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the join of G_1 and G_2, denoted $G_1 * G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set*

$$E = E_1 \cup E_2 \cup \{(v, w) \mid v \in V_1, w \in V_2\}.$$

Theorem 3.2 *[2, 2.1], [18, 1.9] Let G be a graph and $H = G * K_2$. Then H is intrinsically knotted if and only if G has K_5 or $K_{3,3}$ as a minor, that is, if and only if G is non-planar.*

A graph is *n-apex* if it can be turned into a planar graph by deleting n vertices and the edges attached to those vertices. As an immediate consequence of the Theorem, we have the next corollary.

Corollary 3.3 *If the graph G is 2-apex, then it is not intrinsically knotted.*

Theorem 3.2 has consequences for the following types of minors of a graph G.

Definition 3.4 *Let G be a graph in which all vertices have degree at least 3 and let a and b be two distinct vertices in G. The graph $G \setminus \{a, b\}$ is obtained from G by removing the vertices a and b and all edges that are adjacent to a, b, or both a and b. The graph $G(a, b)$ is obtained from $G \setminus \{a, b\}$ by deleting vertices (and their adjacent edges) that have degree 1. The* simplification *of $G(a, b)$, denoted $\widehat{G}(a, b)$, is obtained from $G(a, b)$ by eliminating all degree 2 vertices by contracting one of their adjacent edges. See Figure 3.*

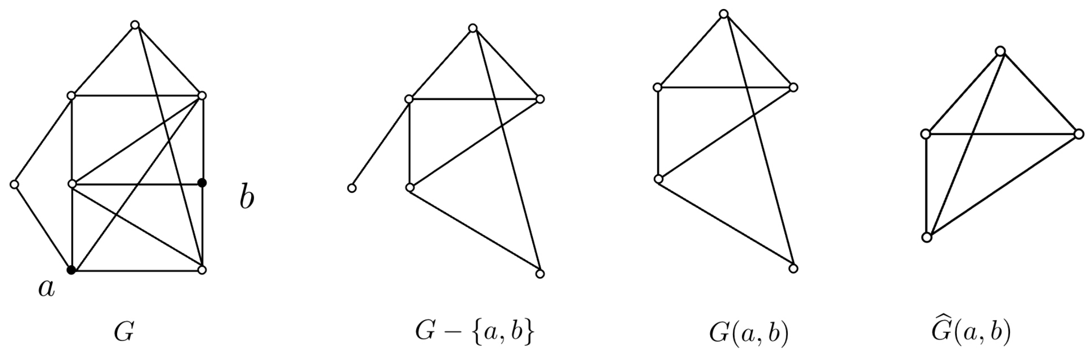

Figure 3: An example of $\widehat{G}(a, b)$

Corollary 3.5 *Let G be a graph and a and b be distinct vertices in G.*

1. *If G is intrinsically knotted, then $G(a, b)$ and $\widehat{G}(a, b)$ must contain K_5 or $K_{3,3}$ as a minor.*

2. *If $G(a, b)$ or $\widehat{G}(a, b)$ have 8 or fewer edges, then G is not intrinsically knotted.*

Proof.

1. Since G is a subgraph of $G \setminus \{a, b\} * K_2$, it follows that if G is intrinsically knotted, then $G \setminus \{a, b\} * K_2$ must be as well. Then Theorem 3.2 implies that $G \setminus \{a, b\}$ must contain K_5 or $K_{3,3}$ as a minor. Deleting a degree 1 vertex or contracting an edge adjacent to a degree 2 vertex does not change this.

2. If $G(a, b)$ or $\widehat{G}(a, b)$ have 8 or fewer edges, then they cannot contain K_5 or $K_{3,3}$ as minors. By (1), G cannot be intrinsically knotted.

□

The preceding results inform the basic strategy of Kim, Mattman, and Oh for determining if a *starting configuration* for a graph, that is, partial information about the degree of the vertices and the edges in a graph, can be used to build an intrinsically knotted graph.

Remark 3.6 The strategy of Kim, Mattman, and Oh for identifying possible intrinsically knotted graphs comprises the following steps [9, 11, §2].

1. Fix a starting configuration C, and assume G is a graph with that starting configuration.
2. Choose two vertices a and b in G and form the graph $\widehat{G}(a,b)$. Determine an upper bound $B(a,b)$ on the numbers of edges in $\widehat{G}(a,b)$.
3. (a) If $B(a,b) \leq 8$, then starting configuration C cannot be used to build an intrinsically knotted graph by Corollary 3.5.
   (b) If $B(a,b) > 8$, identify a collection of edges and vertices in G that can have $K_{3,3}$ or K_5 as a minor, and use that to define a new starting configuration C' from C.
4. Repeat steps until it has been determined that C cannot be used to build an intrinsically knotted graph, or until all edges in a potentially intrinsically knotted graph have been identified.

We note that this strategy does not prove that a graph is intrinsically knotted; instead it is used to eliminate some starting configurations from consideration, and for other starting configurations, it can be used to completely determine the edge structure for G that is necessary (but perhaps not sufficient) for it to be intrinsically knotted. It is this latter outcome, that may involve one or more iterations of step 3(b), that Kim, Mattman, and Oh refer to as the *restoring method*.

3.2 Counting Edges

Before providing an example of the strategy outlined above, we introduce some notation and use it to determine an upper bound on the number of edges in $\widehat{G}(a,b)$. The notation and inequalities that follow are based on the second sections of [9] and [11].

For a graph $G = (V,E)$ and $a,b \in V$, we write $a \sim b$ if a is adjacent to b, that is, if $(a,b) \in E$. For a single vertex a and natural number n,

$$V(a) = \{b \in V \mid b \sim a\}, \text{ and}$$
$$V_n(a) = \{b \in V \mid b \sim a \text{ and } \deg(b) = n\}.$$

For a pair of distinct vertices a and b, and natural number n,

$$V_n^{\cap}(a,b) = V_n(a) \cap V_n(b),$$
$$V_n^{\cup}(a,b) = V_n(a) \cup V_n(b), \text{ and}$$
$$V^{\cup}(a,b) = V(a) \cup V(b).$$

Finally, for a pair of distinct vertices a and b,

$$E(a,b) = \{(a,x) \in E\} \cup \{(b,y) \in E\}.$$

We note that the number of edges in $E(a,b)$ is either the sum of the degrees of a and b if a and b are not adjacent, or 1 less than that sum when they are adjacent.

For a finite set X, we use $|X|$ to indicate the number of elements in X. When $G \setminus \{a,b\}$ is formed from G, the vertices in $V_3^{\cup}(a,b)$, and $V_4^{\cap}(a,b)$ become degree 1 or 2 vertices, which are deleted or contracted away together with an edge in forming $\widehat{G}(a,b)$ from $G \setminus \{a,b\}$. It follows that for a graph $G = (V,E)$, the number of edges in $\widehat{G}(a,b)$ satisfies the inequality

$$|E(\widehat{G}(a,b))| \leq |E| - |E(a,b)| - |V_3^{\cup}(a,b)| - |V_4^{\cap}(a,b)|. \tag{1}$$

It's possible that more edges may be deleted in forming $\widehat{G}(a,b)$. For example, if $v \in V_3^{\cap}(a,b)$ and $w \sim v$ is a degree 3 vertex that is not adjacent or equal to a and b, then w is a degree 2 vertex in $G(a,b)$ and will be contracted away with an edge in forming $\widehat{G}(a,b)$. We use $V_Y(a,b)$ to denote all the vertices of this form. That is,

$$V_Y(a,b) = \{w \in V(G) \mid \deg(w) = 3, w \notin V^{\cup}(a,b), \text{ and } \exists v \in V_3^{\cap}(a,b) \text{ with } v \sim w\}.$$

Using this notation, we have

$$|E(\widehat{G}(a,b))| \leq |E| - |E(a,b)| - |V_3^{\cup}(a,b)| - |V_4^{\cap}(a,b)| - |V_Y(a,b)|. \tag{2}$$

Example 3.7 For our starting configuration, we assume that the graph $G = (A \sqcup B, E)$ is a bipartite graph with 21 edges. As part of this starting configuration, we assume that A has five vertices, $a_1^5, a_1^4, a_2^4, a_3^4, a_4^4$ and B has 6 vertices, $b_1^5, b_1^4, b_1^3, b_2^3, b_3^3$, and b_4^3. The superscripts in the vertex notation indicate the degree of the vertex, that is, vertices a_j^i and b_j^i have degree i.

Consider $\widehat{G}(b_1^5, b_1^4)$. We have $|E(b_1^5, b_1^4)| = 9$ and $|V_4^{\cap}(b_1^5, b_1^4)| = 3$ or 4. If $|V_4^{\cap}(b_1^5, b_1^4)| = 4$, then (1) tells us that $|E(\widehat{G}(b_1^5, b_1^4))| \leq 21 - 9 - 4 = 8$, so $\widehat{G}(b_1^5, b_1^4)$ is planar. Hence, by Corollary 3.5, for G to be intrinsically knotted, we must have $|V_4^{\cap}(b_1^5, b_1^4)| = 3$.

Assuming $|V_4^{\cap}(b_1^5, b_1^4)| = 3$, it must be the case that $b_1^4 \sim a_1^5$, since b_1^5 is adjacent to all of the degree 4 vertices in A. Without loss of generality, we may assume that b_1^4 is also adjacent to a_1^4, a_2^4, and a_3^4. Since a_1^5 has degree 5, it must be adjacent to 3 of the degree 3 vertices in B, say b_1^3, b_2^3, and b_3^3. This new starting configuration is pictured in Figure 4.

Now consider $\widehat{G}(a_1^5, a_1^4)$. Note that $|E(a_1^5, a_1^4)| = 9$, $|V_4^{\cap}(a_1^5, a_1^4)| = 1$, and $|V_3^{\cup}(a_1^5, a_1^4)| \geq 3$. By (1), $|E(\widehat{G}(a_1^5, a_1^4))| \leq 21 - 9 - 1 - 3 = 8$, so by Corollary 3.5, the original starting configuration cannot be part of an intrinsically knotted graph.

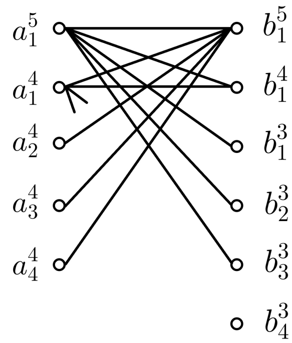

Figure 4: Example 3.7

4 Vertex Degree

As a first step in identifying bipartite graphs with 21 edges that could be intrinsically knotted, we determine the degrees that the vertices in such graphs can have. We begin by eliminating some extreme cases.

Lemma 4.1 *Let G be a graph with 21 edges that has at least one vertex whose degree is less than 3. Then G is not intrinsically knotted.*

Proof. Suppose that G has a vertex v whose degree is 1 or 2.

If the degree of v is 1, then deleting v and its adjacent edge produces a graph that has 20 edges and cannot be intrinsically knotted by Theorem 2.1. Since reinstating v and its adjacent edge does not create any additional cycles, the original graph cannot be intrinsically knotted.

If the degree of v is 2, then one of its adjacent edges can be contracted to produce a graph that has 20 edges and is not intrinsically knotted. Again, reinstating the contracted edge will not produce additional cycles in the graph, so the original graph cannot be intrinsically knotted. □

Lemma 4.2 *If G is a bipartite graph with 21 edges in which each vertex has degree at least 3, then for any vertex v in G, $\deg(v) \leq 7$.*

Proof. Suppose $G = (A \sqcup B, E)$ is a bipartite graph whose vertices all have degree at least 3. If $v \in A$ has degree at least 8, then B must contain at least 8 vertices that are adjacent to v, each of which must have at least 2 additional edges. Since vertices in B are not adjacent to one another this means that G must have at least 24 distinct edges. □

Proposition 4.3 *If $G = (A \sqcup B, E)$ is an intrinsically knotted bipartite graph with 21 edges and v is a vertex in G, then $3 \leq \deg(v) \leq 5$.*

Proof. Lemmas 4.1 and 4.2 guarantee that $3 \leq \deg(v) \leq 7$. We also note that the requirement that each vertex has degree at least 3 implies that $|A| \leq 7$ and $|B| \leq 7$. In the case where $|A| = 7$, respectively $|B| = 7$, each vertex in A, respectively B, must have degree 3 since G only has 21 edges. To proceed, we assume, without loss of generality, that $v \in A$. We show that if $\deg(v) = 7$ or $\deg(v) = 6$, then G cannot be intrinsically knotted.

If $\deg(v) = 7$, then B must have at least 7 vertices. As noted in the previous paragraph, B can have no more than 7 vertices, so it has exactly 7 vertices. Furthermore, each of these must be degree 3. Consider $a \in A$ with $a \neq v$. Then $E(v, a) \geq 10$, $V_3^{\cup}(v, a) = 7$, and $|E(\widehat{G}(v, a))| \leq 21 - 10 - 7 = 4$ by (1). Such a graph is not intrinsically knotted by Corollary 3.5, so G does not have a degree 7 vertex.

Suppose $\deg(v) = 6$. We consider two cases: (i) B has 7 degree 3 vertices or (ii) B has fewer than 7 vertices. In case (i), for $a \in A$ with $a \neq v$, we have $E(v, a) \geq 9$, $V_3^{\cup}(v, a) \geq 6$, and by (1) and Corollary 3.5, $|E(\widehat{G}(a, v)| \leq 21 - 9 - 6 = 6$ so that G cannot be intrinsically knotted.

In case (ii), B has 6 vertices. The condition that G has 21 edges implies that at least 3 vertices in B must have degree 3, and at least one vertex in B must have degree greater than 3. To see this, suppose that B has a degree 3 vertices, b degree 4 vertices, c degree 5 vertices, and d degree 6 vertices. Then $a + b + c + d = 6$ and $3a + 4b + 5c + 6d = 21$ since B has 6 vertices and G has 21 edges. The only nonnegative integer solutions to this system have $a \geq 3$, and at least one of b, c, and d not equal to 0. Let b be the vertex of maximal degree in B. Then $4 \leq \deg(b) \leq 6$.

If $\deg(b) > 4$, then $|E(v, b)| \geq 10$ and $V_3^{\cup}(v, b) \geq 3$. By (1) and Corollary 3.5, $|E(\widehat{G}(v, b))| \leq 21 - 10 - 3 = 8$ and G is not intrinsically knotted.

If $\deg(b) = 4$, then it must be the case that B has exactly 3 degree 3 vertices and 3 degree 4 vertices. Let $a \in A$ such that $a \neq v$ and $a \sim b$. Then we have $|E(v, a)| \geq 9$, $|V_3^{\cup}(v, a)| \geq 3$, and $|V_4^{\cap}(v, a)| \geq 1$. By (1) and Corollary 3.5, $|E(\widehat{G}(v, a))| \leq 21 - 9 - 3 - 1 = 8$ and G is not intrinsically knotted.

Hence $\deg(v)$ cannot exceed 5.

□

Having shown that the vertices in any intrinsically knotted bipartite graph with 21 edges can only be of degrees 3, 4, or 5, we introduce some notation to help us keep track of possibilities.

Definition 4.4 *If V is a set of vertices for a graph, we write $\deg(V) = [c_5, c_4, c_3]$ to indicate that the number of degree 3 vertices in V is c_3, the number of degree 4 vertices in V is c_4, and the number of degree 5 vertices in V is c_5.*

For a bipartite graph $G = (A \sqcup B, E)$ with $|E| = n$, the sum of the degrees of the vertices in each of A and B must equal n. So, for an intrinsically knotted bipartite graph with vertex set $A \sqcup B$ and 21 edges, we can determine all possibilities for $\deg(A)$ and $\deg(B)$ by finding all nonnegative integer solutions to the equation

$$3c_3 + 4c_4 + 5c_5 = 21.$$

Doing so gives us the following.

Proposition 4.5 *Let $G = (A \sqcup B, E)$ be an intrinsically knotted bipartite graph with* 21 *edges. Then* $\deg(A)$ *and* $\deg(B)$ *can equal* $[3, 0, 2]$, $[2, 2, 1]$, $[1, 4, 0]$, $[1, 1, 4]$, $[0, 3, 3]$, *or* $[0, 0, 7]$.

5 Elimination of Values for $\deg(A)$ and $\deg(B)$

In this section, we use the strategy described in Remark 3.6 together with Propositions 4.3 and 4.5 to prove the following.

Proposition 5.1 *If $G = (A \sqcup B, E)$ is an intrinsically knotted bipartite graph with* 21 *edges, then* $\deg(A) = \deg(B) = [0, 0, 7]$.

Proof. Let $G = (A \sqcup B, E)$ be a bipartite graph with 21 edges. We consider three possibilities:

- both A and B contain degree 5 vertices,
- exactly one of A and B has a degree 5 vertex, and
- G has at least one degree 4 vertex, but no degree 5 vertices.

For each of these possibilities, we show that G cannot be intrinsically knotted. By Proposition 4.5, this implies that the only possible value for both $\deg(A)$ and $\deg(B)$ is $[0, 0, 7]$.

We treat these three possibilities in the three subsections that follow. For the vertices in A and B, we continue to use the notation established in Example 3.7: the vertex a_i^j is the ith vertex of degree j in A and b_i^j is the ith vertex of degree j in B.

5.1 Both A and B Contain Degree 5 Vertices

In this case, $\deg(A)$ and $\deg(B)$ can equal $[3, 0, 2]$, $[2, 2, 1]$, $[1, 4, 0]$, $[1, 1, 4]$. Let $D = \{\deg(A), \deg(B)\}$. We consider three subcases:

- $[3, 0, 2]$ or $[2, 2, 1]$ is an element of D,
- $[1, 4, 0] \in D \subseteq \{[1, 4, 0], [1, 1, 4]\}$, and
- $D = \{[1, 1, 4]\}$.

5.1.1 $[3, 0, 2]$ or $[2, 2, 1]$ is an Element of D

Assume $\deg(A) = [3, 0, 2]$ or $[2, 2, 1]$. We will treat the four possibilities for $\deg(B)$ separately.

In case (i), $\deg(B) = [3, 0, 2]$. First suppose $\deg(A) = [3, 0, 2]$. Consider a degree 5 vertex a_i^5 in A and a degree 5 vertex b_j^5 in B. We have $|E(a_i^5, b_j^5)| = 9$ and $|V_3^{\cup}(a_i^5, b_j^5)| = 4$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, b_j^5))| \leq 21-9-4 = 8$ and G is not intrinsically knotted. Otherwise suppose $\deg(A) = [2, 2, 1]$. For any two of the degree five vertices, b_i^5 and b_j^5, in

B, we have $|E(b_i^5, b_j^5)| = 10$, $|V_4^{\cap}(b_i^5, b_j^5)| = 2$, and $|V_3^{\cup}(b_i^5, b_j^5)| = 1$. By (1) and Corollary 3.5, $|E(\widehat{G}(b_i^5, b_j^5))| \leq 21 - 10 - 2 - 1 = 8$ and G is not intrinsically knotted.

In case (ii), $\deg(B) = [2, 2, 1]$. For any two of the degree five vertices a_i^5, a_j^5 in A, we have $|E(a_i^5, a_j^5)| = 10$, $|V_4^{\cap}(a_i^5, a_j^5)| = 2$, and $|V_3^{\cup}(a_i^5, a_j^5)| = 1$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 10 - 2 - 1 = 8$ and G is not intrinsically knotted.

In case (iii), $\deg(B) = [1, 4, 0]$. For any two of the degree five vertices a_i^5, a_j^5 in A, we have $|E(a_i^5, a_j^5)| = 10$ and $|V_4^{\cap}(a_i^5, a_j^5)| = 4$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 10 - 4 = 7$ and G is not intrinsically knotted.

In case (iv), $\deg(B) = [1, 1, 4]$. For any two of the degree five vertices a_i^5, a_j^5 in A, we have $|E(a_i^5, a_j^5)| = 10$, and $|V_3^{\cup}(a_i^5, a_j^5)| \geq 3$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 10 - 3 = 8$ and G is not intrinsically knotted.

5.1.2 $[1, 4, 0] \in D \subset \{[1, 4, 0], [1, 1, 4]\}$

Without loss of generality, assume $\deg(A) = [1, 4, 0]$. There are two cases to consider: $\deg(B) = [1, 4, 0]$ or $\deg(B) = [1, 1, 4]$. The second case was treated in Example 3.7. For the first case, consider the degree five and four vertices, a_1^5, a_1^4, in A. We have $|E(a_1^5, a_1^4)| = 9$ and $|V_4^{\cap}(a_1^5, a_1^4)| \geq 3$. By (1), $|E(\widehat{G}(a_1^5, a_1^4))| \leq 21 - 9 - 3 = 9$. Thus, by Corollary 3.5, for G to be intrinsically knotted, we must have $\widehat{G}(a_1^5, a_1^4) \cong K_{3,3}$. However, in creating $\widehat{G}(a_1^5, a_1^4)$ from G, no degree 1 vertices are created. This implies that the degrees of a_2^4, a_3^4, and a_4^4 are unchanged. Since $\widehat{G}(a_1^5, a_1^4)$ has a degree 4 vertex, it cannot be $K_{3,3}$ and G is not intrinsically knotted.

5.1.3 $D = \{[1, 1, 4]\}$

In this case, $\deg(A) = \deg(B) = [1, 1, 4]$. Consider the degree five vertices a_1^5 and b_1^5. We have $|E(a_1^5, b_1^5)| \geq 9$ and $|V_3^{\cup}(a_1^5, b_1^5)| \geq 6$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, b_1^5))| \leq 21 - 9 - 6 = 6$ and G is not intrinsically knotted.

5.2 Exactly One of A and B has a Degree 5 Vertex

Without loss of generality, assume only A has degree 5 vertices. By Proposition 4.5, $\deg(A)$ can be $[3, 0, 2]$, $[2, 2, 1]$, $[1, 4, 0]$ or $[1, 1, 4]$ and $\deg(B)$ can be $[0, 3, 3]$ or $[0, 0, 7]$. We handle the cases where A has more than one degree 3 vertex first, then consider each remaining possibility separately.

5.2.1 $\deg(A) = [3, 0, 2]$ or $\deg(A) = [2, 2, 1]$

We will consider two cases: (i) $\deg(B) = [0, 3, 3]$ or (ii) $\deg(B) = [0, 0, 7]$. For case (i), consider any two of the degree five vertices, a_i^5, a_j^5, in A. We have $|E(a_i^5, a_j^5)| = 10$, $|V_3^{\cup}(a_i^5, a_j^5)| \geq 2$, and $|V_4^{\cap}(a_i^5, a_j^5)| \geq 1$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 10 - 2 - 1 = 8$ and G is not intrinsically knotted.

In case (ii), $\deg(B) = [0,0,7]$. For any two of the degree five vertices, a_i^5, a_j^5, in A, we have $|E(a_i^5, a_j^5)| = 10$ and $|V_3^{\cup}(a_i^5, a_j^5)| \geq 5$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 10 - 5 = 6$ and G is not intrinsically knotted.

5.2.2 $\deg(A) = [1,4,0]$ and $\deg(B) = [0,3,3]$

Consider the degree 5 vertex a_1^5 in A. We distinguish two cases: (i) $|V_4(a_1^5)| = 3$, and (ii) $|V_4(a_1^5)| = 2$. In the first case, $|V_4(a_1^5)| = 3$. Then we have $a_1^5 \sim b_1^4, b_2^4, b_3^4$. Since a_1^5 has degree five, we may assume $a_1^5 \sim b_1^3, b_2^3$. Since b_3^3 is not adjacent to a_1^5, b_3^3 is adjacent to three degree 4 vertices in A. We may assume $b_3^3 \sim a_1^4, a_2^4, a_3^4$. Consider $\widehat{G}(a_1^5, a_4^4)$. We have $|E(a_1^5, a_4^4)| = 9$, $|V_3^{\cup}(a_1^5, a_4^4)| = 3$, and $|V_4^{\cap}(a_1^5, a_4^4)| \geq 1$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_i^5, a_j^5))| \leq 21 - 9 - 3 - 1 = 8$ and G is not intrinsically knotted.

In the second case, suppose $|V_4(a_1^5)| = 2$. We may assume $a_1^5 \sim b_1^4, b_2^4$. Since a_1^5 has degree five, a_1^5 is adjacent to all three degree 3 vertices, b_1^3, b_2^3, b_3^3. Since $b_3^4 \not\sim a_1^5$, we must have $b_3^4 \sim a_1^4, a_2^4, a_3^4, a_4^4$. Since b_1^4 has degree four, it is adjacent to three degree 4 vertices in A, say a_1^4, a_2^4, a_3^4. See Figure 5(a). Consider $\widehat{G}(a_1^5, b_3^4)$. We have $|E(a_1^5, b_3^4)| = 9$ and $|V_3^{\cup}(a_1^5, b_3^4)| \geq 3$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, b_3^4))| \leq 21 - 9 - 3 = 9$. For G to be intrinsically knotted, we must have $\widehat{G}(a_1^5, b_3^4) \cong K_{3,3}$. In $\widehat{G}(a_1^5, b_3^4)$, $a_1^4, a_2^4, a_3^4, a_4^4, b_1^4$, and b_2^4 have degree three. Since $b_1^4 \sim a_1^4, a_2^4, a_3^4$, the two sets of vertices in $\widehat{G}(a_1^5, b_3^4)$ are $\{a_1^4, a_2^4, a_3^4\}$ and $\{b_1^4, b_2^4, a_4^4\}$. Thus, we must have $b_2^4 \sim a_1^4, a_2^4, a_3^4$ in G. See Figure 5(b). The dotted edges represent the edges deleted to form $G \setminus \{a_1^5, b_3^4\}$ and the blue edges are the edges of G we "restore" to guarantee $\widehat{G}(a_1^5, b_3^4) \cong K_{3,3}$. Now consider $\widehat{G}(a_1^5, a_1^4)$. We have $|E(a_1^5, a_1^4)| = 9$, $|V_3^{\cup}(a_1^5, a_1^4)| = 3$, and $|V_4^{\cap}(a_1^5, a_1^4)| = 2$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, a_1^4))| \leq 21 - 9 - 3 - 2 = 7$ and G is not intrinsically knotted.

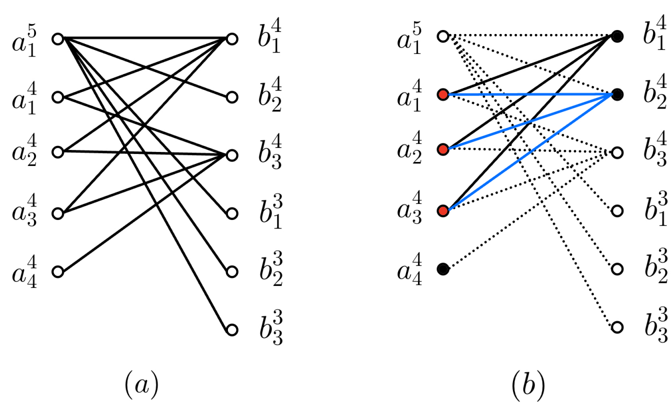

Figure 5: 5.2.2, case (ii)

5.2.3 $\deg(A) = [1,4,0]$ and $\deg(B) = [0,0,7]$

In this case, consider the degree five vertex a_1^5 and a degree four vertex a_1^4 in A. We have $|E(a_1^5, a_1^4)| = 9$, and $|V_3^{\cup}(a_1^5, a_1^4)| \geq 5$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, a_1^4))| \leq$

$21 - 9 - 5 = 7$ and G is not intrinsically knotted.

5.2.4 $\deg(A) = [1, 1, 4]$ and $\deg(B) = [0, 3, 3]$

Consider the degree 5 vertex a_1^5 in A. We distinguish two cases: (i) $|V_4(a_1^5)| = 3$, and (ii) $|V_4(a_1^5)| = 2$.

In case (i), we have $|V_4(a_1^5)| = 3$ and $|V_3(a_1^5)| = 2$. The vertex a_1^4 is adjacent to at least one degree 4 vertex in B. If a_1^4 is adjacent to exactly one degree 4 vertex in B, then it is adjacent to three degree 3 vertices in B and

$$|V_4^{\cap}(a_1^5, a_1^4)| + |V_3^{\cup}(a_1^5, a_1^4)| = 1 + 3 = 4.$$

If a_1^4 is adjacent to 2 or more degree 4 vertices in B, then

$$|V_4^{\cap}(a_1^5, a_1^4)| + |V_3^{\cup}(a_1^5, a_1^4)| \geq 2 + 2 = 4.$$

In either case, since $E(a_1^5, a_1^4) = 9$, we have $|E(\widehat{G}(a_1^5, a_1^4))| \leq 21 - 9 - 4 = 8$ by (1), and G is not intrinsically knotted by Corollary 3.5.

In case (ii), $|V_4(a_1^5)| = 2$ and $|V_3(a_1^5)| = 3$. Suppose b_1^4 is the degree 4 vertex in B that is not adjacent to a_1^5. We have $|E(a_1^5, b_1^4)| = 9$ and $|V_3^{\cup}(a_1^5, b_1^4)| \geq 6$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, b_1^4))| \leq 21 - 9 - 6 = 6$ and G is not intrinsically knotted.

5.2.5 $\deg(A) = A = [1, 1, 4]$ and $\deg(B) = [0, 0, 7]$

In this case, consider the degree five vertex a_1^5 and the degree four vertex a_1^4 in A. We have $|E(a_1^5, a_1^4)| = 9$ and $|V_3^{\cup}(a_1^5, a_1^4)| \geq 5$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^5, a_1^4))| \leq 21 - 9 - 5 = 7$ and G is not intrinsically knotted.

5.3 G has at Least One Degree 4 Vertex, but no Degree 5 Vertices

We divide this subsection into two parts – one to consider the case where both A and B have a degree 4 vertex and the other to consider the case where only one does.

5.3.1 Both A and B Have at Least one Degree 4 Vertex

In this case, $\deg(A) = [0, 3, 3]$ and $\deg(B) = [0, 3, 3]$.

We claim that the starting configuration cannot be part of an intrinsically knotted graph if there exists a degree 4 vertex that is adjacent to three degree 3 vertices. For example, if $|V_3(a_i^4)| = 3$, then there exists a degree 4 vertex b_j^4 such that $b_j^4 \not\sim a_i^4$. Consider $\widehat{G}(a_i^4, b_j^4)$. We have $|E(a_i^4, b_j^4)| = 8$ and $|V_3^{\cup}(a_i^4, b_j^4)| \geq 5$. By (1), we have $|E(\widehat{G}(a_i^4, b_j^4))| \leq 21 - 8 - 5 = 8$. Thus, for G to be intrinsically knotted, we must have $|V_3(a_i^4)| \leq 2$, $|V_4(a_i^4)| \geq 2$, $|V_3(b_i^4)| \leq 2$, and $|V_4(b_i^4)| \geq 2$.

Now consider two degrees 4 vertices in the same set, say a_1^4 and a_2^4. We consider the following cases:

(i) $|V_3^{\cap}(a_1^4, a_2^4)| = 2$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 2$

(ii) $|V_3^{\cap}(a_1^4, a_2^4)| = 2$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 1$

(iii) $|V_3^{\cap}(a_1^4, a_2^4)| = 1$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 3$

(iv) $|V_3^{\cap}(a_1^4, a_2^4)| = 1$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 2$

(v) $|V_3^{\cap}(a_1^4, a_2^4)| = 1$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 1$

(vi) $|V_3^{\cap}(a_1^4, a_2^4)| = 0$

In case (i), $|V_3^{\cap}(a_1^4, a_2^4)| = 2$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 2$. We may assume $a_1^4, a_2^4 \sim b_1^4, b_2^4, b_1^3, b_2^3$. Then b_3^4 must be adjacent to all three degree 3 vertices in A and by our claim, G is not intrinsically knotted.

In case (ii), $|V_3^{\cap}(a_1^4, a_2^4)| = 2$, and $|V_4^{\cap}(a_1^4, a_2^4)| = 1$. We may assume $a_1^4, a_2^4 \sim b_1^4$ and $a_1^4, a_2^4 \sim b_1^3, b_2^3$. Since $|V_4(a_1^4)| \geq 2$, a_1^4 must be adjacent to at least one of b_2^4 or b_3^4, say b_2^4. Note that $b_2^4 \not\sim a_2^4$, since $|V_4^{\cap}(a_1^4, a_2^4)| = 1$. Since $|V_4(b_2^4)| \geq 2$, b_2^4 must be adjacent to a_3^4. Similarly, b_3^4 must be adjacent to a_2^4, a_3^4, since $|V_4(b_4^3)| \geq 2$. This new starting configuration is pictured in Figure 6. Now, we consider two possibilities: (1) $|V_3(b_3^3)| = 3$

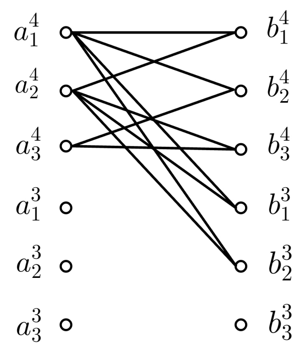

Figure 6: 5.3.1, case (ii)

and (2) $|V_3(b_3^3)| = 2$.

If $|V_3(b_3^3)| = 3$, then $b_3^3 \sim a_1^3, a_2^3, a_3^3$. Since b_3^4 has degree 4, it is adjacent to two degree 3 vertices in A, say a_1^3, a_2^3. See Figure 7(a) for the configuration. Consider $\widehat{G}(a_1^4, b_3^3)$ and Figure 7(b), where dotted edges represent edges removed from the starting configuration to form $\widehat{G}(a_1^4, b_3^3)$. We have $|E(a_1^4, b_3^3)| = 7$ and $|V_3^{\cup}(a_1^4, b_3^3)| = 5$. By (1), we have $|E(\widehat{G}(a_1^4, b_3^3))| \leq 9$. Since $\widehat{G}(a_1^4, b_3^3))$ has degree 4 vertices, it does not contain $K_{3,3}$ as a minor. Thus, $\widehat{G}(a_1^4, b_3^3))$ is planar and G is not intrinsically knotted.

Now consider the case where $|V_3(b_3^3)| = 2$. Then $|V_4(b_3^3)| = 1$ and $b_3^3 \sim a_3^4$. Since $|V_3(b_3^3)| = 2$, we may assume $b_3^3 \sim a_1^3, a_2^3$. See Figure 8(a). Suppose $b_1^4 \sim a_3^4$ as in Figure 8(b). In this case, we have $|E(a_1^4, a_3^4)| = 8$, $|V_4^{\cap}(a_1^4, a_3^4)| = 2$, and $|V_3^{\cup}(a_1^4, a_3^4)| = 3$. By (1) and Corollary 3.5, we have $|E(\widehat{G}(a_1^4, a_3^4))| \leq 8$ and G is not intrinsically knotted.

Otherwise, suppose $b_1^4 \not\sim a_3^4$. Since a_3^4 has degree 4, it must be adjacent to one of b_1^3 and b_2^3. Assume $a_3^4 \sim b_1^3$. Consider $\widehat{G}(a_1^4, b_3^3)$. We have $|E(a_1^4, b_3^3)| = 7$ and $|V_3^{\cup}(a_1^4, b_3^3)| = 4$.

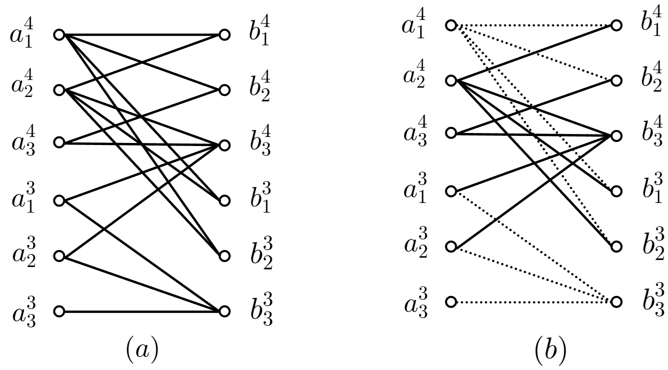

Figure 7: 5.3.1, case(ii), subcase (1)

By (1), we have $|E(\widehat{G}(a_1^4, b_3^3))| \leq 10$. By Corollary 3.5, in order for G to be intrinsically knotted, $\widehat{G}(a_1^4, b_3^3)$ must contain $K_{3,3}$ or K_5 as a minor. Since it doesn't have any degree 5 vertices, it must contain $K_{3,3}$.

We note that there are two degree 4 vertices, a_2^4 and b_3^4, in $\widehat{G}(a_1^4, b_3^3)$, and that they have an edge between them. Thus, it must be the case that $\widehat{G}(a_1^4, b_3^3) - (a_2^4, b_3^4) \cong K_{3,3}$. We also note that a_3^4 is a degree 3 vertex in $\widehat{G}(a_1^4, b_3^3)$ that is adjacent to both a_2^4 and b_3^4. It follows that the two sets of vertices in $\widehat{G}(a_1^4, b_3^3) - (a_2^4, b_3^4)$ are $\{a_2^4, b_2^4, b_3^4\}$ and $\{a_3^4, a_3^3, b_1^4\}$. These sets are colored red and black, respectively, in Figure 8(c). Based on this and what we have established about the edges in G, we know a_3^3 is adjacent to a_2^4, b_2^4, and b_3^4 in $\widehat{G}(a_1^4, b_3^3)$, and we must have a_3^3 adjacent to b_2^3, b_2^4, and b_3^4 in our starting configuration for our graph to be intrinsically knotted. Since b_1^4 has degree 4, we must have $b_1^4 \sim a_1^3, a_2^3$ based on the degree information we have for the vertices in A. The restored edges are colored blue in Figure 8(c). For this configuration, we now consider $\widehat{G}(b_1^4, b_2^3)$. We have $|E(b_1^4, b_2^3)| = 7$, $|V_4^{\cap}(b_1^4, b_2^3)| = 2$, and $|V_3^{\cup}(b_1^4, b_2^3)| = 3$. By (1), we have $|E(\widehat{G}(b_1^4, b_2^3))| \leq 9$. Since $\widehat{G}(b_1^4, b_2^3))$ has degree 4 vertices, it is not isomorphic to $K_{3,3}$. Hence, we have a planar $\widehat{G}(b_1^4, b_2^3))$ and G is not intrinsically knotted.

(iii) Suppose $|V_3^{\cap}(a_1^4, a_2^4)| = 1$ and $|V_4^{\cap}(a_1^4, a_2^4)| = 3$. Then we have $a_1^4, a_2^4 \sim b_1^4, b_2^4, b_3^4$, and we may assume $a_1^4, a_2^4 \sim b_1^3$. This configuration is shown in Figure 9(a). Consider $\widehat{G}(a_1^4, a_2^4)$. We have $|E(a_1^4, a_2^4)| = 8$, $|V_4^{\cap}(a_1^4, a_2^4)| = 3$, and $|V_3^{\cup}(a_1^4, a_2^4)| = 1$. By (1), $|E(\widehat{G}(a_1^4, a_2^4))| \leq 9$.

Observe that if b_1^3 is adjacent to any of the degree 3 vertices in A, then $|V_Y(a_1^4, a_2^4)| = 1$ and by (2), $|E(\widehat{G}(a_1^4, a_2^4)| \leq 8$. So for G to be intrinsically knotted, we must have $b_1^3 \sim a_3^4$. Since a_3^4 has degree 4, it is adjacent to three other vertices. If those three vertices are b_1^4, b_2^4, and b_3^4, then b_2^3 and b_3^3 must be adjacent to a_1^3, a_2^3, and a_3^3 which implies that $|V_3(b_1^4, b_2^4)| = 2$. Then $|E(\widehat{G}(b_1^4, b_2^4))| \leq 8$ and G is not intrinsically knotted. So a_4^3 is adjacent to b_2^3 or b_3^3, say b_2^3. Note that a_3^4 must also be adjacent to two of b_1^4, b_2^4, and b_3^4, since $|V_4(a_3^4)| \geq 2$. We assume, without loss of generality, that $a_3^4 \sim b_1^4, b_2^4$. Then we must have $b_3^3 \sim a_1^3, a_2^3, a_3^3$. See Figure 9(b). If G is intrinsically knotted, then $\widehat{G}(a_1^4, a_2^4)$

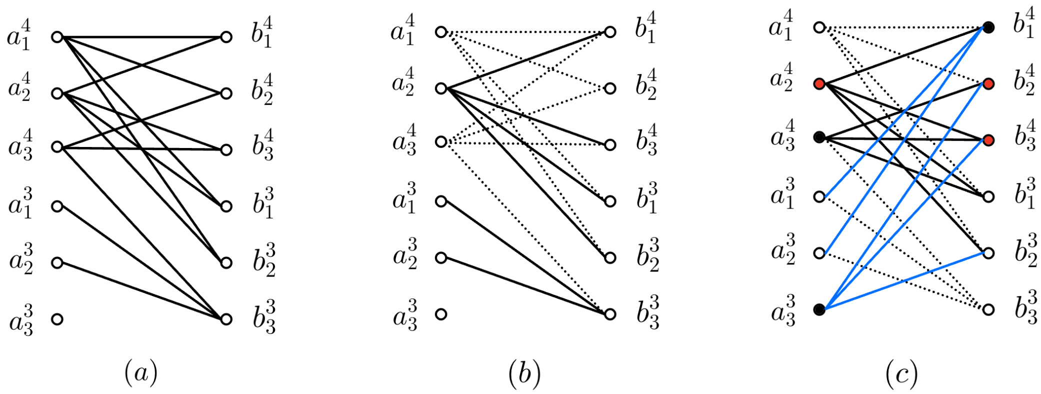

Figure 8: 5.3.1, case(ii), subcase (2)

is isomorphic to $K_{3,3}$. Since $b_3^3 \sim a_1^3, a_2^3, a_3^3$, the vertex sets should be $\{a_1^3, a_2^3, a_3^3\}$ and $\{a_3^4, b_2^3, b_3^3\}$, colored black and red, respectively, in Figure 9(c). However, since a_3^4 and b_2^3 share an edge, $\widehat{G}(a_1^4, a_2^4)$ cannot be $K_{3,3}$.

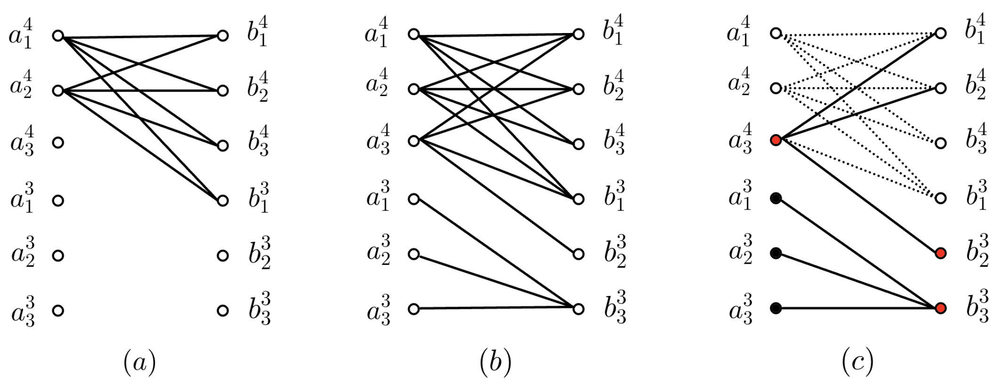

Figure 9: 5.3.1, case (iii)

In case (iv), $|V_3^{\cap}(a_1^4, a_2^4)| = 1, |V_4^{\cap}(a_1^4, a_2^4)| = 2$, and we may assume $a_1^4, a_2^4 \sim b_1^4, b_2^4, b_1^3$. Since $|V_4(b_3^4)| \geq 2$, b_3^4 must be adjacent to a_3^4 and one of a_1^4, a_2^4, say a_1^4. In addition, b_3^4 must be adjacent to two degree 2 vertices, say a_1^3, a_2^3. Since $|V_4^{\cap}(a_1^4, a_2^4)| = 2$, $a_2^4 \not\sim b_3^4$. Thus, a_2^4 is adjacent to a degree 3 vertex in B other than b_1^3, say b_2^3. Since $|V_4(a_3^4)| \geq 2$, a_3^4 is adjacent to at least one of b_1^4, b_2^4. We may assume $a_3^4 \sim b_1^4$. See Figure 10(a) for the new configuration. Consider $\widehat{G}(a_1^4, a_2^4)$. In this case, we have $|E(a_1^4, a_2^4)| = 8$, $|V_4^{\cap}(a_1^4, a_2^4)| = 2$, and $|V_3^{\cup}(a_1^4, a_2^4)| = 2$. By (1), we have $|E(\widehat{G}(a_1^4, a_2^4))| \leq 9$. Observe that $|V_Y(a_1^4, a_2^4)| = 1$, if b_1^3 is adjacent to any of the degree 3 vertex in A, in which case $|E(\widehat{G}(a_1^4, a_1^2))| \leq 8$ by (2). So for G to be intrinsically knotted, we must have $b_1^3 \sim a_3^4$ and $\widehat{G}(a_1^4, a_2^4) \cong K_{3,3}$. Since $b_3^4 \sim a_3^4, a_1^3, a_2^3$, the two sets of vertices in $\widehat{G}(a_1^4, a_2^4)$ are $\{a_3^4, a_1^3, a_2^3\}$ and $\{a_3^3, b_3^4, b_3^3\}$, colored black and red, respectively, in Figure 10(b). Since a_3^3 is adjacent to a_3^4 in $\widehat{G}(a_1^4, a_2^4)$, we have

$a_3^3 \sim b_1^4$. Consider $\widehat{G}(b_1^4, b_3^4)$. We have $|E(b_1^4, b_3^4)| = 8$, $|V_4^{\cap}(b_1^4, b_3^4)| = 2$, and $|V_3^{\cup}(b_1^4, b_3^4)| = 3$. By (1), we have $|E(\widehat{G}(b_1^4, b_3^4))| \leq 8$. Hence, $\widehat{G}(b_1^4, b_3^4)$ is planar and G is not intrinsically knotted.

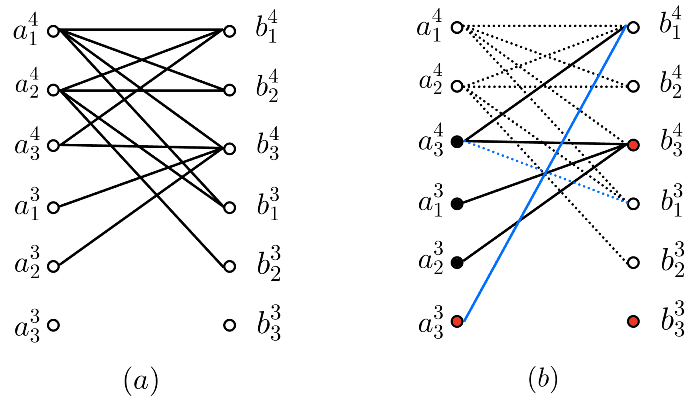

Figure 10: 5.3.1, case (iv)

In case (v), $|V_4^{\cap}(a_1^4, a_2^4)| = 1$ and $|V_3^{\cap}(a_1^4, a_2^4)| = 1$. We may assume $a_1^4 \sim b_1^4, b_2^4, b_1^3$ and $a_2^4 \sim b_1^4, b_3^4, b_1^3$. Since $|V_4(b_2^4)| \geq 2$, we must have $b_2^4 \sim a_3^4$. Similarly, we have $b_3^4 \sim a_3^4$. Since $|V_4^{\cap}(a_1^4, a_2^4)| = 1$, a_1^4 and a_2^4 are adjacent to exactly two degree 4 vertices in B. Therefore, a_1^4 and a_2^4 are adjacent to exactly two degree 3 vertices in B. Since $|V_3^{\cap}(a_1^4, a_2^4)| = 1$, We may assume $a_1^4 \sim b_2^3$ and $a_2^4 \sim b_3^3$. Since b_3^4 has degree 4, we may assume $b_3^4 \sim a_1^3, a_2^3$. See Figure 11(a) for the configuration. Consider $\widehat{G}(a_1^4, a_2^4)$. We have $|E(a_1^4, a_2^4)| = 8$, $|V_4^{\cap}(a_1^4, a_2^4)| = 1$, and $|V_3^{\cup}(a_1^4, a_2^4)| = 3$. By (1), we have $|E(\widehat{G}(a_1^4, a_2^4))| \leq 9$. Observe that $|V_Y(a_1^4, a_2^4)| = 1$, if b_1^3 is adjacent to any of the degree 3 vertices in A. For G to be intrinsically knotted, we must have $b_1^3 \sim a_3^4$. If G is intrinsically knotted, then $\widehat{G}(a_1^4, a_2^4)$ is isomorphic to $K_{3,3}$. Based on the remaining edges, we see that the vertex sets must be $\{a_3^4, a_1^3, a_2^3\}$ and $\{a_3^3, b_2^4, b_3^4\}$ which are colored black and red, respectively in Figure 11(b). It follows that $b_2^4 \sim a_1^3, a_2^3$. Based on the degree and edge information we have for the vertices in B, we must have $a_3^3 \sim b_1^4, b_2^3, b_3^3$. See Figure 11(b). Consider $\widehat{G}(a_1^4, b_3^4)$. We have $|E(a_1^4, b_3^4)| = 8$, and $|V_3^{\cup}(a_1^4, b_3^4)| = 4$. By (1), we have $|E(\widehat{G}(a_1^4, b_3^4)| \leq 9$. If G is intrinsically knotted, then $\widehat{G}(a_1^4, b_3^4)$ is isomorphic to $K_{3,3}$. Based on the remaining edges, we see that the vertex sets must be $\{a_2^4, b_2^4, a_3^3\}$ and $\{b_1^4, a_3^4, b_3^3\}$ which are colored black and red, respectively, in 11(c). Since a_3^4 is adjacent to a_3^3 in $\widehat{G}(a_1^4, a_3^4)$, we have $a_3^4 \sim b_2^3$. By the degree information of vertices in A, b_1^4 and b_3^3 must be adjacent to a_1^3, a_2^3. We may assume $b_1^4 \sim a_1^3$ and $b_3^3 \sim a_2^3$. Now, $|E(\widehat{G}(a_3^4, a_2^3))| \leq 9$, since $|E(a_3^4, a_2^3)| = 7$, $|V_4^{\cap}(a_3^4, a_2^3)| = 2$, and $|V_3^{\cup}(a_3^4, a_2^3)| = 3$. But $\widehat{G}(a_3^4, a_2^3)$ is not isomorphic to $K_{3,3}$, since it has three degree 4 vertices.

In case (vi), $|V_3^{\cap}(a_1^4, a_2^4)| = 0$. It follows that $|V_3^{\cup}(a_1^4, a_2^4)| = 2$ or 3. If $|V_3^{\cup}(a_1^4, a_2^4)| = 2$, then a_1^4 and a_2^4 share 6 edges with the degree 4 vertices in B. Then $|E(\widehat{G}(a_1^4, a_2^4))| \leq 8$, since $|V_4^{\cap}(a_3^4, a_2^3)| = 3$. If $|V_3^{\cup}(a_1^4, a_2^4)| = 3$, then $|V_4^{\cap}(a_1^4, a_2^4)| = 2$ and again, we have $|E(\widehat{G}(a_1^4, a_2^4))| \leq 8$. By Corollary 3.5, G is not intrinsically knotted.

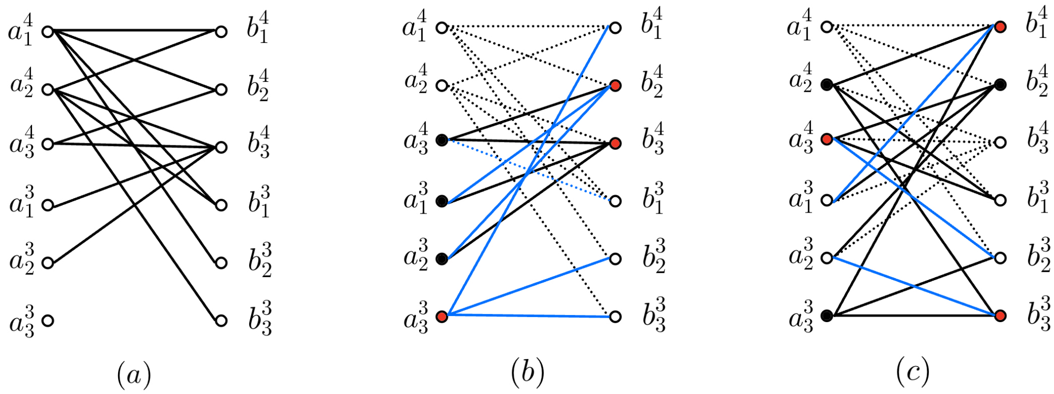

Figure 11: 5.3.1, case (v)

5.3.2 Only one of A and B Contains Degree 4 Vertices

Without loss of generality, suppose only A contains degree 4 vertices. Then $\deg(A) = [0,3,3]$ and $\deg(B) = [0,0,7]$. First, we show that G cannot be intrinsically knotted if any degree three vertex in B is adjacent to more than one degree four vertex in A. Suppose $|V_4(b_1^3)| \geq 2$, then there exists a degree four vertex a_1^4 in A that is not adjacent to b_1^3. Consider $\widehat{G}(a_1^4, b_1^3)$. We have $|E(a_1^4, b_1^3)| = 7$ and $|V_3^{\cup}(a_1^4, b_1^3)| \geq 6$. By (1) and Corollary 3.5, $|E(\widehat{G}(a_1^4, b_1^3))| \leq 21 - 7 - 6 = 8$ and G is not intrinsically knotted.

Since B has 7 vertices, $|V_3^{\cap}(a_i^4, a_j^4)| \geq 1$ for any a_i^4, a_j^4 in A. It follows that at least one of the degree three vertex in B is adjacent to both a_i^4 and a_j^4. Thus, G is not intrinsically knotted. This completes the proof of Proposition 5.1. $\square$

6 The Only Bipartite Intrinsically Knotted Graph With 21 Edges

We use this section to prove our main theorem.

Theorem 6.1 *The only bipartite intrinsically knotted graph with* 21 *edges is the Heawood graph.*

We use the next two lemmas to prove this theorem.

Lemma 6.2 *Let $G = (A \sqcup B, E)$ be a bipartite graph with* $\deg(A) = \deg(B) = [0,0,7]$. *If G contains a 4-cycle, then G is not intrinsically knotted.*

Proof. Suppose G contains a 4-cycle C. We may assume that the vertices of C are a_1^3, b_1^3, a_2^3, and b_2^3. We will consider two cases: (i) $|V_3^{\cup}(a_1^3, a_2^3)| = 3$ or (ii) $|V_3^{\cup}(a_1^3, a_2^3)| = 4$. In case (i), $|V_3^{\cup}(a_1^3, a_2^3)| = 3$ and we may assume $a_1^3, a_2^3 \sim b_3^3$. Since $|V_3^{\cup}(b_1^3, b_2^3, b_3^3)| \leq 5$, at least two vertices in A are not adjacent to b_1^3, b_2^3, b_3^3. Assume $a_3^3 \not\sim b_1^3, b_2^3, b_3^3$. Since a_3^3 has degree three, we may assume $a_3^3 \sim b_4^3, b_5^3, b_6^3$. Since b_7^3 has degree three, it must be adjacent to three vertices in A other than a_1^3, a_2^3, a_3^3. So we may assume $b_7^3 \sim a_4^3, a_5^3, a_6^3$. See Figure 12(a). Consider $\widehat{G}(a_1^3, a_3^3)$. We have $|E(a_1^3, a_3^3)| = 6$, and $|V_3^{\cup}(a_1^3, a_3^3)| = 6$. By (1), $|E(\widehat{G}(a_1^3, a_3^3))| \leq 21 - 6 - 6 = 9$. For G to be intrinsically knotted, we must have $\widehat{G}(a_1^3, a_3^3) \cong K_{3,3}$. Since $b_7^3 \sim a_4^3, a_5^3, a_6^3$ in $\widehat{G}(a_1^3, a_3^3)$, the vertex sets must be $\{a_4^3, a_5^3, a_6^3\}$ and $\{a_2^3, a_7^3, b_7^3\}$, colored red and black, respectively in Figure 12(b). Since a_2^3 is adjacent to a_4^3, a_5^3, a_6^3 in $\widehat{G}(a_1^3, a_3^3)$, we may assume $b_1^3 \sim a_4^3$,$b_2^3 \sim a_5^3$, and $b_3^3 \sim a_6^3$. Since a_7^3 is adjacent to a_4^3, a_5^3, a_6^3 in $\widehat{G}(a_1^3, a_3^3)$, we may assume a_7^3 connects to a_4^3, a_5^3, a_6^3 in $\widehat{G}(a_1^3, a_3^3)$ via b_4^3, b_5^3, b_6^3, respectively. See Figure 12(b). Consider $\widehat{G}(a_4^3, b_7^3)$, as shown in Figure 12(c). It is straightforward to show directly that this graph has a planar embedding, so G is not intrinsically knotted.

In case (ii), $|V_3^{\cup}(a_1^3, a_2^3)| = 4$. We may assume $a_1^3 \sim b_3^3$ and $a_2^3 \sim b_4^3$. If $|V_3^{\cup}(b_1^3, b_2^3)| = 3$, then we are done by case (i). Therefore, $|V_3^{\cup}(b_1^3, b_2^3)| = 4$ and we assume $b_1^3 \sim a_3^3$, and $b_2^3 \sim a_4^3$. Since a_3^3 has degree 3 and $a_3^3 \sim b_1^3$, at least one of b_5^3, b_6^3, b_7^3 is not adjacent to a_3^3, say b_5^3. Consider $\widehat{G}(a_3^3, b_5^3)$. We have $|E(a_3^3, b_5^3)| = 6$, and $|V_3^{\cup}(a_1^3, b_5^3)| = 6$. By (1), $|E(\widehat{G}(a_3^3, b_5^3))| \leq 21 - 6 - 6 = 9$. Since a_1^3, a_2^3,and b_2^3 form a triangle in $\widehat{G}(a_3^3, b_5^3)$, it cannot be isomorphic to $K_{3,3}$. Therefore, G is not intrinsically knotted. See Figure 13. □

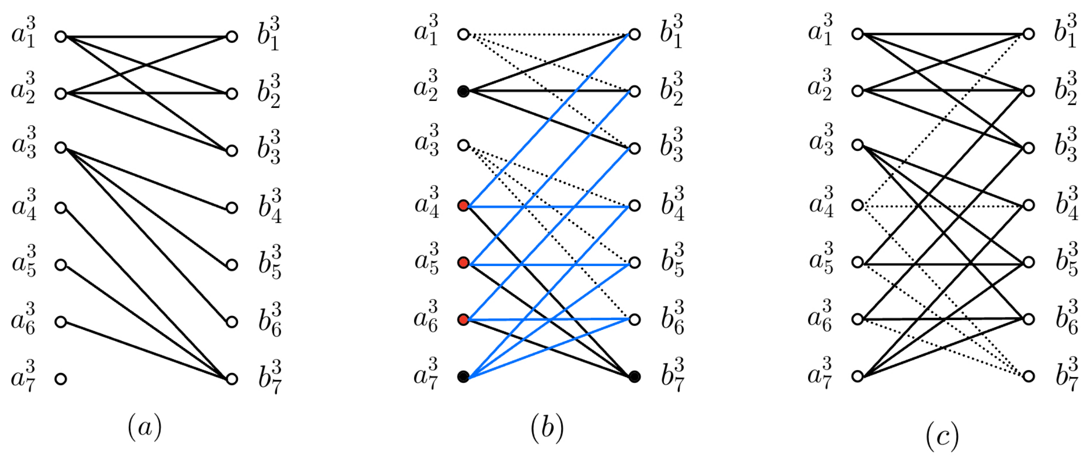

Figure 12: Lemma 6.2, case(i)

Lemma 6.3 *Let $G = (A \sqcup B, E)$ be a bipartite graph with* $\deg(A) = \deg(B) = [0, 0, 7]$. *For any vertices $a_i^3, a_j^3 \in A$ (resp $b_i, b_j \in B$), G is not intrinsically knotted if $V_3^{\cap}(a_i^3, a_j^3) \neq 1$ (resp $V_3^{\cap}(b_i, b_j) \neq 1$).*

Proof. Consider a_1^3, a_2^3 in A. If $V_3^{\cap}(a_1^3, a_2^3) \geq 2$, then G has a 4-cycle, so G is not intrinsically knotted by Lemma 6.2. Now, suppose $V_3^{\cap}(a_1^3, a_2^3) = 0$. We may assume $a_1^3 \sim b_1^3, b_2^3, b_3^3$ and $a_2^3 \sim b_4^3, b_5^3, b_6^3$. Since b_7^3 is not adjacent to a_1^3, a_2^3, we may assume $b_7^3 \sim a_5^3, a_6^3, a_7^3$. Then, $V_3^{\cup}(a_3^3, a_1^3) \leq 4$ or $V_3^{\cap}(a_3^3, a_2^3) \geq 2$. In either case G has a 4-cycle and is not intrinsically knotted by Lemma 6.2.

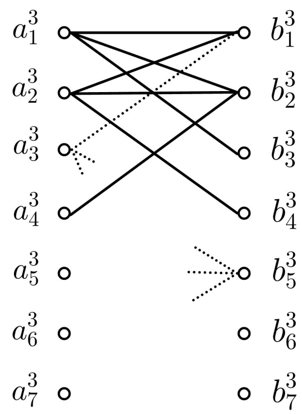

Figure 13: Lemma 6.2, case(ii)

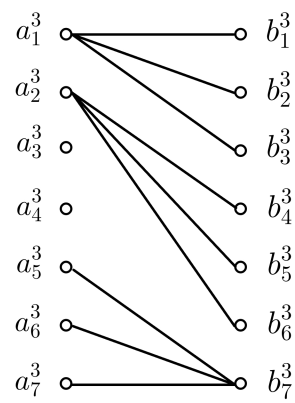

Figure 14: Lemma 6.3

□

We finish with the proof of Theorem 6.1.

Proof. Suppose that $G = (A \sqcup B, E)$ is an intrinsically knotted bipartite graph with 21 edges. Then $\deg(A) = \deg(B) = [0, 0, 7]$ by Proposition 5.1.

Consider a_1^3, a_2^3 in A. For G to be intrinsically knotted, we must have $V_3^{\cap}(a_1^3, a_2^3) = 1$ by Lemma 6.3. We assume $a_1^3 \sim b_1^3, b_2^3, b_3^3$ and $a_2^3 \sim b_3^3, b_4^3, b_5^3$. Since b_3^3 has degree three, we may assume $b_3^3 \sim a_3^3$. For G to be intrinsically knotted, we must have $V_3^{\cap}(a_1^3, a_3^3) = 1$ and $V_3^{\cap}(a_2^3, a_3^3) = 1$. Therefore, $a_3^3 \sim b_6^3, b_7^3$. Since b_6^3 has degree three, we may assume $b_6^3 \sim a_4^3, a_5^3$. For G to be intrinsically knotted, we must have $V_3^{\cap}(b_6^3, b_7^3) = 1$, so $b_7^3 \sim a_6^3, a_7^3$. Since $V_3^{\cap}(b_1^3, b_6^3) = 1$ and $V_3^{\cap}(b_1^3, b_7^3) = 1$, b_1^3 must be adjacent to one of a_4^3, a_5^3, and one of a_6^3, a_7^3. We assume $b_1^3 \sim a_4^3, a_6^3$. Since $V_3^{\cap}(a_4^3, a_2^3) = 1$, a_4^3 must be adjacent to one of b_4^3, b_5^3, say b_4^3. See Figure 15(a). Consider $\widehat{G}(b_6^3, b_7^3)$. we have $|E(b_6^3, b_7^3)| = 6$, $|V_3^{\cup}(b_6^3, b_7^3)| = 5$, and $|V_Y = (b_6^3, b_7^3)| = 1$. By (2), we have $|E(\widehat{G}(b_6^3, b_7^3))| \leq 9$. Thus, for G to be intrinsically knotted, we must have $\widehat{G}(b_6^3, b_7^3) \cong K_{3,3}$. In this case, the two set of vertices of $\widehat{G}(b_6^3, b_7^3)$ are

$\{a_1^3, b_4^3, b_5^3\}$, and $\{b_1^3, b_2^3, a_2^3\}$. Since b_1^3 is adjacent to b_5^3 in $\widehat{G}(b_6^3, b_7^3)$, we must have $a_6^3 \sim b_5^3$. Since $V_3^{\cap}(b_4^3, b_7^3) = 1$, $b_4^3 \sim a_7^3$. Since b_2^3 is adjacent to b_4^3 in $\widehat{G}(b_6^3, b_7^3)$, we have $b_2^3 \sim a_7^3$. Since b_2^3 has degree three, $b_2^3 \sim a_5^3$. Since a_5^3 has degree three, $a_5^3 \sim b_5^3$. See Figure 15(b). The restored graph is the Heawood graph. By Corollary 2.3, the proof is complete.

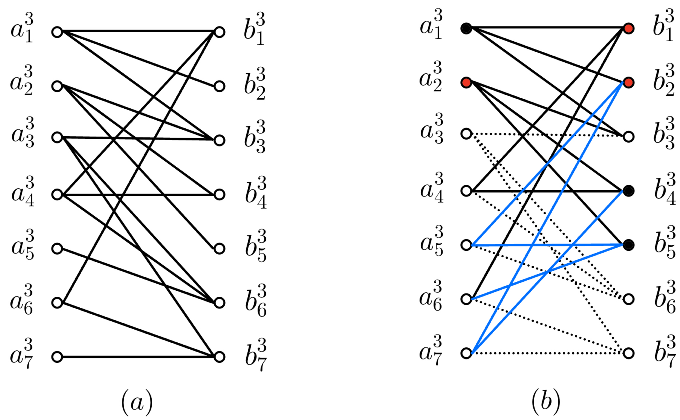

Figure 15: Theorem 6.1

□

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Brenda Johnson
Union College
Department of Mathematics
807 Union Street
Schenectady, NY 12308
E-mail: `johnsonb@union.edu`

Shizhe Li
Department of Statistics
Stanford University
Stanford, CA 94305-4020
E-mail: `shizheli@stanford.edu`

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