Investigations in Patterns in Last Digits of Square Numbers and Higher Powers

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Abstract - It is interesting to see what patterns exist in the final digits of powers of integers; the goal of this work is to introduce some new problems and results, which are ideally suited for interested readers to pursue further. We start our investigation with squares and say a square is a k-square good number if its last k digits base 10 are the same non-zero number, and the last $k+1$ digits are not identical. We completely analyze k-square good numbers; when $k = 1$ we can have final digits of 1, 4, 5, 6 and 9 (with the second to last digit different), when $k = 2$ we can end with 44 but not 444, when $k = 3$ we can end with 444 but not 4444, and there are no 4-square good numbers. We then generalize these arguments to look at k-cube and k-fourth good numbers, completely analyzing these cases. Noting that any even power $2m$ is the same as squaring an m^{th} power, we can extend our results to even powers more easily than to odd, as the behavior has to be a subset of what we have seen for squares. There are complications arising from powers of 2 dividing our numbers, but using results from elementary number theory (in particular concerning the totient function and Euler's theorem), we reduce the problem for even powers to a tractable finite computation, and completely resolve this case, and then end with a list of accessible next problems.

Keywords : patterns in final digits; k-power good numbers; totient function; Euler's theorem

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1 Introduction

One of the oldest problems in mathematics is trying to find patterns in digits; frequently if we look at numbers in the proper way beautiful structure emerges. For example, Euler's constant e base 10 is not particularly appealing, starting off

$$2.71828182845904523536028747135266249775724709369995957496696762772407663035 3,$$

however its continued fraction expansion (see for example [5]) is

$$[2,1,2,1,1,4,1,1,6,\dots] \;=\; 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cdots}}}}}.$$

We could look at decimal expansions of reciprocals of primes (see for more information [6]; by the pigeonhole principle these rational numbers must have either a finite or repeating expansion, and it is an interesting question to find all primes p whose period is the largest possible, $p-1$ (see [2]). For our last motivating example, we consider the distribution of leading digits. Amazingly, for many natural and mathematical data sets the probability of having a first digit d is not the same for all d, but exhibits a pronounced bias where we have d with probability $\log_{10}(1+1/d)$. Thus about 30% of the time the leading digit is one, with the probabilities decreasing down to around 4.6% for a first digit of 9; this is known as Benford's law, and there is an extensive literature on the subject and applications (especially to data integrity and fraud detection); see [4].

These questions and results motivated us to look at the decimal expansion of numbers, and investigate what possible patterns might emerge in arithmetic operations on it. As this is a very general question, we concentrate on a specific case: what can we say about the final digits of squares of integers? More specifically, how often can the final digits all be the same?

Clearly, it is possible to have arbitrary long strings of the same digit in the end, simply by looking at large powers of ten. As it is thus trivial to obtain squares with as many zeros at the end as we wish, we exclude a final digit of zero and investigate what happens for other digits.

In this paper, we examine the patterns of specific numbers of last digits of powers, so we adopt the following definitions.

Definition 1.1 (k-square good number) *We say a square number is a k-square good number if the final k digits are the same non-zero number, but the final $k+1$ are not.*

Definition 1.2 (k-cube good number) *We say a cube number is a k-cube good number if the final k digits are the same non-zero number, but the final $k+1$ are not.*

Definition 1.3 (k-fourth-power good number) *We say a fourth power of a number is a k-fourth-power good number if the final k digits are the same non-zero number, but the final $k+1$ are not.*

Later we consider arbitrary even powers, and leave the odd powers as a future project.

Theorem 1.4 *Consider squares of integers.*

- ***a)*** *There are infinitely many 1-square good numbers, ending with final digit 1, 4, 5, 6 or 9, and these are the only values for which this happens (and all of these cases happen).*
- ***b)*** *There are infinitely many 2-square good numbers, all ending in 44, which is the only value for which this happens.*
- ***c)*** *There are infinitely many 3-square good numbers, all ending in 444, which is the only value for which this happens.*

d) *There are no 4-square good numbers.*

Theorem 1.5 *Consider cubes of integers.*

a) *There are infinitely many 1-cube good numbers, ending with every non-zero possible digit, and these are the only values for which this happens (and all of these cases happen).*

b) *For any* $x \in \{11, 33, 44, 77, 88, 99\}$*, there are infinitely many 2-cube good numbers whose last two digits are* x*, and there are no other 2-cube good numbers. These are the only values for which this happens.*

c) *For any* $x \in \{111, 333, 777, 888, 999\}$*, there are infinitely many 3-cube good numbers whose last three digits are* x*, and there are no other 3-cube good numbers. These are the only values for which this happens.*

d) *For any* $x \in \{1111, 3333, 7777, 8888, 9999\}$*, there are infinitely many 4-cube good numbers whose last four digits are* x*, and there are no other 4-cube good numbers. These are the only values for which this happens.*

e) *For any* $x \in \{11111, 33333, 77777, 88888, 99999\}$*, there are infinitely many 5-cube good numbers whose last five digits are* x*, and there are no other 5-cube good numbers. These are the only values for which this happens.*

f) *There are infinitely many 6-cube good numbers, all ending in 888888, which is the only value for which this happens.*

g) *There are no 7-cube good numbers.*

Theorem 1.6 *Consider fourth powers of integers.*

a) *There are infinitely many 1-fourth-power good numbers, ending with final digit 1, 5, or 6, and these are the only values for which this happens (and all of these cases happen).*

b) *There are no 2-fourth-power good numbers.*

At first it seems quite daunting to investigate all powers of all integers; fortunately some arguments from number theory greatly winnow down the number of cases needed to study if the power is even. In §5 we show it suffices to study integers of the form $b \cdot 2^k$, with $b \leq 1000$ odd and $k \in \{0, \ldots, 103\}$, and we only need to study their m^{th} power where $m \leq 400$ is congruent to 2 modulo 4.

Theorem 1.7 *The only repeated final digits of* x^{2m} *are 44 and 444. The only time the final three digits are 444 is when* $2m = 2$*, and the final three digits of* x *are 38, 462, 538, and 962. There are many cases where the last two digits are 44, but the last three are not 444; see* https://tinyurl.com/3vtshze6.

As the analysis of the squares plays a key role in the proof for general powers, we start with that case. After looking at square numbers and their patterns in last digits, we turn to higher powers, and end with several accessible suggestions for future work (interested readers should contact the first author at sjm1@williams.edu).

2 Squares

2.1 1-Square Good Numbers

Proof of Theorem 1.4.a: **There are infinitely many 1-square good numbers, ending with final digit 1, 4, 5, 6 or 9, and these are the only values for which this happens (and all of these cases happen).**

Proof. Without loss of generality, it suffices to consider numbers of the form $50n + b$. When this is squared, we get $2500n^2 + 100bn + b^2$. Thus the final two digits of $(50n + b)^2$ are just the final two digits of b^2. There are only 49 choices of b to check. Take b to equal 1, 2, 5, 6, and 3. We get final two digits of 01, 04, 25, 36, and 09, respectively; a straightforward computation shows that none of the other possibilities give a different final digit than 1, 4, 5, 6 and 9. Thus, there are infinitely many 1-square good numbers with last digits 1, 4, 5, 6, and 9, and these are the only 1-square good numbers. □

2.2 2-Square Good Numbers

From Theorem 1.4.a, the only possible 2-square good numbers must end in 11, 44, 55, 66, or 99; below we eliminate all but 44.

2.2.1 Digits 11, 55, 66, and 99

Lemma 1.4.b. **There are no squares whose last two digits are 11, 55, 66, or 99.**

Proof. Again it suffices to consider numbers of the form $50n + b$. When this is squared, we get $2500n^2 + 100bn + b^2$, so the final two digits of $(50n + b)^2$ are just the final two digits of b^2. We calculate b^2 with b ranging from 1 through 49. From these calculations, we find the only values of b so that b^2 ends with the same two digits: 00 (which we exclude) and 44. Thus there are no square numbers with last two digits of 11, 55, 66, or 99. □

Since two-square good numbers cannot end with 11, 55, 66, and 99, we now only have to consider the case of 44. To obtain two-square numbers, it suffices to find infinitely many numbers ending with 144, which is 12^2.

Remark 2.1 Before continuing, we briefly remark on our efforts to minimize the computations required. We are writing our numbers modulo 50; we cannot use a smaller modulus and always have the last two digits the same as b^2.

To see that 50 is minimal, consider numbers of the form $(an + b)^2$. Let $b = 12$. When we expand, we get $a^2n^2 + 24an + 144$. In order to make the last two digits of this equal the last two digits of b^2, one way to do this is to ensure that a^2 and $2a$ have two zeros as their last two digits; this happens if $a = 50$. Perhaps, however, we might have $a^2n^2 + 24an$ is a multiple of 100 for all n, which would suffice. Such a result is possible; for example, consider the simpler example of $n^2 + n$, which equals $n(n + 1)$. Note that 2 does not always divide n, 2 does not always divide $n + 1$, but 2 always divides $n(n + 1)$ because multiplying two consecutive numbers results in an even number.

Returning to our analysis, a brute force computation shows that there is no $a < 50$ such that $a^2n^2 + 24an$ is always a multiple of 100. Taking $n = 1$ gives the last two digits are not 00 for all $a < 50$ except for $a = 26$. For $a = 26$, if we take $n = 2$, the last two digits are not 00. Thus 50 is the smallest modulus we can use.

2.2.2 Digits 44

Lemma 1.4.b shows that the only possibility for a 2-square good number is 44.

***Proof of Theorem 1.4.b:* There are infinitely many 2-square good numbers, all ending in 44, which is the only value for which this happens.**

Proof. Consider numbers of the form $500n + 12$, which, when squared, is $250000n^2 + 12000n + 144$. This lets us know immediately that the last three digits are 144. We get infinitely many, one for each n. Thus, there are infinitely many square numbers with last digits of 44 but not 444. □

2.3 3-Square Good Numbers

From Theorem 1.4.b, the only possible 3-square good numbers can end in 444.

***Proof of Theorem 1.4.c:* There are infinitely many 3-square good numbers, all ending in 444, which is the only value for which this happens.**

Proof. Consider numbers of the form $(an + b)^2$, which is $a^2n^2 + 2abn + b^2$. Since we are only looking at the last three digits, if a^2 and $2a$ end in at least three zeros, then we know immediately that the last three digits are dependent only on b^2. Multiples of 500 work for a: $(500n + b)^2 = 250000n^2 + 1000bn + b^2$. We must calculate b^2 for b ranging from 1 to 499. We find that when b is 38 or 462, $(500n + b)^2$ ends in 444 ($38^2 = 1444$ and $462^2 = 213444$), but $(500n + b)^2$ for b might end in 4444. Thus, consider numbers of the form $(5000n + 38)^2$, which expands to $(25 \cdot 10^6)n^2 + 380000n + 1444$, and $(5000n + 462)^2$, which expands to $(25 \cdot 10^6)n^2 + 4620000n + 213444$. For every value of n, both of these numbers will always end in 444 but not 4444, resulting in an infinite number of 3-square good numbers. □

2.4 4-Square Good Numbers

***Proof of Theorem 1.4.d:* There are no 4-square good numbers.**

Proof. From the proof of Theorem 1.4.c, we know that the only squares that end in 444 are of the form $(500n + 38)^2$ and $(500n + 462)^2$ and that when $10|n$, these numbers are not 4-square good.

Consider numbers of the form $(an+b)^2$, which is $a^2n^2+2abn+b^2$. Since we are only looking at the last four digits, if a^2 and $2a$ end in at least four zeros, the last four digits are dependent only on b^2. We may look at multiples of 5000 for a: $(5000n+b)^2 = (25\cdot 10^6)n^2+10000bn+b^2$. We must calculate b^2 for b ranging from 1 to 4999. Thus, to prove that there is never a square number that ends in 4444, we just need to calculate $(500n+38)^2$ and $(500n+462)^2$ for $n=0,1,...,9$ because 5000 divided by 500 is 10. From the calculations, there is no square number whose last four digits are 4444. □

It is worth noting that our previous analysis paid enormous dividends here, as we only had to check 20 possibilities.

3 Cubes

3.1 1-Cube Good Numbers

***Proof of Theorem 1.5.a:* There are infinitely many 1-cube good numbers, ending with every non-zero possible digit (and all of these cases happen).**

Proof. Consider numbers of the form $500n+b$. When this is cubed, we get $(125\cdot 10^6)n^3+750000bn^2+1500b^2n+b^3$. Thus the final two digits of $(500n+b)^3$ are just the final two digits of b^3. So there are only 499 choices of b to check. Take b to equal 1, 8, 7, 4, 5, 6, 3, 2, and 9. We get final two digits of 01, 12, 43, 64, 25, 16, 27, 08, and 29, respectively. Therefore for these choices, the only last digit of b^3 is 1, 2, 3, 4, 5, 6, 7, 8, and 9. Thus, every non-zero possible digit occurs. □

3.2 2-Cube Good Numbers

From Theorem 1.5.a, 2-cube good numbers can end in anything (i.e., 11, 22, 33, 44, 55, 66, 77, 88, or 99 are all possible); we eliminate three of these and show it must end with 11, 33, 44, 77, 88, or 99.

3.2.1 Digits 22, 55, and 66

***Lemma 1.5.b:* There are no cubes whose last two digits are 22, 55, or 66.**

Proof.

We can consider numbers of the form $5000n+b$, whose cubes are

$$(125\cdot 10^9)n^3+(75\cdot 10^6)bn^2+15000b^2n+b^3,$$

and thus the final two digits of $(5000n+b)^3$ are just the final two digits of b^3. Let b range from 1 through 4999. From calculations of b^3 for these numbers, we find values of b so that b^3 ends with the same two digits for 6 digit pairs, 11, 33, 44, 77, 88, and 99. Thus there are no cubes with the last two digits of 22, 55, or 66 (and the six options are potential 2-good cube numbers). □

3.2.2 Digits 11, 33, 44, 77, 88, and 99

Lemma 1.5.b shows that the only possibilities for a 2-cube good number are 11, 33, 44, 77, 88, and 99. We now show all these possibilities are realized.

Proof of Theorem 1.5.b: **For any $x \in \{11, 33, 44, 77, 88, 99\}$, there are infinitely many 2-cube good numbers whose last two digits are x, and there are no other 2-cube good numbers.**

Proof. Consider numbers of the form $5000n + b$, which, when cubed, is

$$(125 \cdot 10^9)n^3 + (75 \cdot 10^6)bn^2 + 15000b^2n + b^3,$$

and the last three digits are the last three digits of b^3. The last three digits of 71^3, 77^3, 14^3, 53^3, 42^3, and 99^3 are 911, 533, 744, 877, 088, and 299, respectively. When $b = 71$, 77, 14, 53, 42, and 99, we get infinitely many 2-cube good numbers, one for each n. Thus, there are infinitely many cube numbers with last digits of 11, 33, 44, 77, 88, and 99, but not 111, 333, 444, 777, 888, or 999. □

3.3 3-Cube Good Numbers

From Theorem 1.5.b, the only possible 3-cube good numbers can end in 111, 333, 444, 777, 888, or 999, and here we eliminate 444.

3.3.1 Digits 444

Lemma 1.5.c: **There are no cubes whose last three digits are 444.**

Proof. Consider numbers of the form $5000n + b$, whose cubes are

$$(125 \cdot 10^9)n^3 + (75 \cdot 10^6)bn^2 + 15000b^2n + b^3.$$

As always, the final three digits of $(5000n + b)^3$ are just the final three digits of b^3. Take b to range from 1 through 4999. From calculations of b^3 for these numbers, we final values of b so that b^3 ends with the same last three digits only for 111, 333, 777, 888, and 999. Thus, there are no cubes with the last three digits of 444. □

3.3.2 Digits 111, 333, 777, 888, and 999

Lemma 1.5.c shows that the only possibilities for a 3-cube good number are 111, 333, 777, 888, and 999. We show all of these work.

Proof of Theorem 1.5.c: **For any $x \in \{111, 333, 777, 888, 999\}$, there are infinitely many 3-cube good numbers whose last three digits are x, and there are no other 3-cube good numbers. These are the only values for which this happens.**

Proof. Consider numbers of the form $(an+b)^3$, which is $a^3n^3+3a^2bn^2+3ab^2n+b^3$. Since we are only looking at the last three digits, if a^3, $3a^2$, and $3a$ end in at least three zeros, then we know immediately that the last three digits are dependent only on b^3. Multiples of 5000 work for a:

$$(5000n+b)^3 \;=\; (125\cdot 10^9)n^3+(75\cdot 10^6)bn^2+15000b^2n+b^3.$$

We must calculate b^3 for b ranging from 1 to 4999. From the calculations, the last four digits of 471^3, 477^3, 1753^3, 192^3, and 999^3 are 7111, 1333, 4777, 7888, and 2999, respectively, but $(5000n+b)^3$ for b might end in 1111, 3333, 7777, 8888, or 9999 as our analysis only cared about the final three digits. Thus, we increase the modulus a and consider numbers of the form $(50000n+471)^3$, which expands to

$$(125\cdot 10^{12})n^3+(75\cdot 10^8)bn^2+150000b^2n+b^3.$$

For every value of n, all such numbers with b equal to one of 471, 477, 1753, 192, and 999, will respectively always end in 111 but not 1111, 333 but not 3333, 777 but not 7777, 888 but not 8888, and 999 but not 9999, resulting in an infinite number of 3-cube good numbers. $\square$

3.4 4-Cube Good Numbers

From Theorem 1.5.c, we know that the only possible 4-cube good numbers can end in 1111, 3333, 7777, 8888, or 9999; we show all happen.

Proof of Theorem 1.5.d: **For any $x \in \{1111, 3333, 7777, 8888, 9999\}$, there are infinitely many 4-cube good numbers whose last four digits are x, and there are no other 4-cube good numbers. These are the only values for which this happens.**

Proof. Consider numbers of the form $(an+b)^3$, which is $a^3n^3+3a^2bn^2+3ab^2n+b^3$. Since we are only looking at the last four digits, if a^3, $3a^2$, and $3a$ end in at least four zeros, then we know immediately that the last four digits are dependent only on b^3. Multiples of 50000 work for a:

$$(50000n+b)^3 \;=\; (125\cdot 10^{12})n^3+(75\cdot 10^8)bn^2+150000b^2n+b^3.$$

We must calculate b^3 for b ranging from 1 to 49999. From the calculations, the last five digits of 8471^3, 6477^3, 753^3, 4442^3, and 9999^3 are 71111, 53333, 57777, 18888, and 29999, respectively, but $(50000n+b)^3$ for b might end in 11111, 33333, 77777, 88888, or 99999. Thus, consider numbers in the form $(5\cdot 10^5n+b)^3$, which expands to

$$(125\cdot 10^{15})n^3+(75\cdot 10^{10})bn^2+(15\cdot 10^5)b^2n+b^3.$$

For every value of n, all such numbers with b equal to 8471, 6477, 753, 4442, or 9999, will respectively always end in 1111 but not 11111, 3333 but not 33333, 7777 but not 77777, 8888 but not 88888, and 9999 but not 99999, resulting in an infinite number of 4-cube good numbers. $\square$

3.5 5-Cube Good Numbers

From Theorem 1.5.d, we know that the only possible 5-cube good numbers can end in 11111, 33333, 77777, 88888, or 99999; we show all happen.

Proof of Theorem 1.5.e: **For any $x \in \{11111, 33333, 77777, 88888, 99999\}$, there are infinitely many 5-cube good numbers whose last five digits are x, and there are no other 5-cube good numbers. These are the only values for which this happens.**

Proof. Consider numbers in the form $(an + b)^3$, which is $a^3n^3 + 3a^2bn^2 + 3ab^2n + b^3$. Since we are only looking at the last five digits, if a^3, $3a^2$, and $3a$ end in at least five zeros, then we know immediately that the last five digits are dependent only on b^3. Multiples of $5 \cdot 10^5$ work for a:

$$(5 \cdot 10^5 n + b)^3 \;=\; (125 \cdot 10^{15})n^3 + (75 \cdot 10^{10})bn^2 + (15 \cdot 10^5)b^2n + b^3.$$

We must calculate b^3 for b ranging from 1 to 499999. From the calculations, the last six digits of 88471^3, 46477^3, 60753^3, 1942^3, and 99999^3 are 511111, 533333, 577777, 988888, and 299999, respectively, but $(5 \cdot 10^5 n + b)^3$ for b might end in 11111, 33333, 77777, 88888, or 99999. Thus, consider numbers in the form $(5 \cdot 10^6 n + b)^3$, which expands to

$$(125 \cdot 10^{18})n^3 + (75 \cdot 10^{12})bn^2 + (15 \cdot 10^6)b^2n + b^3.$$

For every value of n, all such numbers with b equal to 88471, 46477, 60753, 1942, or 99999, will always respectively end in 11111 but not 111111, 33333 but not 333333, 77777 but not 777777, 88888 but not 888888, and 99999 but not 999999, resulting in an infinite number of 5-cube good numbers. □

3.6 6-Cube Good Numbers

From Theorem 1.5.e, we know that the only possible 6-cube good numbers can end in 111111, 333333, 777777, 888888, or 999999, and here we eliminate all but 888888.

3.6.1 Digits 111111, 333333, 777777, and 999999

Lemma 1.5.f: **There are no cubes whose last six digits are 111111, 333333, 777777, or 999999.**

Proof. Consider numbers in the form $5 \cdot 10^6 n + b$. When this is cubed, we get

$$(125 \cdot 10^{18})n^3 + (75 \cdot 10^{12})bn^2 + (15 \cdot 10^6)b^2n + b^3,$$

and thus the final six digits of $(5 \cdot 10^6 n + b)^3$ are just the final six digits of b^3. Take b to equal 1 through 4999999. From calculations of b^3 for these numbers, we find only one value of b so that b^3 ends with the same last six digits, and those digits are 888888. Thus, there are no cubes with the last six digits of 111111, 333333, 777777, and 999999. □

3.6.2 Digits 888888

Lemma 1.5.f shows that the only possibility for a 6-cube good number is 888888.

Proof of Theorem 1.5.f: **There are infinitely many 6-cube good numbers, all ending in 888888, which is the only value for which this happens.**

Proof. Consider numbers of the form $(an+b)^3$, which is $a^3n^3 + 3a^2bn^2 + 3ab^2n + b^3$. Since we are only looking at the last six digits, if a^3, $3a^2$, and $3a$ end in at least six zeros, then we know immediately that the last six digits are dependent only on b^3. Multiples of $5 \cdot 10^6$ work for a:

$$(5 \cdot 10^6 n + b)^3 \;=\; (125 \cdot 10^{18})n^3 + (75 \cdot 10^{12})bn^2 + (15 \cdot 10^6)b^2n + b^3.$$

We must calculate b^3 for b ranging from 1 to 4999999. From the calculations, the last seven digits of 76942^3 are 0888888, but $(5 \cdot 10^6 n + b)^3$ for b might end in 8888888. Thus, consider numbers in the form $(5 \cdot 10^7 n + b)^3$, which expands to

$$(125 \cdot 10^{21})n^3 + (75 \cdot 10^{14})bn^2 + (15 \cdot 10^7)b^2n + b^3.$$

For every value of n, all numbers with $b = 76942$ will always end in 888888 but not 8888888, resulting in an infinite number of 6-cube good numbers. □

3.7 7-Cube Good Numbers

Proof of Theorem 1.5.g: **There are no 7-cube good numbers.**

Proof. From Theorem 1.5.f, we know that the only numbers that end in 888888 are of the form $(5 \cdot 10^6 n + 76942)^3$. Consider numbers of the form $(an+b)^3$, which is $a^3n^3 + 3a^2bn^2 + 3ab^2n + b^3$. Since we are only looking at the last seven digits, if a^3, $3a^2$, and $3a$ end in at least seven zeros, then we know immediately that the last seven digits are dependent only on b^3. We may look at multiples of $5 \cdot 10^7$ for a:

$$(5 \cdot 10^7 n + b)^3 \;=\; (125 \cdot 10^{21})n^3 + (75 \cdot 10^{14})bn^2 + (15 \cdot 10^7)b^2n + b^3.$$

We must calculate b^3 for b ranging from 1 to 49999999. Thus, to prove that there is never a cube that ends in 8888888, we just need to check $(5 \cdot 10^6 n + 76942)^3$ for $n = 0, 1, \ldots, 9$ because $5 \cdot 10^7$ divided by $5 \cdot 10^6$ is 10, which thanks to the analysis above is now a very short calculation.

$$\begin{gathered}
(5 \cdot 10^6 n + 76942)^3 \\
n = 0 : (5 \cdot 10^6 \cdot 0 + 76942)^3 = 455502130888888 \\
n = 1 : (5 \cdot 10^6 \cdot 1 + 76942)^3 = 130859906572590888888 \\
n = 2 : (5 \cdot 10^6 \cdot 2 + 76942)^3 = 1023260657643050888888 \\
n = 3 : (5 \cdot 10^6 \cdot 3 + 76942)^3 = 3427202708713510888888 \\
n = 4 : (5 \cdot 10^6 \cdot 4 + 76942)^3 = 8092686059783970888888 \\
n = 5 : (5 \cdot 10^6 \cdot 5 + 76942)^3 = 15769710710854430888888
\end{gathered}$$

$$
\begin{aligned}
n = 6 &: (5 \cdot 10^6 \cdot 6 + 76942)^3 = 27208276661924890888888\\
n = 7 &: (5 \cdot 10^6 \cdot 7 + 76942)^3 = 43158383912995350888888\\
n = 8 &: (5 \cdot 10^6 \cdot 8 + 76942)^3 = 64370032464065810888888\\
n = 9 &: (5 \cdot 10^6 \cdot 9 + 76942)^3 = 91593222315136270888888.
\end{aligned}
$$

Thus, there is no cube whose last seven digits are 8888888. □

4 Fourth Powers

4.1 1-Fourth-Power Good Numbers

Proof of Theorem 1.6.a: **There are infinitely many 1-fourth-power good numbers, ending with final digit 1, 5, or 6, and these are the only values for which this happens (and all of these cases happen).**

Proof. Consider numbers of the form $50n + b$. When this is raised to the power of four, we get

$$6250000n^4 + 500000n^3n + 15000n^2b^2 + 200nb^3 + b^4,$$

thus the final two digits of $(50n + b)^4$ are just the final two digits of b^4 and there are only 49 choices of b to check. When 1 through 49 is raised to the fourth power, the last digits of those results only end in 1, 5, and 6, so 2, 3, 4, 7, 8, and 9 cannot be the last digit of a fourth power. Taking b to equal 1, 5, and 2, we get final two digits are 01, 25, and 16, respectively. Thus, there are infinitely many 1-square good numbers with last digits 1, 5, and 6. □

4.2 2-Fourth-Power Good Numbers

From Theorem 1.6.a, the only possible 2-fourth-power good numbers can end in 11, 55, or 66, and here we eliminate all. This result will play a key role in our later analysis of higher powers.

Proof of Theorem 1.6.b **There are no 2-fourth-power good numbers.**

Proof. Consider numbers in the form $50n + b$. When this is raised to the power of four, we get

$$6250000n^4 + 500000n^3n + 15000n^2b^2 + 200nb^3 + b^4,$$

and thus the final two digits of $(50n+b)^4$ are just the final two digits of b^4. Take b to equal 1 through 49. From calculations of b^4 for these numbers, we find no values of b so that b^4 ends with the same last two digits of 11, 55, or 66. Thus, there are no fourth powers with last two digits of 11, 55, or 66, and therefore there are no 2-good fourth powers. □

5 General Even Powers

We now turn to looking for how often, and when, the last d digits of n^p are the same but not the last $d+1$. We shall study only even powers here, and leave the odd powers for future work. From our earlier analysis we know that the only possible repeated final digits for squares are 44 and 444, and there are no repeated final digits for fourth powers. Thus if we look at n^{2m} there cannot be repeated digits at the end if $2m$ is a multiple of four; thus it suffices to consider $2m \equiv 2 \bmod 4$. As $n^{2m} = (n^m)^2$ is a square, we know the only possible repeated final digits are 44 and 444, and thus it suffices to see when either of these happen.

We recall the definition of *Euler's totient function*, $\phi(n)$, which is the number of non-negative integers at most n which are relatively prime to n; this is a multiplicative function, with $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime and for p prime we have $\phi(p^r) = p^{r-1}(p-1)$. Of its many important properties, for us the most important is that if a is a positive integer relatively prime to m, then $a^{\phi(m)} \equiv 1 \bmod a$. See for example [2, 5] for proofs. We can now prove our main result on even powers, noting we have already analyzed squares and fourth powers.

Proof of Theorem 1.7.a: **The only repeated final digits of x^{2m} are 44 and 444. The only time the final three digits are 444 is when $2m = 2$, and the final three digits of x are 38, 462, 538, and 962. There are many cases where the last two digits are 44, but the last three are not 444; see https://tinyurl.com/3vtshze6.**

Proof. By using Euler's Totient Theorem and the pigeonhole principle, we can reduce the calculations and analyses for even powers to a very reasonable number of computations. We can write any n^{2m} as $(b \cdot 2^k)^{2m}$, where b is odd and k is a non-negative integer. As we only need to study the final three digits, it suffices to look at $n \bmod 1000$.

If b is a multiple of 5 then we know the final digit must be a 5 and thus we cannot end with 44 or 444, and thus we may exclude all such odd b. Thus it suffices to study odd $b \in \{1, 3, 7, 9, 11, \dots, 999\}$. As

$$\phi(1000) \;=\; \phi(8)\phi(125) \;=\; 2^2(2-1) \cdot 5^2(5-1) \;=\; 400,$$

we only need to look at 400 powers of the odd number b before the final three digits cycle.

For powers of 2, we notice

$$2^1 \;=\; 2, \quad 2^2 \;=\; 4, \quad 2^3 \;=\; 8, \quad \dots, 2^{103} \bmod 1000 \;=\; 008;$$

thus the last three digits of powers of 2 cycle with a period of 100, but starting not with the zeroth or first power but with the third. Fortunately 100 is a divisor of 400, and combining these two observations we see it suffices to study $(b \cdot 2^k)^{2m} \bmod 1000$ for odd b relatively prime to 5 that are less than 1000 and $k \in \{0, 1, 2, \dots, 102\}$.

Case I: **n is relatively prime to 10.**

By the above analysis, we need to study $b^{2m} \bmod 1000$ for $2m \in \{2, 6, 10, \dots, 398\}$ and odd b relatively prime to 5 and at most 999. This leads to 400 choices of b and 200 choices of m, or only 80,000 calculations. As these numbers are always odd, they cannot end in 44 or 444, so there are no odd numbers whose even power ends with repeated digits. All that is left is seeing what first digits are possible, and a quick analysis shows that the powers of odd integers relatively prime to 5 always end in a 1 or a 9 (the interested reader can write down a formula for which odd numbers and which odd powers end with a 1 versus a 9).

***Case II:* n is a multiple of 2 but not 5.**

From the above analysis, we have to study $(b \cdot 2^k)^{2m}$ mod 1000 where $2m \equiv 2 \bmod 4$, $b \leq 999$ is an odd number relatively prime to 5, and k is a positive integer at most 102. The possible final digits of even numbers to even powers are just 4 and 6; note we cannot end in 66 so we just have to study the n ending with a 4 (the interested reader can write down a formula for when the final digit is a 6). The calculation is 102 times larger than the previous, but this can be handled in less than a minute on a reasonable laptop. We find that the only power where the final three digits are 444 is 2, coming from our integer n being 38, 462, 538 or 962 modulo 1000 (all of these are divisible by 2 but not by 4).

There are more situations where the final two digits are 44 but not 444; see `https://tinyurl.com/3vtshze6` to view the cases[1]. There are 64 numbers that, when raised to an even power, have inputs ending in 44 but not 444, and they are 2, 8, 12, 22, 28, 38, 42, 48, 52, 58, 62, 72, 78, 88, 92, 98, 102, 108, 112, 122, 128, 138, 142, 148, 152, 158, 162, 172, 178, 188, 192, 198, 202, 208, 212, 222, 228, 238, 242, 248, 252, 258, 262, 272, 278, 288, 292, 298, 302, 308, 312, 322, 328, 338, 342, 348, 352, 358, 362, 372, 378, 388, 392, and 398.

For all of these 64 numbers, starting at 2, the corresponding even powers cycle every four numbers. The corresponding even powers for the first number are 18, 38, 58, 78, 98, 118, 138, 158, 178, 198, 218, 238, 258, 278, 298, 318, 338, 358, 378, 398. The powers for the second number are 6, 26, 46, 66, 86, 106, 126, 146, 166, 186, 206, 226, 246, 266, 286, 306, 326, 346, 366, 386. The powers for the third number are 2, 22, 42, 62, 82, 102, 122, 142, 162, 182, 202, 222, 242, 262, 282, 302, 322, 342, 362, 382. The powers for the fourth number are 14, 34, 54, 74, 94, 114, 134, 154, 174, 194, 214, 234, 254, 274, 294, 314, 334, 354, 374, 394. The next four numbers have the same powers, but the order is flipped so that the fifth number's first power is 2, and the eighth number's first power is 18. The next four powers flip again so that their powers are the same as the first four numbers. This cycle happens all the way through 398. □

6 Future Work

One of the main purposes of this research was to propose accessible problems for future students to explore; if interested contact the first named author at sjm1@williams.edu.

1. We analyzed the case of even powers, what happens for odd powers? In particular, we saw for cubes there were a lot more possibilities for repeated final digits (both

[1]Email the authors for the code if interested.

in terms of number and length). What happens for other odd powers? Is there a maximum number of repeated final digits one can have as we vary the power and integer, or can we find arbitrarily long strings of the same last digits when raised to an appropriate power?

2. The above questions are for base 10; what happens if we look at other bases? The simplest generalization is to look at base b expansions, starting with $b-2$, but there are other possible extensions. For example, one can study the Zeckendorf expansion of a number[2], and look at what happens to the "digits" when we raise to various powers. See for example [1, 3] for an introduction to these types of decompositions.

3. We only looked at how often the last digits are the same, but one could (and should) explore all the possibilities. What fraction of the final digit tuples arise for squares? If we look at just the last digit, it is one, four, five, six, nine, which is 5/9. If we include zero, it is 60 percent. What fraction of the possible final two digits are realized for squares? How often do each of these occur? What happens if we look at other powers and other bases?

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[2] If we define the Fibonacci numbers by $F_{n+1} = F_n + F_{n-1}$ with $F_1 = 1$ and $F_2 = 2$, then every integer has a unique representation as a sum of non-adjacent Fibonacci numbers; thus

$$2023 \ = \ 1597 + 377 + 34 + 13 + 2 \ = \ F_{16} + F_{13} + F8 + F6 + F2 \ = \ 1001000010100010_{\rm Z}.$$

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