Covering Area in a Disc by Finitely Many Triangles

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Abstract - Consider how to cover the most area inside of a circular disc by using a fixed finite number of triangles. Even when the number of triangles is small, it is not clear what is the maximum area that can be covered. However, as the number of triangles tends to infinity, under and over estimates can be given for the rate that the uncovered area goes to zero.

Keywords : triangles; area covering; optimal

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1 Introduction

The general question is "How much area can be covered by a finite number of triangles inside of a region $\mathcal{D}$ in the plane?" If the region is a simple geometric one like a closed, convex set, or more generally the closed region whose boundary is a simple closed curve, *triangulation* is a standard method in geometry that covers the interior as best possible with many small triangles overlapping only on their boundaries. Then, in the limit as the number of triangles goes to infinity, the area covered by the triangles will give the planar area of $\mathcal{D}$.

But what can be said about the amount of area covered if the number N of triangles is fixed? Specifically, what if N is not very large? Moreover, what is the shape and position of the triangles that achieves the optimal covering? Finally, letting N get larger, what is the fraction of the area, as a function of N, not covered by the triangles?

This article focuses on obtaining the answers to these questions in one of the simplest possible cases, where $\mathcal{D}$ is a closed circular disc D. Take a finite number N of triangles all contained in a circular disc of radius 1.

Question: What is the maximum value $\mathcal{A}(N)$ of area in D covered by N triangles contained in D , and what are the shapes and arrangements of the triangles that achieve this?

Remark 1.1 a) It can be shown, using a continuity argument, that the value $\mathcal{A}(N)$ is uniquely determined. But it is not obvious what are the triangles and their arrangements that achieve this, up to isometric rearrangement of the triangles.

b) It is not clear if the optimal covering could have non-trivial overlap of the triangles. This might seem reasonable, but without a proof of this it is an open issue. For this reason, in this article all covering of area by triangles in D will be where the triangles overlap only at most at their edges. □

When N is small, some examples are given in Section 2. But even when N is small, the exact value of $\mathcal{A}(N)$ is unknown. Nonetheless, in Section 3, using various examples, some computations, and some literature, estimates are found for the *minimal error* $\mathcal{E}(N)$, as N goes to infinity. where $\mathcal{E}(N)$ is the area in the disc not covered by an optimal placement of disjoint triangles. See Corollary 3.3 and Proposition 4.2 for the main results used for getting upper and lower bounds on the minimal error. They show that there are universal constants C_1 and C_2 such that

$$\frac{C_1}{N^2} \geq \mathcal{E}(N) \geq \frac{C_2}{N^2}.$$

This research fits into the general topic of optimal triangulations of surfaces. See Section 5 at the end of this article for a short discussion of general problems related to triangulations. Also, see for example the articles by Bronstein [1] and Heckbert and Garland [3].

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2 Optimal Covering of Area by a Few Triangles in a Disc

Consider the closed unit disc in the plane $D = \{(x,y) : x^2 + y^2 \leq 1\}$. Take a small number N triangles whose interiors are disjoint and whose union is a subset of D. Allow the shape and areas of these triangles to vary so that as much area $\mathcal{A}(N)$ as possible is covered.

It is not clear what is the solution to the optimal area that is covered when N is relatively small. However, when $N = 1$, it is easy to argue that the best solution is to inscribe an equilateral triangle in the disc; this gives $\mathcal{A}(1) = \frac{3\sqrt{3}}{4}$. But when $N = 2$, more area can be covered. For example, to obtain a proxy for $\mathcal{A}(2)$, take a inscribed quadrilateral consisting of two isosceles triangles, both with a common edge that is a diameter of the disc, and with their third vertices on the boundary of D. This arrangement should give the maximum covered area of 2.

Now take the case that $N = 3$. Consider the following two scenarios **A)** and **B)**.

A) A first natural configuration is where the triangles each have one vertex at the center of the disc and the other vertices on the boundary ∂D of the disc D . Say the angle at the vertices at the origin are all the same α. Then the total area is $T(\alpha) = 3(2(\frac{1}{2}\cos(\alpha/2)\sin(\alpha/2))) = \frac{3}{2}\sin(\alpha)$. This is a maximum when $\alpha/2 = \pi/4$.

So the maximum area is $3/2$. Because $\alpha < 2\pi/3$, there is room to rotate these triangles and get different positions of the triangles with the some total area being covered.

In contrast to this, what happens if three different vertex angles $\alpha_i, i = 1, 2, 3$ are allowed, with the constraint $\alpha_1 + \alpha_2 + \alpha_3 \leq 2\pi$. Now the total area $T(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^{3} \cos(\alpha_i/2)\sin(\alpha_i/2)$. But then the maximum is where each term has $\alpha_i = \pi/2$ and the total area is $3/2$ again.

Remark 2.1 It is interesting that the solution in this case is not to take $\alpha = 2\pi/3$, which would cover the central region of the disc entirely. Note that in this case the three triangles together give an inscribed equilateral triangle whose area $A(\alpha) = \frac{3}{2}\sin(2\pi/3) = \frac{3\sqrt{3}}{4}$ which is approximately $1.299 < 3/2$. Also, this value of $A(\alpha)$ equals $\mathcal{A}(1)$. So it is not surprising that this is not the optimal solution with three triangles because $\mathcal{A}(N)$ is strictly increasing as N increases. Hence, we know that $\mathcal{A}(1) < \mathcal{A}(2) < \mathcal{A}(3)$ without doing any computation at all. □

B) The results in **A)** are not as good as can be derived with another arrangement of three triangles. Indeed, the optimal value may perhaps only be achieved when all of the vertices of the triangles are on the boundary ∂D of D. This is not the case with the arrangements considered above when $N = 3$.

Here is an example of how to cover more area than the above with the triangles having all of their vertices on ∂D. Place an isosceles triangle $T(x)$ in the disc with vertices on ∂D. Say one bottom vertex is at $(0, -1)$ and the other two top vertices are at $(\cos(x), \sin(x))$ and $(-\cos(x), \sin(x))$. Then this is an equilateral triangle if $x = \frac{\pi}{6}$.

Now let the equal length lower edges be E_1 and E_2. Then on each E_i, one places an isosceles triangle with its vertices on ∂D and one edge the same as E_i. Call this arrangement a $3D$.

Next vary the top right vertex position of T, that is vary x, to maximize the total area of the corresponding $3D$. Actually, the solution is not with T being an equilateral triangle. Indeed, after looking at the critical points of the total area of the $3D$ with respect to x, it can be seen that the solution for maximum total area is with $x = \frac{3}{10}\pi > \frac{\pi}{6}$. One is giving up some area in the central triangle to get more area in the two side isosceles triangles, and in this way getting a larger total area. The total area in this case is $\frac{5}{2}\sin(2\pi/5)$, which is approximately 2.37764 and is significantly larger than the $3/2$ area above. Note: if this $3D$ is to give $\mathcal{A}(3)$, then we expect it to have a larger than $\mathcal{A}(2)$, which it likely to be 2 as noted above.

Remark 2.2 An interesting point is that in fact the union of the triangles on any $3D$ is a pentagon $\mathcal{P}(3)$ inscribed in D. Also, the optimal area $3D$ is where $\mathcal{P}(3)$ is a regular pentagon. Note: when $\mathcal{P}(3)$ is a regular pentagon, it corresponds to the union of the triangles in what is called a 5-array in Section 3. So the area for the optimal $3D$ is $5\cos(\pi/5)\sin(\pi/5)$, which is the value obtained above. □

Question: Is the value of $\mathcal{A}(3)$ the area $5\cos(\pi/5)\sin(\pi/5)$ obtained above?

Remark 2.3 As a final remark here, ask what happens if a third isosceles triangle is placed on the unoccupied edge of T with its vertices on ∂D and its longer edge the edge of T. If one calculates the maximum area covered for this configuration of 4 triangles, the solution has T being an equilateral triangle. □

3 Covering Area by More Triangles in a Disc

Now consider what happens as N gets larger, and in the end goes to infinity. It may not be obvious what is the actual value $\mathcal{A}(N)$. But there is still hope for getting an estimate of the asymptotic value of the minimal error, the area that is not covered by the triangles. Let us denote the minimal error $\pi - \mathcal{A}(N)$ by $\mathcal{E}(N)$.

Let us first see what can be given by a standard arrangement, even though it may not be optimal. Take the triangles to be congruent isosceles triangles with vertices at the origin, and the angle at that vertex being $2\pi/N$ for each of the triangles. Assume that the other vertices of these triangles are on the boundary of D. Call this the N*-array.* Take the meaning of an N-array to be the set of triangles themselves and also their union, unless it causes clarity issues.

Proposition 3.1 *The area $A(N)$ covered by the triangles in the N-array satisfies $A(N) = N sin(2\pi/N)/2$. The error $E(N) = \pi - A(N) \sim \frac{2\pi^3}{3N^2}$. as $N \to \infty$.*

Proof. The vertex angle in the component isosceles triangles in the N-array is $\alpha = 2\pi/N$. So the area of each of these triangles is $\sin(\alpha)/2$. This makes $A(N) = N\sin(2\pi/N)/2$. The error value then follows from the Taylor polynomial approximation of $\sin(2\pi/N)$. So $A(N) = N(2\pi/N - \frac{1}{6}(2\pi/N)^3)/2 + O(1/N^4) = \pi - \frac{8}{12}(\pi^3/N^2) + O(1/N^4)$. So $E(N) = \pi - A(N) \sim \frac{2}{3}(\pi^3/N^2)$ as N goes to infinity. □

We should check one aspect of N-arrays. Say a general N-array is like the standard N-array but the vertex angles at center of D are allowed to be different.

Proposition 3.2 *The maximum area for a general N-array is given by the standard N-array.*

Proof. Take the general N-array with vertex anges $\alpha_1, \ldots, \alpha_N$. Compute the area in the usual fashion, and apply a Lagrange multiplier computation with the α_k as variables and the constraint being $\sum_{k=1}^{N} \alpha_k = 2\pi$. This will show that the maximum area is when the α_k are all equal to $2\pi/N$, and so the maximum area for a general N-array is given by the standard N-array. □

This result on N-arrays gives us an upper bound on the minimal error.

Corollary 3.3 *The minimal error $\mathcal{E}(N)$ is bounded above by $\frac{2\pi^3}{3N^2}$.*

Let us consider a technique for adding new triangles to the N-array arrangement that we already have, which seems to be a fairly good choice for being close to the optimal choice. As above, for four triangles in Remark 2.3, a good arrangement to cover area consists of a central equilateral triangle T_1 with vertices on ∂D, and three isosceles triangles T_2, T_3, T_4 on the edges of T_1 with one edge of each of them coincident with the corresponding edge of T_1 and the vertices of the isosceles on ∂D again. So then one has for the union $\bigcup_{i=1}^{4} T_i$ is a 6-array A_1. Now inductively add new isosceles triangles as in the first step on each edge of this 6-array to get a 12-array A_2, and so on generating arrays $A_k, k \geq 1$. The numbers of edges on the array A_k then can be seen to be $N_k = 3(2^k)$. So this is giving us an area covered inside of the disc consisting of N_k isosceles triangles with one vertex at the origin. But it is actually also covering area using $M_k = 1 + 3(2^k - 1)$ triangles consisting of a central equilateral triangle and isosceles triangles attached on the outside in the inductive fashion described above. So $N_k = M_k + 2$. This gives the following.

Proposition 3.4 *For k, there is a covering of area in the disc with $M_k = 1 + 3(2^k - 1)$ triangles that covers an area $A_k < \pi$ with an error $E_k = \pi - A_k \sim \frac{2\pi^3}{3(M_k+2)^2}$.*

Proof. Use the discussion above and Proposition 3.1. □

Continuing to look closely at N-arrays, there is a worthwhile observation that might give insights into what is the optimal covering with triangles. Take an N-array $\mathcal{A}$ with total area $A(N)$. Then along $K \leq N$ exterior edges $E_1, \ldots, E_K$ add an isosceles triangle with the same edge E_i and all its vertices on ∂D. This gives a set of $N + K$ triangles $\mathcal{B}(N, K)$ which covers an area $B(N, K)$. Now modify the shape of the triangles in $\mathcal{B}(N, K)$ so that their union is the same as an $N + K$-array $\mathcal{C}$ with area $C(N, K)$. This is similar to the process observed in **B**), Remark 2.3, where $N = 3$ and $K = 2$, except that now more isosceles triangles are being added.

This set up gives the following proposition. Since this is not used directly in this paper, for brevity's sake the details will be left to the reader. But at least observe that the formula for the areas can be derived using the formulas in Proposition 3.1.

Proposition 3.5 *The area $B(N, K) = (N - K)\frac{\sin(2\pi/N)}{2} + 2K\frac{\sin(2\pi/2N)}{2}$ and the area $C(N, K) = (N + K)\frac{\sin(2\pi/(N+K))}{2}$. Moreover $C(N, K) > B(N, K)$.*

Remark 3.6 Keeping all the triangles in $\mathcal{B}(N, K)$ in their relative positions, with their vertices on ∂D, then the maximum area case could be, as in the case where $N = 3$ and $K = 2$, when the union of the triangles is an $N + K$ array. This suggests, but certainly does not prove, that N-arrays hold the key for what is the optimal choice of triangles to cover area in a disc. □

3.1 The Gauss Circle Theorem

Before ending this section, it is worth comparing this results of the placement of congruent isosceles triangles in the disc, that have been considered above, with the error obtained from the asymptotics of the Gauss circle problem. See Lax and Phillips [4] for background on the Gauss circle problem.

The Gauss circle problem is to count the number of lattice points in $\mathbb{Z}^2$ inside a disc of radius R, centered at the origin. Each square can be divided by a diagonal into two isosceles triangles. So the number of these triangles is asymptotically $2\pi R^2$. Now rescale. This means pack N_R congruent isosceles triangles into D with $N_R \sim 2\pi R^2$. Each of these triangles have an area $1/2R^2$.

Now the error in the Gauss circle problem is still not known precisely, but the literature shows that it is near to $R^{1/2}$. The current thinking is that it is $O(R^{1/2+\epsilon})$ for all $\epsilon > 0$. It is known that one cannot decrease the exponent $1/2$. Now $R^{1/2+\epsilon}/R^2 = 1/R^{3/2-\epsilon}$.

But $R \sim CN_R^{1/2}$ for some constant C. So this gives an error for the covering of area in D with the N_R congruent isosceles triangles that is on the order of $\frac{C}{N_R^{1/2(3/2-\epsilon)}}$ for some constant C. That is the error is $1/R^{3/4-\epsilon}$, perhaps for all $\epsilon > 0$. But note that the basic exponent here is then $3/4$, which is good but not as good as the exponent 2 that is given in Proposition 3.3.

4 Under Estimates for the Minimal Error

While it is not evident what $\mathcal{A}(N)$ and $\mathcal{E}(N)$ are, there is an upper bound for $\mathcal{E}(N)$ from Corollary 3.3. However, the optimal covering could potentially give a minimal error that is much less than this bound. In this section, an argument is given that this is not the case, as far as the order of magnitude of the error goes.

Given a triangle T with two vertices V_1 and V_2 on ∂D, the *cap* determined by T is the set in D bounded by ∂D and the edge between the two vertices V_1 and V_2. If in fact the three vertices of T are all on ∂D then there are three caps determined by T.

Lemma 4.1 *Suppose there are N disjoint triangles in D and some subset of them have vertices on ∂D with caps covering ∂D. Assume the caps themselves do not contain any subsets of the other triangles. Then the area $A(N)$ covered by the N triangles has an error $E(N) = \pi - A(N) \geq C/N^2$ for some universal constant C.*

Proof. The number of caps here is no more than $M \leq 3N$. In this case, $M = 3N$ because all the vertices of all the T_i are on ∂D, and the vertices are all distinct points as i varies. Associate to the caps an array of triangles S_i with a vertex at the origin, and two vertices at the ends of the caps, which is not necessarily an N-array, but rather one in which the angles at the origin might be different values α_i. The assumptions about the caps imply that $\sum_{i=1}^{M} \alpha_i = 2\pi$, and the error of the original triangles $E(N) \geq \pi - \sum_{i=1}^{M} A_i$ where A_i is the area of S_i. As above, the area $A_i = \sin(\alpha_i/2)\cos(\alpha_i/2) = \frac{1}{2}\sin(\alpha_i)$. In minimizing this underestimate, without loss of generality it can be assumed that all the angles $\alpha_i < \pi/2$.

Use Lagrange multipliers to minimize $\pi - \sum_{i=1}^{M} \frac{1}{2}\sin(\alpha_i) = \frac{1}{2}\sum_{i=1}^{M} \alpha_i - \sum_{i=1}^{M} \frac{1}{2}\sin(\alpha_i)$ subject to the constraint $\sum_{i=1}^{M} \alpha_i = 2\pi$. That is, find the critical points of $L(\alpha_1, \ldots, \alpha_M, \lambda) =$

$\pi - \sum_{i=1}^{M} \frac{1}{2}\sin(\alpha_i) + \lambda(\sum_{i=1}^{M} \alpha_i - 2\pi)$. The immediate result is that for all i, $\frac{\partial L}{\partial \alpha_i} = 0$ gives $\frac{1}{2}(1 - \cos(\alpha_i) + \lambda = 0$. Hence, all of the α_i are equal to a fixed value α, since $0 \leq \alpha_i < \pi$ for all i, with $M\alpha = 2\pi$. Now one can check that this solution is giving the minimum, which can be done using standard second order derivative methods.

This calculation tells us that the error can be bounded below by the error $E(M)$ for a standard M-array. What now follows is that $E(N) \geq E(M) \geq 2\pi^3/3M^2 \geq 2\pi^3/3(9N^2)$ □

The error for the N-array has been found above. Lemma 4.1 shows that for certain types of covering by N triangles, a lower bound for the minimal error can be directly computed, and it is the same order as the upper bound for the minimal error given by the N-array. In order to extend this, use the ideas in Lemma 4.1 to get a lower bound on the error $\mathcal{E}(N)$ in general.

Proposition 4.2 *Suppose there are N disjoint triangles in D. Then the maximum area $\mathcal{A}(N)$ covered by the N triangles has an error $\mathcal{E}(N) = \pi - \mathcal{A}(N) \geq C/N^2$ with $C = \frac{1}{9}(\frac{2}{3}\pi^3)$.*

Proof. Let $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \mathbf{a}_{i3}$ be respectively the nearest points on ∂D to the vertices $\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}$ of T_i. Obtain $\mathbf{a}_{ij}$ by extending a ray from the origin through $\mathbf{v}_{ij}$ for $j = 1, 2, 3$. By a small change in the triangles T_i, it may be assumed without loss of generality that the points $\mathbf{a}_{ij}$, for $i = 1, \ldots, N$ and $j = 1, 2, 3$, are distinct. Now label all the $\mathbf{a}_{ij}$ in counterclockwise fashion from $(1, 0)$ as $\mathbf{p}_k, k = 1, \ldots, 3N$. For each pair $\mathbf{p}_k, \mathbf{p}_{k+1}$, take a cap C_k. In this fashion these boundary points determine $3N$ caps $C_1 \ldots, C_{3N}$.

For large N and any choice of the triangles that makes the error small, the caps that are formed will all have small maximum width along rays to the origin. That is, there is a low value of N to consider. Indeed, with one triangle contained in the disc but to one side of a diameter, then there would be two smaller caps and one very large cap corresponding to it. But this configuration gets avoided as N gets larger and the error is smaller.

With this understood, none of the triangles T_i intersect any of the caps C_k . Indeed, if one did, then there would be a boundary point $\mathbf{a}_{ij}$ between $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$, contrary to the construction. So the construction of the caps C_k tells us that the error $\mathcal{E}(N)$ is larger than the sum of the areas of the C_k. But as in Lemma 4.1, this sum of cap areas is bounded below by $2\pi^3/3(9N^2)$ □

Remark 4.3 These results raise the possibility that there is actually a limit value for $N^2\mathcal{E}(N)$ as $N \to \infty$. A suggestion that this might hold comes from the asymptotic results in McClure and Vitale [6]. In particular, their Theorem 5 (ii) suggests what the constant might be. In [6], if one takes the domain K to be D, a disc of radius 1, then one has $r_K(\theta) = 1$ for all θ and so $n^2 \inf\{D_A(P_n, K) : P_n \in \mathcal{P}_n, P_n \subset K\} \to \frac{1}{12}(2\pi)^3 = \frac{2}{3}\pi^3$ as $n \to \infty$. This is the limiting value for $N^2\mathcal{E}(N)$ in the case of the N-array being used as a proxy to calculate $\mathcal{E}(N)$. So perhaps the same result holds for covering area by triangles inside D as it does for optimal area covered by polygons inscribed in D. □

Remark 4.4 Curiously enough, as opposed to finding the largest polygon inside of a disc, the covering result in Theorem 1 in McClure and Vitale [6] has a different asymptotic constant in the case of $K = D$, namely $\frac{1}{3}\pi^3$. Of course, this suggests that all of the work above could be changed to finding the smallest area for a union U of N triangles such that $D \subset U$. Now the miniml error would be the difference of the area of U and the smaller area π of D. The minimal error, say it is denoted by $\mathcal{E}_O(N)$ in this case, can be seen to be $O(1/N^2)$ like with the error $\mathcal{E}(N)$ above. But it is not clear if $\lim_{N\to\infty} N^2\mathcal{E}_O(N)$ exists, and it is not clear if in fact $\lim_{N\to\infty} N^2\mathcal{E}_O(N) = \lim_{N\to\infty} N^2\mathcal{E}(N)$. □

The bounds on $\mathcal{E}(N)$ have been obtained in this section in an elementary fashion. Proposition 3.1 is a standard calculation and Proposition 4.2 has a reasonably short proof, although it requires some discussion. But if one is willing to do more extensive and difficult arguments, then the bounds can be improved. For example, the lower bound of $\frac{1}{9}(\frac{2}{3}\pi^3)$ in Theorem 4.2 can be improved using McClure and Vitale [6].

Theorem 4.5 *There is an underestimate:*

$$\liminf_{N\to\infty} N^2\mathcal{E}(N) \geq \frac{1}{2}(\frac{2}{3}\pi^3).$$

Proof. Take N large and disjoint triangles $T_1, \ldots, T_N$ in D. Let P be the convex hull of $\bigcup_{n=1}^{N} T_n$. It is not hard to see that P has at most $2N$ edges. So Theorem 5 (ii) in McClure and Vitale [6] gives the underestimate. □

Remark 4.6 It seems likely that with more complications the results in this section can be extended from D to more general sets K in the plane. As part of this, McClure and Vitale [6] would be used. One basic question will be whether or not the constant they have in Theorem 5 (ii) is the same for covering an area with triangles in a general convex region K. □

5 Optimal Triangulations of a Surface

The area covering problem studied in this article fits into a wide class of triangulation problems. For example, a number of authors have sought to extend to higher dimensions the optimal approximation of curves by polygons, as in Gleason [2]. In this article, we have considered a very specialized version of this type of question where we are approximating the area of a disc using inscribed triangles

For surfaces this type of optimization problem would start with a bounded surface $\mathcal{S}$ in $\mathbb{R}^3$. Then consider the surface $cl(\mathcal{S})$ consisting of $\mathcal{S}$ together with its boundary. Fix $N \geq 1$. Now consider a polygonal surface $\mathcal{P}_N$ with N triangular faces whose vertices are on $cl(\mathcal{S})$. It is important to note that this would give only a *triangulation* of a subset $cl(\mathcal{S})$. But for simplicity, still refer to the choice of $\mathcal{P}_N$ as an *approximate triangulation.*

The challenge is to describe the optimal approximate triangulations of surfaces with a fixed number N of triangles. Optimality means having the area of the polyhedral surface given by the approximate triangulation to be as close to the surface area as possible. Describing such a triangulation means (at best) determining explicitly where the vertices are positioned. This can be difficult because all the vertices of the approximate triangulation have to change position as N changes. So determining the distribution of the vertices as N increases may be the best option for describing them. It might be possible to carry this process out using the ideas and techniques in Rosenblatt [7].

Describing the optimal approximate triangulation of a surface with a finite number of triangles is a difficult geometrical problem. But even in the case that the surface is flat (i.e. lies in a plane), but it cannot be tiled by triangles, there are issues in identifying an optimal approximate triangulation as we have seen in this article.

The results in Section 3 suggest that the error in an optimal approximate triangulation of a surface with N triangles could be on the order of $1/N^2$. To know this in more generality it probably would be necessary to have a uniform bound on the curvature of the surface.

Even if this is the case, it leaves open at least two questions. First, what is the optimal approximate triangulation? Second, what is the asymptotic distribution of the vertices of its triangles as N goes to infinity? One can hope to get limiting distribution results such as the ones in Rosenblatt [7].

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