Self and Mixed Delta-Moves on Algebraically Split Links

A. Bosman, D. Garcia, J. Goode,
Y. Kas-Danouche, and D. Smith*

Abstract - A Δ-move is a local move on a link diagram. The Δ-Gordian distance between links measures the minimum number of Δ-moves needed to move between link diagrams. A self Δ-move only involves a single component of a link whereas a mixed Δ-move involves multiple (2 or 3) components. We prove that two links are mixed Δ-equivalent precisely when they have the same pairwise linking numbers and components; we also give a number of results on how (mixed/self) Δ-moves relate to classical link invariants including the Arf invariant and crossing number. This allows us to produce a graph showing links related by a self Δ-move for algebraically split links with up to 9-crossings. For these links we also introduce and calculate the Δ-splitting number and mixed Δ-splitting number, that is, the minimum number of Δ-moves or mixed Δ-moves needed to separate the components of the link.

Keywords : delta-move; delta-Gordian distance; linking number; algebraically split links

Mathematics Subject Classification (2020) **:** 57K10

1 Introduction

A Δ-move is a local move on a knot or link diagram as in Figure 1. We call two links Δ-equivalent if we can move from a diagram for one to the other with a series of Δ-moves and ambient isotopy. It is well-known that all knots are Δ-equivalent to the unknot and that two links are Δ-equivalent precisely when their pairwise linking numbers are all the same [6]. The Δ-move can be defined with any particular orientation, therefore we only consider unoriented links. Additionally, there is a Δ-pathway between L and L' exactly when there is a Δ-pathway between mL and mL'.

The minimal number of Δ-moves needed to deform one link L into another link L' is called the Δ-Gordian distance, denoted $d_G^{\Delta}(L, L')$. A link is called algebraically split if all of its pairwise linking numbers equal zero. It is well-known that a link L is Δ-equivalent to a trivial link if and only if it is algebraically split [6]. The Δ-unlinking number, denoted $u^{\Delta}(L)$, is the Δ-Gordian distance between L and the trivial link. Similarly, the Δ-unknotting number, $u^{\Delta}(K)$, is the Δ-Gordian distance between a knot K and the unknot. This has been well-studied [8] with many findings extended to algebraically split links [2].

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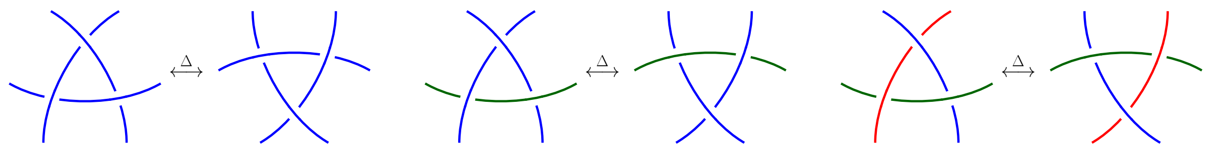

Figure 1: Left: A self Δ-move with all arcs of the same component. Center: Mixed Δ-move containing arcs of two components. Right: Mixed Δ-move containing arcs of three components.

In the case of links with at least two components, we can classify Δ-moves according to the number of components involved in the Δ-move. In particular, if all three strands belong to the same component of the link, the move is called a self Δ-move. If the strands belong to either two or three components, then we will refer to the move as a mixed Δ-move. See Figure 1. We can then introduce the notion of self or mixed Δ-equivalence and self or mixed Δ-Gordian distance, denoted by $d^{s\Delta}$(L,L') and $d^{m\Delta}(L, L')$, by restricting the permitted Δ-moves between to either only self or mixed Δ-moves, respectively.

Links that are Δ-equivalent have been classified using the coefficients of the Conway polynomial [7, 9]. Under this classification, we exhibit in Section 2 graphs of two component algebraically split links with up to 9 crossings by self Δ-moves. This lets us tabulate the Δ-Gordian distance between such pairs of links.

In the case of mixed Δ-moves, we prove the following result:

Theorem 1.1 *Two links are mixed Δ-equivalent if and only if they have the same components and pairwise linking numbers.*

It immediately follows:

Corollary 1.2 *A link L is mixed Δ-equivalent to a split union of its components if and only if L is an algebraically split link.*

Given this result, it is natural to consider the fewest number of mixed Δ-moves needed to deform an algebraically split link into a split union of its components. We call this the mixed Δ-splitting number and denote it $sp^{m\Delta}(L)$. If we instead permit any kind of Δ-move, then we have the Δ-splitting number, denoted $sp^{\Delta}(L)$. In Section 3 we relate $sp^{m\Delta}(L)$ and $sp^{\Delta}(L)$ to the Arf invariant, Δ-unknotting number, and splitting number (as defined in [3]), then we use these relations to determine the (mixed) Δ-splitting number for algebraically split links with up to 9 crossings.

Throughout the paper we follow the Rolfsen naming convention for knots and links as used in Knot Atlas [1].

2 Self Delta-Pathways

The Arf invariant is a binary invariant of a knot that is changed by a Δ-move on a knot. Since a self Δ-move on a link is a Δ-move on a single component of the link, it will change the Arf invariant of that component. Thus, a self Δ-move changes the knot type of the component. A self Δ-pathway between links is a series of transformations by self Δ-moves and ambient isotopy. We call the number of self Δ-moves in a self Δ-pathway the length of the self Δ-pathway.

Proposition 2.1 *If there exist two self Δ-pathways of length n_1 and n_2 between the links L and L', then $n_1 \equiv n_2 \equiv \sum_{i=1}^{m} \operatorname{arf}(L_i) + \sum_{i=1}^{m} \operatorname{arf}(L'_i) \pmod 2$ where each L_i (resp. L'_i) is a component of L (resp. L').*

Proof. A self Δ-move on L changes the parity of $\sum_{i=1}^{m} \operatorname{arf}(L_i)$. Therefore,

$$\sum_{i=1}^{m} \operatorname{arf}(L_i) + n_1 \equiv \sum_{i=1}^{m} \operatorname{arf}(L'_i) \pmod 2$$

and similarly,

$$\sum_{i=1}^{m} \operatorname{arf}(L_i) + n_2 \equiv \sum_{i=1}^{m} \operatorname{arf}(L'_i) \pmod 2.$$

Thus,

$$n_1 \equiv \sum_{i=1}^{m} \operatorname{arf}(L_i) + \sum_{i=1}^{m} \operatorname{arf}(L'_i) \equiv n_2 \pmod 2.$$

□

In particular, since the self Δ-Gordian distance is defined to be the length of the minimal self Δ-pathway between two links, we have:

Corollary 2.2 *Given self Δ-equivalent links L and L',*

$$d_G^{s\Delta}(L, L') \equiv \sum_{i=1}^{m} \operatorname{arf}(L_i) + \sum_{i=1}^{m} \operatorname{arf}(L'_i) \pmod 2.$$

Remark 2.3 The Arf invariant can be extended to proper links [4, 11], that is, links L such that

$$\sum_{1 \le i < j \le m} lk(L_i, L_j) \equiv 0 \pmod 2.$$

In particular, the Arf invariant is well defined for algebraically split links (i.e. links with vanishing pairwise linking number). It has been shown that a Δ-move changes the parity of the Arf invariant of a proper link and therefore $d^{\Delta}(L, L') \equiv \operatorname{arf}(L) + \operatorname{arf}(L') \pmod 2$ [2]. Thus restricting to just self Δ-moves, we have $d^{s\Delta}(L, L') \equiv \operatorname{arf}(L) + \operatorname{arf}(L') \pmod 2$. It then follows that if there exist two Δ-pathways (of self and/or mixed Δ-moves) between the links L and L', they must have the same parity.

Example 2.4 Since both the links $L8a2$ and $L8a4$ can be deformed into the trivial link with a single self Δ-move, $1 \le d_G^{s\Delta}(L8a2, L8a4) \le 2$. See Figure 2. Further

$$d_G^{s\Delta}(L8a2, L8a4) \equiv \operatorname{arf}(L8a2) + \operatorname{arf}(L8a4) \equiv 0 \pmod 2$$

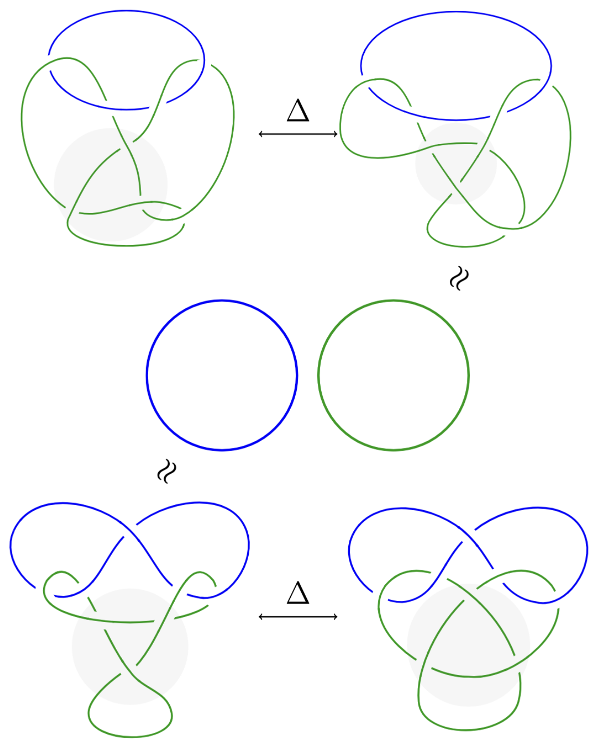

Figure 2: Self Δ-moves connecting $L8a2$ and $L8a4$ via the trivial link.

and therefore $d_G^{s\Delta}(L8a2, L8a4) = 2$.

Additionally, we have the following:

Proposition 2.5 *Given self Δ-equivalent links L and L',*

$$d_G^{s\Delta}(L, L') \geq \sum_{i=1}^{m} \left| u^{\Delta}(L_i) - u^{\Delta}(L_i') \right|.$$

Proof. If L is self Δ-equivalent to L', then (renumbering components, if necessary) there exists a self Δ-pathway taking each component L_i to the component L_i'. Since each self Δ-move only changes the knot type of a single component, we have:

$$d_G^{s\Delta}(L, L') \geq \sum_{i=1}^{m} d_G^{s\Delta}(L_i, L_i') \geq \sum_{i=1}^{m} \left| u^{\Delta}(L_i) - u^{\Delta}(L_i') \right|.$$

□

Example 2.6 There is a self-Δ pathway from $mL5a1$ to $L7n2$ to $L9n3$ to $L9n6$. Since

$$\mathrm{arf}(mL5a1) + \mathrm{arf}(L9n6) = 1 + 0 = 1,$$

(δ_1,δ_2)=(0,0)			
	Trivial	L8a2	L8a4
Trivial	0	1	1
L8a2	-	0	2
L8a4	-	-	0

Table 1: Self Δ-Gordian distance between links.

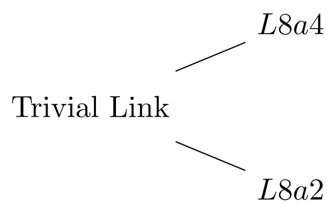

Figure 3: (δ_1,δ_2)=(0,0)

$d^{s\Delta}(mL5a1, L9n6)$ is either 1 or 3. However, since $mL5a1$ has as its components two unknots, $L9n6$ has as its components an unknot and 5_1, and $u^{\Delta}(5_1) = 3$ [10], we conclude $d^{s\Delta}(mL5a1, L9n6) = 3$.

Given a link L, define

$$\delta_1(L) = a_1(L) \text{ and}$$

$$\delta_2(L) = a_3(L) - a_2(L) \times (a_2(L_1) + a_2(L_2))$$

where the a_i are coefficients of the Conway polynomial (see [5] for definition).

Nakanishi-Ohyama showed that two 2-component links are self Δ-equivalent precisely when they have the same values for δ_1 and δ_2 [7][9]. The sign of δ_i is changed by mirroring the link. The below graphs in Figures 3, 4, and 5 exhibit self Δ-pathways for prime links with up to 9 crossing in the families $(\delta_1, \delta_2) = (0, 0)$, $(0, \pm 1)$, and $(0, \pm 2)$. Moreover, for each pair of links L and L', we will use the above restrictions to tabulate in Tables 1, 2, and 3 the possible values for $d_G^{s\Delta}(L, L')$.

Note that if there exists a self Δ-pathway between L and L', then there exists such a pathway of equal length between their mirrors mL and mL', as well. Therefore

$$d_G^{s\Delta}(L, L') = d_G^{s\Delta}(mL, mL').$$

Values in Tables 2 and 3 with asterisks denote upper bounds to $d_G^{s\Delta}(L, L')$; the exact values may be less but must be positive with the same parity as the provided bound.

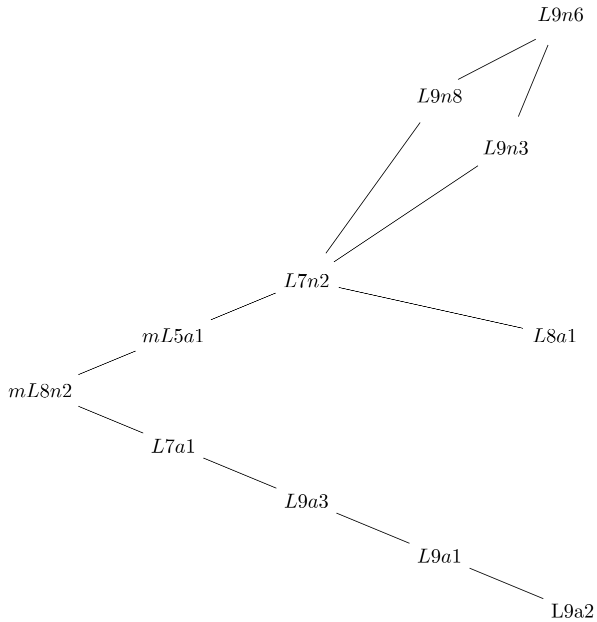

Figure 4: $(\delta_1,\delta_2)=(0,\pm1)$

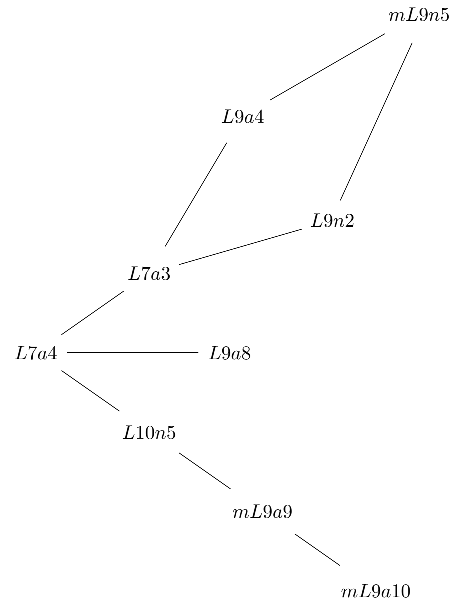

Figure 5: $(\delta_1,\delta_2)=(0,\pm2)$

$(\delta_1,\delta_2)=(0,\pm1)$										
	L7n2	mL8n2	L9n3	L7a1	L9n8	L9a3	L8a1	L9a1	L9n6	L9a2
mL5a1	1	1	2	2	2	3*	2	4*	3	5*
L7n2	0	2	1	3*	1	4*	1	1-5	2	6*
mL8n2	-	0	3	1	2	2	3*	3*	4	4*
L9n3	-	-	0	4*	2	5*	2	6*	1	7*
L7a1	-	-	-	0	4*	1	4*	2	5*	3*
L9n8	-	-	-	-	0	5*	2	6*	1	7*
L9a3	-	-	-	-	-	0	5*	1	6*	2
L8a1	-	-	-	-	-	-	0	6*	3	7*
L9a1	-	-	-	-	-	-	-	0	7*	1
L9n6	-	-	-	-	-	-	-	-	0	8*

Table 2: Self Δ-Gordian distance between links. *Upper bound on the self Δ-Gordian distance.

(δ_1,δ_2)=(0,±2)								
Blank	L7a3	L7a4	L9a4	L9a8	mL9a9	mL9a10	L9n2	mL9n5
L7a3	0	1	1	2	3*	4*	1	2
L7a4	-	0	2	1	2	3*	2	3
L9a4	-	-	0	3	4*	5*	2	1
L9a8	-	-	-	0	3*	4*	3	4
mL9a9	-	-	-	-	0	1	4*	5*
mL9a10	-	-	-	-	-	0	5*	6
L9n2	-	-	-	-	-	-	0	1
mL9n5	-	-	-	-	-	-	-	0

Table 3: Self Δ-Gordian distance between links. *Upper bound on the self Δ-Gordian distance.

3 Delta-Splitting Number

The splitting number of a link is defined to be the minimal number of crossing changes involving different components needed to deform a link into a split union of the components of the link. The splitting number has been tabulated for links with up to 9 crossings [3].

Similarly, as any algebraically split link is Δ-equivalent to the trivial link, we can define the Δ-splitting number $sp^{\Delta}(L)$ to be the minimum number of Δ-moves needed to deform L into the split union of its components. As the split union is not necessarily the trivial link, we have $sp^{\Delta}(L) \leq u^{\Delta}(L)$. The mixed Δ-splitting number $sp^{m\Delta}(L)$ is defined to be the minimum number of mixed Δ-moves needed to deform L into the split union of its components. The following theorem and its corollary show that $sp^{m\Delta}(L)$ is well-defined for algebraically split links.

Theorem 3.1 *Two links are mixed Δ-equivalent if and only if they have the same pairwise linking numbers and components.*

Proof. One direction follows immediately from the fact that mixed Δ-moves preserve linking numbers and does not change knot type of the components. For the converse, suppose m-component links L and L' have the same pairwise linking numbers and equal components $L_i = L'_i$ for $1 \leq i \leq m$. Let λ_{ij} denote $lk(L_i, L_j) = lk(L'_i, L'_j)$ for all $1 \leq i < j \leq m$. There exists a link S with the same linking information as L with the same m components $S_i = L_i$ such that any two components S_i and S_j intersect in λ_{ij} clasps as in Figure 6. In general, such a link S can be formed by taking the split union of the components L_i then band summing with Hopf links with appropriate orientation to achieve the desired linking numbers λ_{ij}. We will argue that L is mixed Δ-equivalent to S, thus L' is also mixed Δ-equivalent to S, and hence L and L' are mixed Δ-equivalent to one another.

First observe that given the components L_i and L_j in L, we can deform them so that the components intersect in pairs of clasps such that the difference between the number

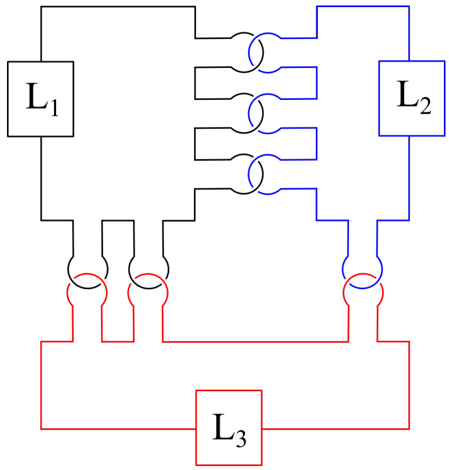

Figure 6: A link S with linking numbers $\lambda_{12} = 3$, $\lambda_{13} = 2$, and $\lambda_{23} = -1$.

of clasps contributing +1 to the linking number and those clasps contributing -1 to the linking number is λ_{ij}. If there exist both positive and negative clasps, we can arrange the diagram such that two clasps with opposite contributions to linking number are adjacent to each other, possibly with other clasps representing intersections of the component with itself or other components between the claps, as in Figure 7.a.

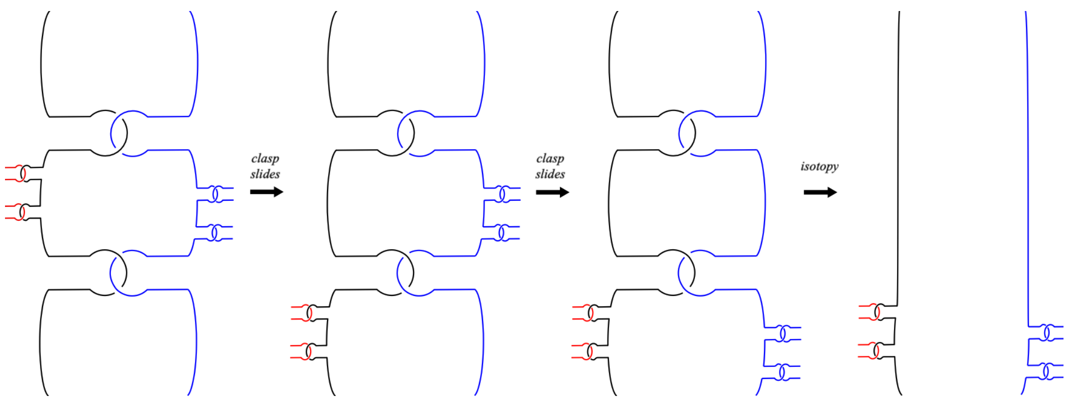

Figure 7: Separating two components of an algebraically split link with claps slides (mixed-Δ moves).

We can move one clasp past another using a single Δ-move, as in Figure 8. Observe that if at least two components are included in one of the clasps, this will be a mixed Δ-move. Repeated application of this move allows us to move any pairs of positive and negative clasps adjacent to each other with no other clasps between them as in Figure 7.b and 7.c. The two clasps then cancel, as in Figure 7.d.

We can repeat this process until all clasps between L_i or L_j are negative or positive. But then there are exactly $|\lambda_{ij}|$ such clasps, the sign of which is determined by the sign of λ_{ij}. Hence, we have transformed L into S via mixed Δ-moves. $\square$

Recall that an algebraically split link is a link with pairwise vanishing linking numbers.

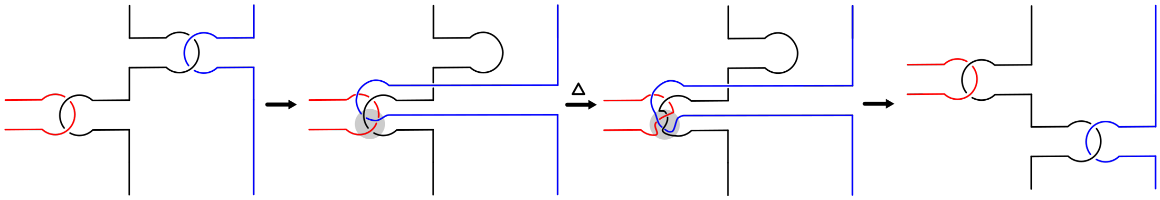

Figure 8: Moving a clasp over another with a mixed-Δ move.

Thus it follows:

Corollary 3.2 *A link L is mixed Δ-equivalent to a split union of its components if and only if L is an algebraically split link.*

Hence the mixed Δ-splitting number, denoted $sp^{m\Delta}(L)$, is well-defined for an algebraically split link L. And since mixed Δ-moves are a subset of Δ-moves more generally (self or mixed), we have $sp^{m\Delta}(L) \geq sp^{\Delta}(L)$. Also, since any two Δ-pathways between L and the split union of the components of L have the same parity (see Remark 2.3), it follows that $sp^{m\Delta}(L) \equiv sp^{\Delta}(L) \pmod 2$. Additionally we have:

Proposition 3.3 *Given an algebraically split link L,*

$$sp^{m\Delta}(L) \equiv \mathrm{arf}(L) + \sum_{i=1}^{m} \mathrm{arf}(L_i) \pmod 2.$$

Proof. Let S denote the split union of the components of L. Then, making use of Remark 2.3, we have:

$$sp^{m\Delta}(L) = d_G^{m\Delta}(L,S) \equiv \mathrm{arf}(L) + \mathrm{arf}(S) \equiv \mathrm{arf}(L) + \sum_{i=1}^{m} \mathrm{arf}(L_i) \pmod 2.$$

$\square$

Example 3.4 The link $L8a2$ has as its components the unknot and 4_1. There exists a length two mixed Δ-pathway to the split union of its components: $L8a4 \xrightarrow{\Delta} mL8n2 \xrightarrow{\Delta} 0_1 \sqcup 4_1$. Therefore $1 \leq sp^{m\Delta}(L8a4) \leq 2$. However, as

$$sp^{m\Delta}(L8a4) \equiv \mathrm{arf}(L8a2) + (\mathrm{arf}(0_1) + \mathrm{arf}(4_1)) = 1 + (0+1) \equiv 0 \pmod 2$$

we have $sp^{m\Delta}(L8a4) = 2$.

Moreover, we have the following relationship:

Proposition 3.5 *Given an algebraically split link L with components $L_1, \ldots, L_m$, we have:*

$$sp^{m\Delta}(L) \geq u^{\Delta}(L) - \sum_{i=1}^{m} u^{\Delta}(L_i).$$

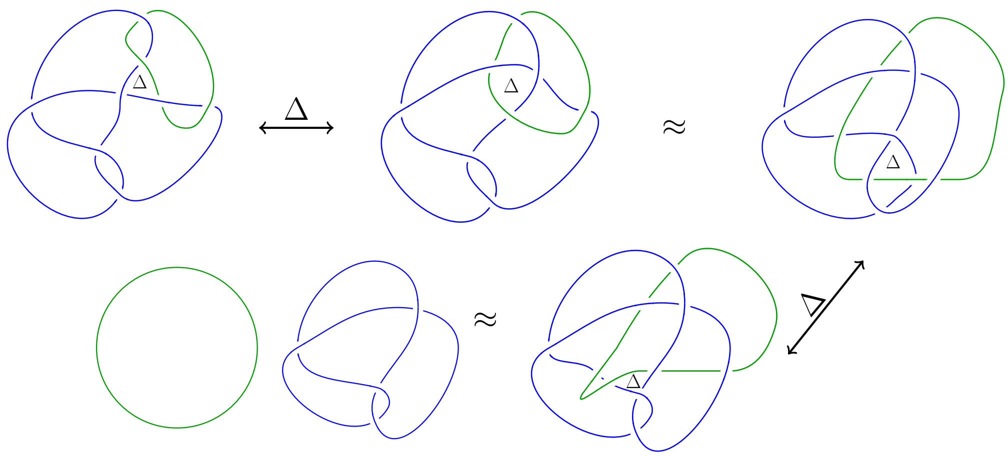

Figure 9: $L8a2$ is two mixed Δ-moves away from the split union of the unknot and 4_1.

Proof. Let S denote the link composed of the split union of $L_1, ..., L_m$. Then L can be deformed into S with $sp^{m\Delta}(L)$ mixed Δ-moves and each component of S can be deformed into the unknot in $u^{\Delta}(L_i)$ Δ-moves. Since mixed Δ-moves preserve the knot type of the components, regardless of if the mixed Δ-move involves 2 or 3 components, and $u^{\Delta}(L)$ is the minimal number of Δ-moves needed to deform L into the trivial link, we have

$$u^{\Delta}(L) \leq sp^{m\Delta}(L) + \sum_{i=1}^{m} u^{\Delta}(L_i).$$

$\square$

Finally, a mixed Δ-move can be achieved with two crossing changes, each between different components, as in Figure 10. Thus $2sp^{m\Delta}(L) \geq sp(L)$ or equivalently:

Proposition 3.6 $sp^{m\Delta}(L) \geq \frac{1}{2} sp(L)$.

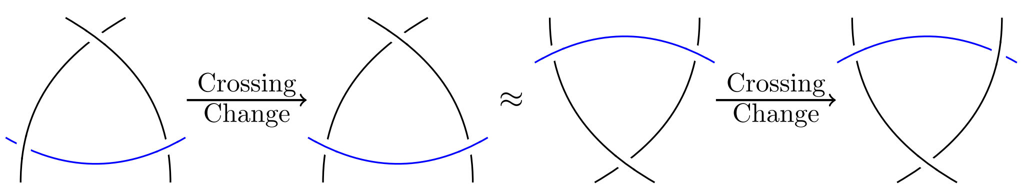

Figure 10: A mixed Δ-move as two crossing changes.

Together, these bounds help us determine the Δ-splitting number and mixed Δ-splitting number for algebraically split links with up to 9 crossings. See Figure 11 and Table 4.

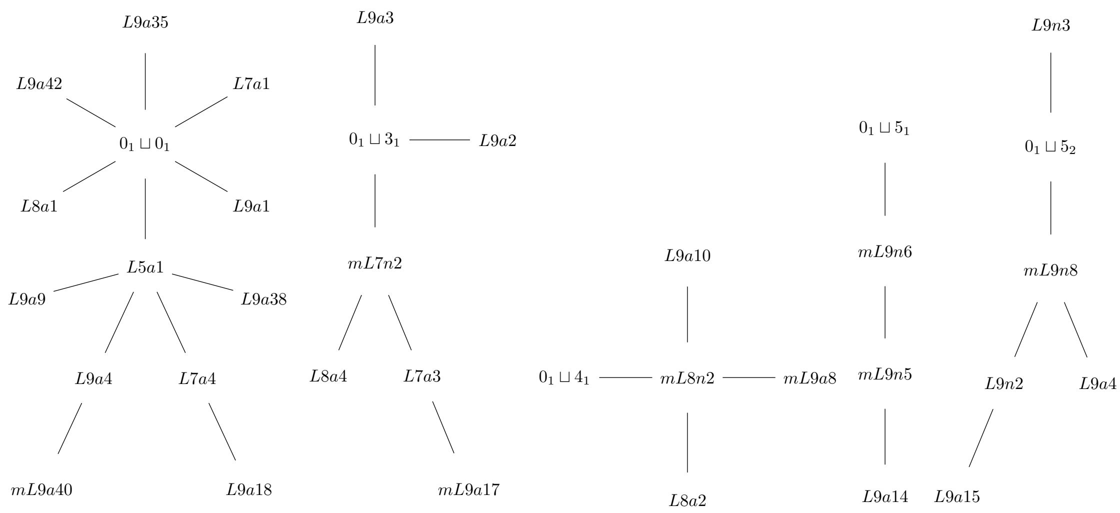

Figure 11: Edges represent a mixed Δ-move.

Link	Components	$sp^{\Delta}(L)$	$sp^{m\Delta}(L)$	$u^{\Delta}(L)$	$\Sigma u^{\Delta}(L_i)$	$sp(L)$	Arf(L)	ΣArf(L_i)
L5a1	$0_1, 0_1$	1	1	1	0	2	1	0
L6a4	$0_1, 0_1, 0_1$	1	1	1	0	2	1	0
L7a1	$0_1, 0_1$	1	1	1	0	2	1	0
L7a3	$0_1, 3_1$	2	2	3	1	2	1	1
L7a4	$0_1, 0_1$	2	2	2	0	2	0	0
L7n2	$0_1, m3_1$	1	1	2	1	2	0	1
L8a1	$0_1, 0_1$	1	1	1	0	2	1	0
L8a2	$0_1, 4_1$	1	2	1	1	2	1	1
L8a4	$0_1, m3_1$	2	2	1	1	2	1	1
L8n2	$0_1, 4_1$	1	1	2	1	2	0	1
L9a1	$0_1, 0_1$	1	1	1	0	2	1	0
L9a2	$0_1, 3_1$	1	1	2	1	2	0	1
L9a3	$0_1, 3_1$	1	1	2	1	2	0	1
L9a4	$0_1, 5_2$	2	2	4	2	2	0	0
L9a8	$0_1, 4_1$	2	2	3	1	2	1	1
L9a9	$0_1, 0_1$	2	2	2	0	2	0	0
L9a10	$0_1, 4_1$	2	2	3	1	2	1	1
L9a14	$0_1, 5_1$	1 or 3	1 or 3	4 or 6	3	2	0	1
L9a15	$0_1, 5_2$	1 or 3	1 or 3	3 or 5	2	2	1	0
L9a17	$0_1, m3_1$	1 or 3	1 or 3	2 or 4	1	2	0	1
L9a18	$0_1, 0_1$	3	3	3	0	2	1	0
L9a35	$0_1, 0_1$	1	1	1	0	2	1	0
L9a38	$0_1, 0_1$	2	2	2	0	2	0	0
L9a40	$0_1, 0_1$	3	3	3	0	4	1	0
L9a42	$0_1, 0_1$	1	1	1	0	2	1	0
L9a53	$0_1, 0_1, 0_1$	1	1	1	0	2	1	0
L9a54	$0_1, 0_1, 0_1$	3	3	3	0	4	1	0
L9n2	$0_1, 5_2$	2	2	4	2	2	0	0
L9n3	$0_1, m5_2$	1	1	3	2	2	1	0
L9n5	$0_1, m5_1$	2	2	5	3	2	1	1
L9n6	$0_1, m5_1$	1	1	4	3	2	0	1
L9n8	$0_1, m5_2$	1	1	3	2	2	1	0
L9n25	$0_1, 0_1, 0_1$	2	2	2	0	2	0	0
L9n27	$0_1, 0_1, 0_1$	2	2	2	2	4	0	0

Table 4: Determine (mixed) Δ-splitting number.

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Anthony Bosman
Andrews University
8975 Old 31
Berrien Springs, MI 49104
E-mail: `bosman@andrews.edu`

Devin Garcia
Andrews University
8975 Old 31
Berrien Springs, MI 49104
E-mail: `gdevin@andrews.edu`

Justyce Goode
Andrews University
8975 Old 31
Berrien Springs, MI 49104
E-mail: `justyce@andrews.edu`

Yamil Kas-Danouche
Andrews University
8975 Old 31
Berrien Springs, MI 49104
E-mail: yamil@andrews.edu

Davielle Smith
Andrews University
8975 Old 31
Berrien Springs, MI 49104
E-mail: davielle@andrews.edu

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