Symmetry of f-Vectors of Toric Arrangements in General Position and Some Applications

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Abstract - A toric hyperplane is the preimage of a point $b \in S^1$ of a continuous surjective group homomorphism $\theta : \mathbb{T}^n \to S^1$. A toric hyperplane arrangement is a finite collection of such hyperplanes. In this paper, we study the combinatorial properties of toric hyperplane arrangements on $\mathbb{T}^n$ which are spanning and in general position. Specifically, we describe the symmetry of f-vectors arising in such arrangements and a few applications of the result to count configurations of hyperplanes.

Keywords : hyperplane arrangements; toric arrangements; deletion-restriction

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1 Introduction

1.1 Hyperplane Arrangements

Real hyperplane arrangements have a rich history, with [1] serving as a standard reference. In recent years, a variation in the form of toric hyperplane arrangements has been studied more prominently, from combinatorial, algebraic, and topological point of views; see [2, 3] for examples. Some applications of the theory of toric hyperplane arrangements to toric varieties have been presented in [4, 5].

In this paper, we study toric hyperplane arrangements, predominantly from a combinatorial perspective. The motivation of this paper is to enumerate toric arrangements of hyperplanes with fixed normal vectors subject to an equivalence relation. A natural invariant to study are the k-dimensional faces arising from a toric hyperplane arrangement $\mathcal{A}$. The enumerative information of k-dimensional faces, shortly k-faces, can be encoded in an f-vector, denoted by $\mathbf{f}(\mathcal{A})$. That is, for a toric hyperplane arrangement $\mathcal{A}$ on the n-dimensional torus $\mathbb{T}^n$, $\mathbf{f}(\mathcal{A})$ is an $n+1$-dimensional vector whose k-th entry is the number of k-faces in $\mathcal{A}$. Specifically, we choose to consider a special class of toric hyperplane arrangement which we will call spanning and in general position (Definition 2.9 and Definition 2.8, respectively). We prove the following relation between the faces of spanning toric hyperplane arrangements in general position.

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Theorem A (Theorem 4.8) *For a spanning toric hyperplane arrangement* $(\mathcal{A}, \mathbb{T}^n)$ *in general position, the* f*-vector* $\mathbf{f}(\mathcal{A})$ *is symmetric.*

In other words, the number of k-faces in this special class toric hyperplane arrangement $\mathcal{A}$ is equal to the number of $(n-k)$-faces of $\mathcal{A}$.

We also present a few applications of Theorem A. We say that two toric hyperplane arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, $\mathcal{A}_1 \sim \mathcal{A}_2$, if we can continuously translate a subset of hyperplanes in $\mathcal{A}_1$ to those of $\mathcal{A}_2$ without ever creating a triple intersection.

Theorem B (Theorem 6.9, Corollary 5.1) *Let* $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$ *be a set of toric hyperplanes on* $\mathbb{T}^2$*. Let* $\mathcal{M}$ *denote the space of all arrangements* $(\mathcal{A}, \mathbb{T}^2)$*. Then we are able to construct a toric hyperplane arrangement* $(\mathcal{M}', \mathbb{T}^{m-2})$*, which contains the space* $\mathcal{M}/\sim$ *parametrizing the translation of hyperplanes in* $\mathcal{A}$*. The* $(m-2)$*-faces of the parameter space* $\mathcal{M}'$ *then correspond to equivalence classes of arrangements of the* m *lines in general position under* $\sim$*.*

1.2 Outline

The paper is structured as follows. After the introduction, we establish the background for toric hyperplane arrangements in Section 2. We exhibit the main results of this paper on an example of toric hyperplane arrangements on $\mathbb{T}^2$ in Section 3. Next, in Section 4, we also generalise the deletion-restriction relation for n-dimensional tori and k-dimensional faces for arrangements in general position in Lemma 4.6. The generalised deletion-restriction then allows us to prove Theorem 4.8, which states that the number of k-faces is equal to the number of $(n-k)$-faces on arrangements on $\mathbb{T}^n$, for which we list a few applications. One of the applications of the theorem is to count the connected components of the complement of a toric hyperplane arrangement as shown in Section 5. We end the paper with Section 6, where we focus on classifying toric hyperplane arrangements on $\mathbb{T}^2$ under the equivalence relation. We show a few examples of constructing geometric spaces classifying toric hyperplane arrangements and give an upper bound for the number of arrangements for a set of hyperplanes with fixed normal vectors.

2 Background

Definition 2.1 *We define an* n*-torus* $\mathbb{T}^n$ *to be the topological group* $\mathbb{R}^n/\mathbb{Z}^n$*.*

There are plenty of ways we can think about tori, and we will use them interchangeably throughout the paper. For example, the n-dimensional torus $\mathbb{T}^n$ can be represented by an n-dimensional hypercube with the opposing faces identified. We also have

$$\mathbb{T}^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n}.$$

Another way to look at $\mathbb{T}^n$ is to consider the quotient map

$$\pi : \mathbb{R}^n \to {}^{\mathbb{R}^n}\!/\!_{\approx} \cong \mathbb{T}^n,$$

where for $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbb{R}^n$, we have $\tilde{\mathbf{x}} \approx \tilde{\mathbf{y}}$ if and only if for all $i \in \{1, 2, \ldots, n\}$ we have $\tilde{x_i} - \tilde{y_i} \in \mathbb{Z}$.

As $\mathbb{T}^n$ is a quotient of $\mathbb{R}^n$, many structures in the affine space $\mathbb{R}^n$ have analogues in the torus $\mathbb{T}^n$. We will denote points in $\mathbb{T}^n$ as $\mathbf{x}$, and their lifts in $\mathbb{R}^n$ as $\tilde{\mathbf{x}}$; therefore $\pi(\tilde{\mathbf{x}}) = \mathbf{x}$.

We now adapt the classical definition of a real hyperplane to $\mathbb{T}^n$. Fix a basis of $\mathbb{R}^n$. Then any surjective continuous group homomorphism $\theta : \mathbb{T}^n \to S^1$ can be lifted as a map $\tilde{\theta} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\tilde{\mathbf{x}} \mapsto \langle a_1, a_2, \ldots, a_n \rangle \cdot \tilde{\mathbf{x}} = a_1\tilde{x_1} + a_2\tilde{x_2} + \ldots + a_n\tilde{x_n},$$

where $a_1, a_2, \ldots, a_n$ are integer scalars and at least one a_i is non-zero. A toric hyperplane H is then given by a set of all vectors $\tilde{\mathbf{x}} \in \mathbb{R}^n$ such that

$$a_1\tilde{x_1} + a_2\tilde{x_2} + \ldots + a_n\tilde{x_n} = b \mod \mathbb{Z}$$

for a fixed arbitrary constant $b \in \mathbb{R}$ and their integer scalar multiples. Hence, we can characterise hyperplanes by their normal vector $\langle a_1, a_2, \ldots, a_n \rangle = \mathbf{a} \in \mathbb{Z}^n$ and their intercept b.

Definition 2.2 *Let $\theta : \mathbb{T}^n \to S^1$ be a non-zero surjective group homomorphism. Then the preimage $\theta^{-1}(b)$ of a point $b \in S^1$ defines an affine toric hyperplane H on $\mathbb{T}^n$. We call the point b the intercept of H.*

We will refer to affine toric hyperplanes as hyperplanes. Note that the definition of an affine toric hyperplane does not pose any restrictions on the map θ, so, unlike the case of real hyperplane arrangements, it is possible that some hyperplanes may appear with multiplicities if the normal vector entries have $\gcd(a_1, a_2, \ldots, a_n) > 1$.

Example 2.3 Let $(\mathcal{A}, \mathbb{T}^1)$ be an arrangement of one hyperplane H given by the normal vector $\mathbf{a} = \langle 5 \rangle$, as shown in Fig. 1. The normal vector $\mathbf{a}$ can also be expressed as $5 \cdot \langle 1 \rangle$ and as a set $\mathcal{A}$ consists of 5 equally spaced points on $\mathbb{T}^1$.

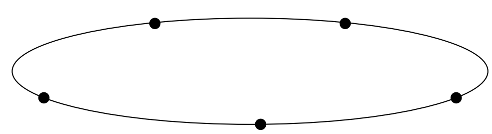

Figure 1: An example of an arrangement on $\mathbb{T}^1$ containing a non-simple hyperplane.

This is unlike real hyperplane arrangements for which non-primitive normal vectors just scale the hyperplane without disconnecting it.

Definition 2.4 *Let H be a hyperplane on $\mathbb{T}^n$ defined by a normal vector $\langle a_1, a_2, \ldots, a_n \rangle$. Then we say that H is a simple hyperplane if and only if $\gcd(a_1, a_2, \ldots, a_n) = 1$.*

Due to the following proposition, with a standard proof provided in [10], we will assume all hyperplanes to be simple unless otherwise stated. We supply our own proof for completeness.

Proposition 2.5 *Let H denote a hyperplane on $\mathbb{T}^n$. Then H is connected if and only if H is simple.*

Proof. Suppose H is not simple, so we assume $\gcd(a_1, a_2, \ldots, a_n) = m$ with $m > 1$. The hyperplane H is defined as the kernel of the map $\theta : \mathbb{T}^n \to S^1;\ \mathbf{x} \mapsto a_1x_1 + a_2x_2 + \ldots + a_nx_n$. We may rewrite the image of $\mathbf{x}$ under θ as $m(b_1x_1 + b_2x_2 + \ldots + b_nx_n)$. Therefore we can consider maps $\theta' : \mathbb{T}^n \to S^1$ given by $\mathbf{x} \mapsto b_1x_1 + b_2x_2 + \ldots + b_nx_n$, and a map $\theta_m : S^1 \to S^1$, multiplication by m, sending $\mathbf{x}$ to $m\mathbf{x}$. We factor θ as in the following commutative diagram.

$$\begin{tikzcd}
\mathbb{T}^n \arrow[rr, "\theta"] \arrow[dr, "\theta'"'] & & S^1 \\
& S^1 \arrow[ur, "\theta_m"'] &
\end{tikzcd}$$

Then $H = \theta'^{-1}(\theta_m^{-1}(c))$, for an intercept $c \in S^1$. Since $\theta_m^{-1}(c)$ is disconnected, it follows that H is disconnected as well.

Conversely, suppose H is simple where H is given by the kernel of the map

$$\theta : \mathbb{T}^n \to S^1; \mathbf{x} \mapsto \langle a_1, a_2, \ldots, a_n \rangle \cdot \mathbf{x}$$

where $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let $\mathbf{v} \in H$ be a vector such that $\langle a_1, a_2, \ldots, a_n \rangle \cdot \mathbf{v} = 1$, which exists by the Bézout identity. We may take two points of H, say

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \text{and} \quad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

By definition of H, we have $\theta(\mathbf{x}) = 0$ and $\theta(\mathbf{y}) = 0$. Denote by π the covering space $\pi : \mathbb{R}^n \to \mathbb{T}^n$ and let $\pi^{-1}(\mathbf{x}), \pi^{-1}(\mathbf{y}) \in \mathbb{R}^n$ be the lifts of $\mathbf{x}$ and $\mathbf{y}$ respectively.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\tilde{\theta}"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi'"] \\
\mathbb{T}^n \arrow[r, "\theta"] & S^1
\end{tikzcd}$$

The diagram above commutes. Fix specific points $\tilde{\mathbf{x}} \in \pi^{-1}(\mathbf{x}), \tilde{\mathbf{y}} \in \pi^{-1}(\mathbf{y})$. To show that H is path-connected, and therefore connected, we will show that there exists a choice of a translation to $\tilde{\mathbf{y}} \in \pi^{-1}(\mathbf{y})$ and a path γ from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$ such that $\gamma \subseteq \pi^{-1}(H)$, where $\pi^{-1}(H)$ is a collection of affine hyperplanes in $\mathbb{R}^n$. Let $\tilde{H}$ be a hyperplane in $\mathbb{R}^n$ so that $\pi(\tilde{H}) = H$, and we fix points $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y_0}}$. We will call $\tilde{H}$ a representative of $\pi^{-1}(H)$. We try a path given by

$$\gamma_k(t) : [0, 1] \to \mathbb{R}^n;\ t \mapsto (1 - t)\tilde{\mathbf{x}} + t(\tilde{\mathbf{y_0}} + k\mathbf{v}), \quad \text{for some } k \in \mathbb{Z}.$$

We want to find a choice of an integer k so that

$$\tilde{\theta}(\tilde{\mathbf{y_0}} + k\mathbf{v}) - \tilde{\theta}(\tilde{\mathbf{x}}) = 0$$

which will subsequently imply that $\gamma_k \subseteq \tilde{H}$. By assumption, we have $\tilde{\theta}(\mathbf{v}) = 1$, so by linearity, $\tilde{\theta}(\tilde{\mathbf{y_0}} + k\mathbf{v})$ hits every connected component of $\pi^{-1}(H)$, one of which must satisfy the relation $\tilde{\theta}(\tilde{\mathbf{y_0}} + k\mathbf{v}) - \tilde{\theta}(\tilde{\mathbf{x}}) = 0$. Therefore, we have found a unique representative $\tilde{\mathbf{y}} = \tilde{\mathbf{y_0}} + k\mathbf{v}$ in $\pi^{-1}(\mathbf{y})$. Now suppose that our choice of k satisfies $\tilde{\theta}(\tilde{\mathbf{x}}) - \tilde{\theta}(\tilde{\mathbf{y_0}} + k\mathbf{v}) = 0$. It is left to check that $\pi(\gamma_k) \subseteq H$. Because $\pi^{-1}(H)$ can be thought of as the union of integer translates of $\tilde{H}$ in $\mathbb{R}^n$, we have that $\tilde{\theta}(\gamma_k(t)) = \tilde{\theta}(\tilde{\mathbf{x}}) \in \mathbb{Z}$. So, $\pi'(\tilde{\theta}(\gamma_k)) = \pi'(\mathbb{Z}) = 0$. This is equivalent to $\theta(\pi(\gamma_k)) = 0$, which implies $\pi(\gamma_k) \in \ker(\theta)$, hence $\pi(\gamma_k) \subseteq H$. □

We illustrate the idea in the proof above for $\mathbb{R}^2$ in Fig. 2.

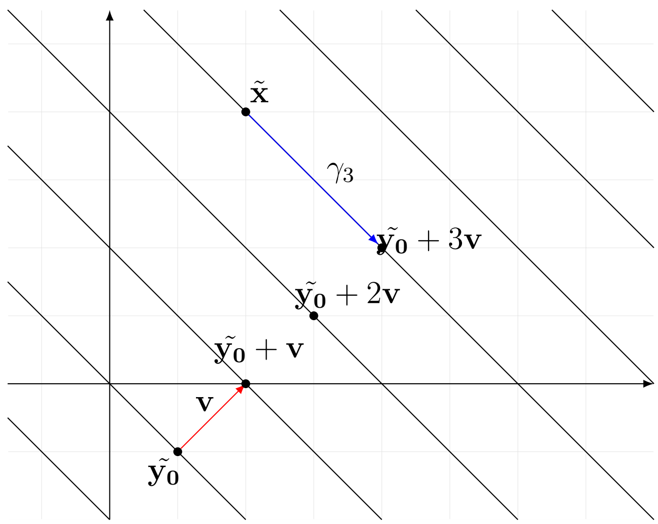

Figure 2: The lift of a hyperplane H of $\mathbb{T}^2$ into $\mathbb{R}^2$.

Remark 2.6 It is possible to view a non-simple hyperplane H in $\mathbb{T}^n$ as a collection of simple hyperplanes instead. Supposing that H is given by a normal vector $\langle a_1, a_2, \ldots, a_n \rangle$ with $\gcd(a_1, a_2, \ldots, a_n) = m$ and an intercept b, the data of a collection of m hyperplanes $H_1, H_2, \ldots, H_m$ with intercepts $b, b + \frac{1}{m}, \ldots, b + \frac{m-1}{m}$, respectively, and all with the same normal vector $\frac{1}{m}\langle a_1, a_2, \ldots, a_n \rangle$ is equivalent.

In the setting of $\mathbb{R}^n$, we define a real hyperplane arrangement $(\mathcal{A}, \mathbb{R}^n)$ to be a finite set of hyperplanes $\mathcal{A}$ in the ambient space $\mathbb{R}^n$. In complete analogy, we now define a toric hyperplane arrangement.

Definition 2.7 *A toric hyperplane arrangement $(\mathcal{A}, \mathbb{T}^n)$ is a finite set $\mathcal{A}$ of affine toric hyperplanes in the ambient space $\mathbb{T}^n$. Denote the number of hyperplanes in $\mathcal{A}$ by $|\mathcal{A}|$.*

For the rest of the paper, we will refer to toric hyperplane arrangements as hyperplane arrangements or just arrangements. By a standard abuse of notation, we often omit the ambient space $\mathbb{T}^n$ and denote arrangements as $\mathcal{A}$ whenever the dimension of the ambient space is clear.

Let an arrangement $(\mathcal{A}, \mathbb{T}^n)$ consist of hyperplanes given by maps $\theta_1, \theta_2, \ldots, \theta_m$ and intercepts $b_1, b_2 \ldots, b_m$. After fixing a basis, denote by Θ the system of hyperplanes of

$\mathcal{A}$ which we define to be a matrix in $\mathrm{Mat}_{m\times n}(\mathbb{Z})$. We can encode the information about hyperplanes in $\mathcal{A}$ using a pair of matrices,

$$(\Theta, \mathbf{b}) = \left(\begin{pmatrix} \leftarrow \mathbf{a_1} \rightarrow \\ \leftarrow \mathbf{a_2} \rightarrow \\ \vdots \\ \leftarrow \mathbf{a_m} \rightarrow \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right).$$

Definition 2.8 ([8]) *Let $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$ be a set of hyperplanes. An arrangement $(\mathcal{A}, \mathbb{T}^n)$ is in general position if*

- $\{H_1, H_2, \ldots, H_k\} \subseteq \mathcal{A}, k \le n$: $\quad \dim(H_1 \cap H_2 \cap \ldots \cap H_k) \le n - k$,
- $\{H_1, H_2, \ldots, H_k\} \subseteq \mathcal{A}, k > n$: $\quad H_1 \cap H_2 \cap \ldots \cap H_k = \emptyset$.

In the case of arrangements on $\mathbb{T}^2$, an arrangement in general position means arrangements without triple intersections. Unless otherwise stated, every arrangement will be assumed to be in a general position. We also wish to exclude the arrangements where all hyperplanes parallel to each other.

Definition 2.9 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement for which $|\mathcal{A}| \ge n$. We say that $\mathcal{A}$ is spanning if there exists a subset $I \subseteq \mathcal{A}$ of size n such that $\bigcap_{i\in I} H_i \neq \emptyset$.*

Equivalently, after fixing a basis, we call an arrangement $\mathcal{A}$ spanning if there exists a subset of hyperplanes whose normal vectors span $\mathbb{R}^n$.

Example 2.10 We may define an arrangement $(\mathcal{I}, \mathbb{T}^n)$ where $\mathcal{I} = \{H_1, H_2, \ldots, H_n\}$ and each hyperplane $H_i = \ker(\theta_i)$, which is equivalent to the intercept being 0, is defined by the map

$$\theta_i : \mathbb{T}^n \to S^1;\ \theta(\mathbf{x}) = \mathbf{e}_i \cdot \mathbf{x}.$$

We call this arrangement the standard arrangement $\mathcal{I}$ since each hyperplane H_i is given by a standard basis vector. The standard arrangement is clearly spanning and in general position, since the n hyperplanes intersect in a single point.

Definition 2.11 *A partially ordered set (or a poset) is a set P and a relation $\le$ satisfying the following axioms: for all $x, y, z \in P$*

1. *(reflexivity) $x \le x$,*
2. *(anti-symmetry) if $x \le y$ and $y \le x$, then $x = y$,*
3. *(transitivity) if $x \le y$ and $y \le z$, then $x \le z$.*

Definition 2.12 ([8]) *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement, and let $L(\mathcal{A})$ be the set of connected components of non-empty intersections of hyperplanes in $\mathcal{A}$, including $\mathbb{T}^n$ itself as the intersection over the empty set. Define $x \le y$ in $L(\mathcal{A})$ if $x \supseteq y$. We call $L(\mathcal{A})$ the intersection poset of $\mathcal{A}$.*

Example 2.13 Let $(\mathcal{A}, \mathbb{T}^2)$ be an arrangement of two hyperplanes H_1 and H_2 given by vectors $\langle 1, 2\rangle$, and, $\langle 1, -2\rangle$ respectively, as shown in Fig. 5. The arrangement $\mathcal{A}$ is in general position, since $\dim(H_1 \cap H_2) = 0 = \dim(\mathbb{T}^2) - |\mathcal{A}|$. Fig. 3 shows the intersection poset $L(\mathcal{A})$ of the arrangement $\mathcal{A}$. Note that the maximum element $\emptyset$, the intersection over the empty set, is usually omitted from $L(\mathcal{A})$.

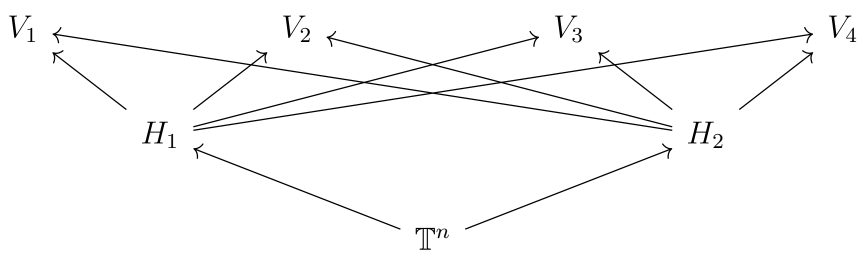

Figure 3: An arrangement $(\mathcal{A}, \mathbb{T}^2)$ and its intersection poset $L(\mathcal{A})$.

Definition 2.14 *Let I be the indexing set for hyperplanes in an arrangement $\mathcal{A}$. The complement $\mathbb{T}^n \setminus \bigcup_{i\in I} H_i$ consists of a disjoint union of finitely many connected open regions called regions of an arrangement $\mathcal{A}$, where the arrangement is not necessarily spanning or in general position. We will denote the set of such connected components $R(\mathcal{A})$. Set $r(\mathcal{A}) := |R(\mathcal{A})|$.*

Unlike real hyperplane arrangements, where every region is simply connected, we have seen examples of arrangements having regions with an infinite cyclic fundamental group, namely any non-spanning arrangements. However, the following lemma asserts that when we assume arrangements to be in general position and spanning, the regions are always simply connected in spanning arrangements in general position.

Lemma 2.15 *Suppose $(\mathcal{A}, \mathbb{T}^n)$ is a spanning arrangement in general position with $|\mathcal{A}| \geq n$. Then every element of $R(\mathcal{A})$ is simply connected.*

Proof. Suppose a spanning subset of hyperplanes $H_1, H_2, \dots, H_n$ is given by maps $\theta_1, \theta_2, \dots, \theta_n$ respectively. Since we assume $|\mathcal{A}| \geq n$, the set of hyperplanes must span the torus $\mathbb{T}^n$. In other words,

$$\Theta : \mathbb{T}^n \xrightarrow{(\theta_1,\, \theta_2,\, \dots\, ,\, \theta_n)} (S^1)^n$$

is a covering space because, by assumption, the spanning set of hyperplanes in $\mathbb{T}^n$ is locally homeomorphic to $(S^1)^n$. Then Θ induces maps

$$\Theta_* : \mathbb{T}^n \setminus \bigcup_{i\leq n} H_i \xrightarrow{(\theta_1,\, \theta_2,\, \dots\, ,\, \theta_n)} (S^1 \setminus \bigcup_{i\leq n} b_i)^n,$$

which is also a covering space. Furthermore, $(S^1 \setminus \bigcup_{i\leq n} b_i)^n \cong I^n$ where I is the unit interval, so the connected components of $\mathbb{T}^n \setminus \bigcup_{i\in I} H_i$ are convex polytopes, since $\pi_1(I^n) \cong$

$\{e\}$. The proof is complete after noting that removing more than n hyperplanes leaves the regions convex. $\square$

Therefore, we can get away with considering simply connected polytopes which arise in $\mathcal{A}$.

Definition 2.16 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then $(\mathcal{B}, \mathbb{T}^n)$ is called a subarrangement. For $X \in L(\mathcal{A})$ define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by*

$$\mathcal{A}_X = \{H \in \mathcal{A} \colon X \subseteq H\}.$$

Define an arrangement $(\mathcal{A}^X, X)$ in X by

$$\mathcal{A}^X = \{X \cap H \colon H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

We call $\mathcal{A}^X$ the restriction of $\mathcal{A}$ to X.

Choose a hyperplane $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ be the deletion of H_0 and we call $\mathcal{A}'' = \mathcal{A}^{H_0}$ to be the restriction to H_0. The deletion $\mathcal{A}'$ and the restriction $\mathcal{A}''$ are also toric arrangements. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a triple of arrangements with distinguished hyperplane H_0. See an example of such a triple on $\mathbb{T}^2$ in Fig. 6.

Example 2.17 Suppose we have a line arrangement $(\mathcal{A}, \mathbb{T}^2)$, with $|\mathcal{A}| \geq n$. We want to describe the sets of vertices, edges, and faces using intersections of hyperplanes. Firstly, we have from Definition 2.14 the 2-dimensional objects which we will call 2-faces. So we define $\Sigma_2(\mathcal{A})$ to be the set

$$\pi_0\left(\mathbb{T}^2 \setminus \bigcup_{i \in I} H_i\right),$$

where $\pi_0(X)$ denotes the connected components of a topological space X. Similarly, we can also describe the 0-dimensional objects which we will call 0-faces,

$$\Sigma_0(\mathcal{A}) = \bigcup_{i,j \in I} \pi_0(H_i \cap H_j),$$

because a 0-faces arise as intersections of 2 hyperplanes in $\mathcal{A}$. Finally, the 1-dimensional objects of $\mathcal{A}$ must be subsets of the hyperplanes themselves. Moreover, we want them to be the edges which we get from the intersections of hyperplanes. Therefore, we write

$$\Sigma_1(\mathcal{A}) = \bigcup_{i \in I} \pi_0(H_i \setminus \Sigma_0(\mathcal{A})).$$

Definition 2.18 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement, I its indexing set, and $L(\mathcal{A})$ its intersection poset. We define an n-face of $\mathcal{A}$ to be an element of the set*

$$\Sigma_n(\mathcal{A}) = \pi_0\left(\mathbb{T}^n \setminus \bigcup_{i \in I} H_i\right).$$

For $0 \le k < n$, let a k-face be an element of the set

$$\Sigma_k(\mathcal{A}) = \pi_0 \left(\bigcup_{\substack{X \in L(\mathcal{A}) \\ \dim(X) = k}} \Sigma_k(\mathcal{A}^X) \right).$$

Let us denote the number of k-faces $|\Sigma_k|$ of an arrangement $\mathcal{A}$ by $f_k(\mathcal{A})$.

We take $R(\mathcal{A})$ to mean the same objects as the n-faces Σ_n when $\mathcal{A}$ is spanning and in general position by Lemma 2.15. Since this paper concerns mostly spanning arrangements in general position, we will frequently use the following corollary.

Corollary 2.19 *Suppose $(\mathcal{A}, \mathbb{T}^n)$ is a spanning arrangement in general position. Then every k-face in $\mathcal{A}$ is simply-connected.*

Proof. Any k-face σ_k can be considered as a region of the restriction on $\mathbb{T}^k$ which is given by the intersection of hyperplanes containing σ_k. Hence it suffices to show that the restriction of $\mathcal{A}$ to $\mathcal{A}^{|\sigma_k|}$, where $|\sigma_k|$ is the affine linear span of σ_k, is spanning whenever $\mathcal{A}$ is spanning in order to apply Lemma 2.15. By definition, $\mathcal{A}$ is a collection of hyperplanes $(H_i)_{i \in I}$ given by maps $\theta_i : \mathbb{T}^n \to S^1$. Each map θ_i lifts to a map $\tilde{\theta}_i : \mathbb{R}^n \to \mathbb{R}$. Note that $\tilde{\theta}_i$ is an element of the dual space $(\mathbb{R}^n)^*$. Since $(\tilde{\theta}_i)_{i \in I}$ span $\mathbb{R}^n$, we can construct a spanning set $\{\tilde{\theta}^j\}_{j \in I}$ for $(\mathbb{R}^n)^*$ which satisfies $\tilde{\theta}^j(\tilde{\theta}_i) = \delta_i^j$. Then, $\mathcal{A}^{|\sigma_k|}$ is essentially an arrangement on $\mathbb{T}^n$, which gives rise to an inclusion map $\iota : \mathbb{T}^k \hookrightarrow \mathbb{T}^n$ and its lift $\tilde{\iota} : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$. By taking the dual, we have a surjective map $\tilde{\iota}^* : (\mathbb{R}^n)^* \twoheadrightarrow (\mathbb{R}^k)^*$. Moreover, the lifts of the hyperplanes in the arrangement $(\mathcal{A}^{\sigma_k}, \mathbb{T}^k)$ are given by maps $\tilde{\iota}^*(\tilde{\theta}_i)$. But since the collection $(\tilde{\theta}_i)_{i \in I}$ spans $\mathbb{R}^n$ and $\tilde{\iota}^*$ is surjective, $\tilde{\iota}^*(\tilde{\theta}_i)$ spans $\mathbb{R}^k$. $\square$

We will also use the convention that there are no (-1)-faces when considering any arrangement $(\mathcal{A}, \mathbb{T}^n)$, in other words $f_{-1}(\mathcal{A}) = 0$.[1]

Definition 2.20 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement. The f-vector $\mathbf{f}(\mathcal{A})$ of an arrangement $\mathcal{A}$ is defined as*

$$\mathbf{f}(\mathcal{A}) = (f_0(\mathcal{A}), f_1(\mathcal{A}), \ldots, f_n(\mathcal{A})).$$

We have previously defined an intersection poset $L(\mathcal{A})$ for an arrangement $(\mathcal{A}, \mathbb{T}^n)$. Since faces arise as intersections of hyperplanes, we define the face poset $\mathcal{L}(\mathcal{A})$.

Definition 2.21 ([1]) *The face poset $\mathcal{L}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is the poset of faces of $\mathcal{A}$ ordered by reverse boundary inclusion, that is, $\tau \lesssim \sigma$ if $\overline{\tau} \subseteq \sigma$.*

The face poset is a sharper invariant than the intersection poset. We may define a map $\zeta : \mathcal{L}(\mathcal{A}) \to L(\mathcal{A})$ by $P \mapsto |P|$, where $|P|$ is the support (i.e. the affine linear span) of P. The map ζ is order-preserving and surjective. It is true that if two arrangements $\mathcal{A}_1, \mathcal{A}_2$ have the same face poset, then $L(\mathcal{A}_1) = L(\mathcal{A}_2)$. The converse does not hold in general.

[1] Note that the convention, such as in [6], is to consider the empty set as a (-1)-face of $\mathcal{A}$, so $f_{-1}(\mathcal{A}) = 1$. Our adjustment is justified by the fact that the empty set is not simply-connected.

Example 2.22 We continue the example of a line arrangement $\mathcal{A}$ with $H_1 = \langle 1, 2\rangle$ and $H_2 = \langle 1, -2\rangle$. The faces of the arrangement are vertices, edges and faces which we will label V, E, and F respectively. Ordering the faces by reverse boundary inclusion, we get the face poset in Fig. 4. In this example, we leave out the maximal element $\emptyset$.

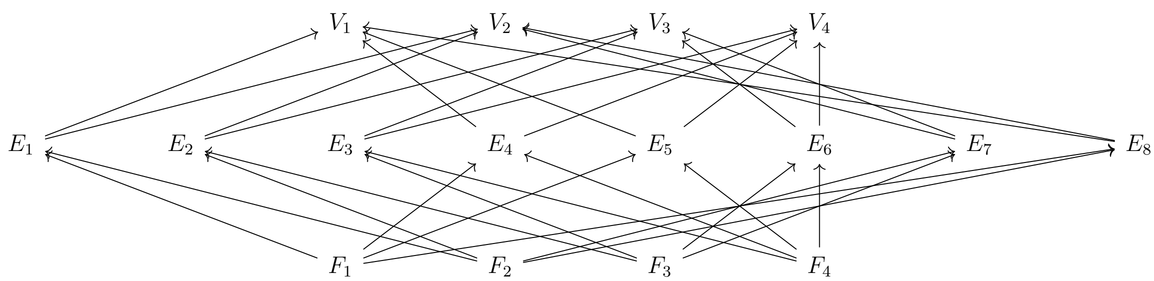

Figure 4: An example of a face poset $\mathcal{L}(\mathcal{A})$.

3 The Two-Dimensional Case

To concretely present the results of this paper, we begin by looking at the case of $\mathbb{T}^2$ which provides intuition for the general case. We sometimes refer to hyperplanes as lines when working on $\mathbb{T}^2$. We will provide proofs to general results later.

3.1 Symmetry of f-Vectors

Example 3.1 Consider the arrangement $(\mathcal{A}, \mathbb{T}^2)$ of hyperplanes H_1 and H_2 given by the normal vectors $\langle 1, 2\rangle$ and $\langle 1, -2\rangle$, respectively, shown in Fig. 5. Then, $\mathcal{A}$ is in general position and it is spanning. The arrangement $\mathcal{A}$ comes with vertices, edges and regions. Denote the number of vertices by f_0, the number of edges by f_1, and the number of regions by f_2. Then the f-vector $\mathbf{f}(\mathcal{A})$ for $\mathcal{A}$ is the vector with the i-th entry being f_i. In our example, we have $\mathbf{f}(\mathcal{A}) = (4, 8, 4)$, in particular, $f_0 = f_2$ for the arrangement $\mathcal{A}$.

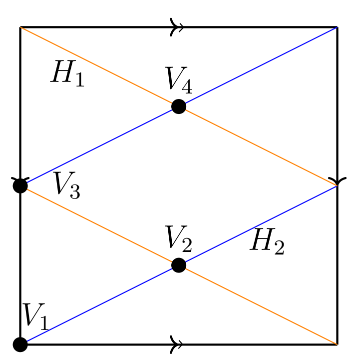

Figure 5: A toric arrangement on $\mathbb{T}^2$ given by hyperplanes $H_1 = \langle 1, 2\rangle$ and $H_2 = \langle 1, -2\rangle$.

In fact, for a line arrangement in general position, $\mathcal{A}$ on $\mathbb{T}^2$, the number of regions f_2 is equal to the number of vertices f_0. The claim can be proven using basic graph theory. For any arrangement of lines $\mathcal{A}$, let $f_0(\mathcal{A})$ be the number of vertices in $\mathcal{A}$ and $f_1(\mathcal{A})$ be the number of edges in $\mathcal{A}$. By a standard graph-theoretic result,

$$\sum_{i=1}^{n} \deg(v_i) = 2f_1(\mathcal{A}).$$

Since $\mathcal{A}$ is in general position, any vertex is an intersection of exactly two lines, so the degree of any vertex is 4. Thus, we have $4f_0(\mathcal{A}) = 2f_1(\mathcal{A})$, implying $2f_0(\mathcal{A}) = f_1(\mathcal{A})$. Since the Euler characteristic of $\mathbb{T}^2$ is given by $f_0(\mathcal{A}) - f_1(\mathcal{A}) + f_2(\mathcal{A}) = 0$, substituting our previous result yields $f_0(\mathcal{A}) = f_2(\mathcal{A})$ as required. However, such reasoning explicitly relies on the Euler characteristic, which is only applicable to arrangements with contractible faces. Hence, this reasoning is not applicable to cases with non-simply connected regions, like arrangements with all lines parallel. We also need to assume, for this argument, that not all hyperplanes are parallel and there are no triple intersections in our arrangements.

3.2 Counting Regions of Arrangements

The symmetry of f-vectors can be utilised to count the number of regions of an arrangement $\mathcal{A}$ from the normal vectors of hyperplanes. Let $\mathbf{a_i}$ denote the normal vector of a line H_i. A standard result in algebraic topology ([9], or Corollary 4.3 for the general result) states that the number of intersections arising between two lines H_1 and H_2 is the absolute value of the determinant

$$f_0(\mathcal{A}) = |\det(\mathbf{a_1}|\mathbf{a_2})| \ .$$

Generalising to $m > n$ lines, and assuming the arrangement is still in general position, we have that the number of intersections is given by

$$f_0(\mathcal{A}) = \sum_{H_i, H_j \in \mathcal{A}} |\det(\mathbf{a_i}|\mathbf{a_j})|.$$

When our arrangement is in general position, we use the symmetry $f_0(\mathcal{A}) = f_2(\mathcal{A})$, so we only need to count the number of line intersections. Using the topological fact that $f_0(\mathcal{A}) - f_1(\mathcal{A}) + f_2(\mathcal{A}) = 0$, we also obtain $f_1(\mathcal{A}) = 2f_0(\mathcal{A})$. Therefore, we have reduced the problem of counting the regions $f_2(\mathcal{A})$ of toric arrangements to counting intersections between the hyperplanes.

3.3 Counting Arrangements Under Equivalence

We take two line arrangements to be equivalent if and only if we can continuously translate a subset of lines in one arrangement to get the other arrangement without creating a triple intersection at any point. Denote an equivalence of two arrangements by $\sim$. The question is: how many distinct line arrangements are there given a set of lines with fixed slope?

Suppose that we have an arrangement $\mathcal{A}'$ on $\mathbb{T}^2$ such that $|\mathcal{A}'| = 2$. There is only one such arrangement, as there is no possibility of a triple intersection when translating either of the hyperplanes.

We give an intuitive way of counting the maximum number of distinct arrangements of 3 lines. We consider adding a new hyperplane H_3 to $\mathcal{A}'$. Since we want to enumerate distinct arrangements subject to $\sim$, we look for arrangements containing triple intersections. We are able to give an upper bound for the possible triple intersections by looking at complete (i.e. double) intersections of the original arrangement $\mathcal{A}'$. Through each complete intersection, we place the new hyperplane H_3, therefore we can have at most $f_0(\mathcal{A})$ possible ways of creating a triple intersection via the addition of H_3. Note that H_3 may pass through more than one complete intersection, which disregards the symmetry of $\mathbb{T}^2$, hence we only get an upper bound. However, to obtain a better bound, we need to consider this process with every hyperplane in $\mathcal{A}$. Using the global translation of the torus, we may fix an intersection of a pair of hyperplanes H_i and H_j at the origin. Let $\mathcal{A}_{H_i}$ denote the deletion of a line H_i from $\mathcal{A}$, symbolically $\mathcal{A}_{H_i} = \mathcal{A} \setminus H_i$. Therefore, we find that the number of distinct toric arrangements of the three hyperplanes is at most

$$\min_{H_i \in \mathcal{A}} (f_0(\mathcal{A}'_{H_i})).$$

Thus, we have established an upper bound for the number of distinct toric arrangements of three lines with fixed normal vectors on $\mathbb{T}^2$. However, we will later see that there is only one arrangement of three lines so that $\mathcal{A}$ is in general position, once we factor in the symmetries of $\mathbb{T}^2$. Hence, the bound is not the tightest, but we address this in the last section.

4 Deletion-Restriction Relations

In this section, we discuss the toric analogue of the deletion-restriction relations ([8]) and its application to proving the symmetry between the number of k-faces and the number of $(n-k)$-faces of an arrangement $(\mathcal{A}, \mathbb{T}^n)$. As will be shown, it is possible to enumerate the faces of spanning arrangements in general position and thus determining their f-vector. Before doing so, we prove the following lemma.

Lemma 4.1 *Consider the covering $\pi : \mathbb{R}^n \to \mathbb{T}^n$ and let $\hat{\varphi}$ be a map $\hat{\varphi} : \mathbb{R}^n \to \mathbb{R}^n$ that sends $\boldsymbol{v} \in \mathbb{Z}^n$ to $\hat{\varphi}(\boldsymbol{v}) \in \mathbb{Z}^n$. Then there exists a map $\varphi : \mathbb{T}^n \to \mathbb{T}^n$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \mathbb{R}^n \arrow[d, "\pi"] \\
\mathbb{T}^n \arrow[r, "\varphi"'] & \mathbb{T}^n
\end{tikzcd}$$

Proof. It suffices to check that the maps are well-defined, that is, the action of $\mathrm{GL_n}(\mathbb{Z})$ is well-defined. Let $A \in \mathrm{GL}_n(\mathbb{Z})$ be an $n \times n$ invertible matrix with integer entries, and define

$\varphi : \boldsymbol{v} \mapsto A\boldsymbol{v}$. Let $[\boldsymbol{v}] \subset \mathbb{T}^n$ be an equivalence class of matrices. Consider two distinct representatives $\boldsymbol{v_1}, \boldsymbol{v_2} \in [\boldsymbol{v}]$, so $\boldsymbol{v_1} - \boldsymbol{v_2} \in \mathbb{Z}^n$. Then $[\varphi(\boldsymbol{v_1})] = [\varphi(\boldsymbol{v_2})]$ is equivalent to $A\boldsymbol{v_1} - A\boldsymbol{v_2} = A(\boldsymbol{v_1} - \boldsymbol{v_2})$. Hence, we have $\boldsymbol{v_1} - \boldsymbol{v_2} \in \mathbb{Z}^n$ by definition and multiplying by A will result with a vector with integer entries. □

For the proof of the following lemma, we make use of the Smith normal form. The Smith normal form takes an arbitrary matrix $A \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$ and factorises it as $A = NDM$ where $N \in \mathrm{GL}_n(\mathbb{Z})$, $M \in \mathrm{GL}_m(\mathbb{Z})$, and D is a diagonal rectangular matrix. We state without a proof that any square matrix with entries from a principal integral domain admits a Smith normal form. Lemma 4.1 says, in essence, that it is legal to apply Smith normal form.

Lemma 4.2 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement such that $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ where each hyperplane is defined by its normal vector. Then we have*

$$f_k(\mathcal{A}) = |\det(\Theta)|\binom{n}{k},$$

where Θ is the system of hyperplanes of $\mathcal{A}$.

Proof. First, we check the lemma for the standard arrangement $(\mathcal{I}, \mathbb{T}^n)$, so we define each hyperplane H_i by the kernel of the map

$$\theta_i : \mathbb{T}^n \to S^1;\ \mathbf{x} \mapsto \mathbf{e}_i \cdot \mathbf{x}.$$

Every k-face in $\mathcal{I}$ is uniquely determined by choosing $n-k$ hyperplanes because any subset of hyperplanes in $\mathcal{A}$ intersect uniquely. Hence,

$$f_k(\mathcal{A}) = \binom{n}{n-k} = \binom{n}{k}.$$

In case the hyperplanes are not given by coordinate vectors, fix a basis and consider the system of hyperplanes Θ of $\mathcal{A}$. Using Smith normal form, we can write $\Theta = N\Theta' M$. Then the rows of Θ' are the normal vectors of multiples of coordinate hyperplanes. This also describes a covering map $q : \mathbb{T}^n \to \mathbb{T}^n$ given by $(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}) \mapsto (m_1\mathbf{e_1}, m_2\mathbf{e_2}, \ldots, m_n\mathbf{e_n})$, where m_i is the i-th diagonal entry of Θ'. The fibre $q^{-1}(0)$ has cardinality $|\det(\Theta')|$, therefore q is a covering of degree $|\det(\Theta')|$, so

$$f_k(\mathcal{A}) = |\det(\Theta')|\binom{n}{k} = |\det(\Theta)|\binom{n}{k}.$$

□

Corollary 4.3 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement of n hyperplanes $H_1, H_2, \ldots, H_n$ defined by their normal vector. Then the number of regions is given by the determinant of the system of hyperplanes* $\det(\Theta)$.

4.1 Generalised Deletion-Restriction and the Symmetry of f-Vectors

Recall the following relation in case of real hyperplane arrangements between the regions of $\mathcal{A}, \mathcal{A}'$, and $\mathcal{A}''$.

Theorem 4.4 ([8]) *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane H_0. Then*

$$f_n(\mathcal{A}) = f_n(\mathcal{A}') + f_{n-1}(\mathcal{A}'').$$

The proof of Theorem 4.4 provided in [8] is adaptable to arrangements in general position on $\mathbb{T}^n$ as well. However, instead of only studying the deletion-restriction relation for regions of $\mathcal{A}$, we would like to generalise the theorem for other dimensional faces as well.

Example 4.5 We continue with the arrangement from Example 3.1 to which we have added a hyperplane $H_0 = \langle 1, 0\rangle$. We let H_0 be a distinguished hyperplane and consider $\mathcal{A}', \mathcal{A}''$, see Fig. 6. The objective is to count the number of 1-dimensional faces recursively using $\mathcal{A}'$ and $\mathcal{A}''$. We can rewrite the set of 1-faces $\Sigma_1(\mathcal{A})$ of $\mathcal{A}$ as

$$\{\sigma \in \Sigma_1(\mathcal{A})\colon \sigma \cap H_0 = \emptyset\} \sqcup \{\sigma \in \Sigma_1(\mathcal{A})\colon \sigma \cap H_0 \neq \emptyset, \sigma \nsubseteq H_0\} \sqcup \{\sigma \in \Sigma_1(\mathcal{A})\colon \sigma \subseteq H_0\},$$

that is, we distinguish the 1-faces of $\mathcal{A}$ which do not intersect H_0, the 1-faces which intersects H_0 in codimension 1 and the 1-faces which are properly contained in H_0.

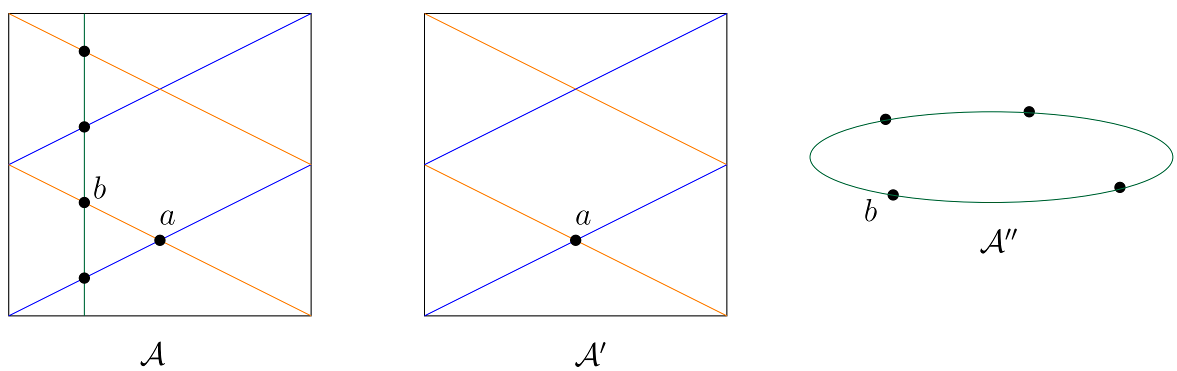

Figure 6: The triple of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ with $H_0 = \langle 1, 0\rangle$ (green), $H_1 = \langle -1, 2\rangle$ (blue), $H_2 = \langle 1, 2\rangle$ (orange).

Clearly, the 1-faces of $\mathcal{A}''$ form a subset of 1-faces of $\mathcal{A}$ as these faces are not contained in $\mathcal{A}'$. We can see that the other 1-faces of $\mathcal{A}$ arise as 1-faces of $\mathcal{A}'$ which are either preserved or cut up by H_0. We can thus write the set of 1-faces of $\mathcal{A}'$ as

$$\Sigma_1(\mathcal{A}') = \{\sigma \in \Sigma_1(\mathcal{A}')\colon \sigma \cap H_0 = \emptyset\} \sqcup \{\sigma \in \Sigma_1(\mathcal{A}')\colon \sigma \cap H_0 \neq 0\}.$$

The set of 1-faces of $\mathcal{A}'$ which have an empty intersection with H_0 are also 1-faces of $\mathcal{A}$. On the other hand, the 1-faces of $\mathcal{A}'$ with a non-empty intersection with H_0 are the faces of $\mathcal{A}''$ which get "cut up" by H_0. Since a 1-face is homeomorphic to an (open) interval I in $\mathbb{R}$, we know that m points, corresponding to m intersections with H_0, divide I into $m+1$ intervals. The intersection information is encoded as 0-faces of $\mathcal{A}''$ by definition. Also, because we now have $m+1$ intervals, we can identify m intervals with a m subset of 0-faces of $\mathcal{A}''$ and the remaining interval is identified with the original 1-face of $\mathcal{A}'$. This holds for every face of $\mathcal{A}'$ which has a non-empty intersection, so finally we identify

$$\Sigma_1(\mathcal{A}) \cong \Sigma_1(\mathcal{A}') \sqcup \Sigma_1(\mathcal{A}'') \sqcup \Sigma_0(\mathcal{A}''),$$

which implies

$$f_1(\mathcal{A}) = f_1(\mathcal{A}') + f_1(\mathcal{A}'') + f_0(\mathcal{A}'').$$

Lemma 4.6 (Generalised deletion-restriction) *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *a triple of spanning arrangements in general position, such that every face is simply connected, with a distinguished hyperplane* H_0 *and assume that* $|\mathcal{A}| \geq n+1$*. Then*

$$f_k(\mathcal{A}) = f_k(\mathcal{A}') + f_k(\mathcal{A}'') + f_{k-1}(\mathcal{A}'').$$

Proof. Let the distinguished hyperplane H_0 be defined by the group homomorphism $\theta : \mathbb{T}^n \to S^1$. Then we can decompose the k-faces of $\mathcal{A}$, $\Sigma_k(\mathcal{A})$, as a disjoint union,

$$\begin{aligned}\Sigma_k(\mathcal{A}) = &\overbrace{\{\sigma \in \Sigma_k(\mathcal{A})\colon \sigma \cap H_0 = \emptyset\}}^{(1)} \sqcup \\ &\sqcup \overbrace{\{\sigma \in \Sigma_k(\mathcal{A})\colon \sigma \cap H_0 \neq \emptyset, \sigma \nsubseteq H_0\}}^{(2)} \sqcup \overbrace{\{\sigma \in \Sigma_k(\mathcal{A})\colon \sigma \subseteq H_0\}}^{(3)},\end{aligned} \tag{$*$}$$

where we keep track of the sets by numbering throughout the proof.

Since $\mathcal{A}$ is in general position, it follows that any k-face $\sigma \in \Sigma_k(\mathcal{A})$ belongs to exactly one of the sets listed in $(*)$.

We immediately have that a k-face properly contained in H_0 must be a k-face of $\mathcal{A}''$ by definition, so

$$\overbrace{\{\sigma \in \Sigma_k(\mathcal{A})\colon \sigma \subseteq H_0\}}^{(3)} = \Sigma_k(\mathcal{A}'').$$

The k-faces of $\mathcal{A}$ which are not contained in $\mathcal{A}''$ must be contained in $\mathcal{A}'$. So we examine $\Sigma_k(\mathcal{A}')$. By adding H_0 to $\mathcal{A}'$, a k-face of $\mathcal{A}'$ is either preserved or gets cut up by H_0 into more k-faces. Thus, the set $\Sigma_k(\mathcal{A}')$ can be written as a disjoint union

$$\Sigma_k(\mathcal{A}') = \overbrace{\{\sigma \in \Sigma_k(\mathcal{A}')\colon \sigma \cap H_0 = \emptyset\}}^{(4)} \sqcup \overbrace{\{\sigma \in \Sigma_k(\mathcal{A}')\colon \sigma \cap H_0 \neq \emptyset\}}^{(5)}.$$

The set of k-faces of $\mathcal{A}'$ with an empty intersection with H_0 is in bijection with the set of k-faces of $\mathcal{A}$ with an empty intersection with H_0, therefore

$$\overbrace{\{\sigma \in \Sigma_k(\mathcal{A}')\colon \sigma \cap H_0 = \emptyset\}}^{(4)} = \overbrace{\{\sigma \in \Sigma_k(\mathcal{A})\colon \sigma \cap H_0 = \emptyset\}}^{(1)}.$$

Now, it is left to investigate the set (2), the k-faces of $\mathcal{A}$ which have a non-empty intersection with H_0 but are not properly contained in H_0. We claim that such faces can be labelled by either a k-face of $\mathcal{A}'$ or a $(k-1)$-face of $\mathcal{A}''$. To see this, let $\pi : \mathbb{T}^n \to \mathbb{R}^n$ be a covering space and $\tilde{\theta} : \mathbb{R}^n \to \mathbb{R}$ be the lift of θ. For a k-face σ, we will let τ be its boundary if $\sigma \lneq \tau$ in $\mathcal{L}(\mathcal{A})$ and $\dim(\tau) = \dim(\sigma) - 1$. Define a boundary τ of a lift of a face $\pi^{-1}(\sigma)$ in H_0 to be positive if for all paths $\gamma : [0,1] \to \mathbb{R}^n$ contained in $\pi^{-1}(\sigma)$ ending at τ with $\gamma' = 0$ we have $(\tilde{\theta}(\gamma))' > 0$.

Fix a face $\sigma \in \Sigma_k(\mathcal{A})$ and suppose, for sake of contradiction, that its lift $\pi^{-1}(\sigma)$ has two distinct positive boundaries, say τ_1 and τ_2. Since σ is convex by Lemma 2.15, $\pi^{-1}(\sigma)$ is convex and there exists a positive linear path γ such that $\gamma(0) \in \tau_1$ and $\gamma(1) \in \tau_2$ which is contained in $\pi^{-1}(\sigma)$. But then the opposite path γ' such that $\gamma'(0) \in \tau_2$ and $\gamma'(1) \in \tau_1$ cannot be positive, so τ_1 is not positive which is a contradiction. Thus, there is at most one positive boundary of σ in H_0.

Each k-face σ in $\{\sigma \in \Sigma_k(\mathcal{A}') \colon \sigma \cap H_0 \neq \emptyset\}$ can be written as a disjoint union of a finite chain of k-faces $\sigma^1, \sigma^2, \ldots, \sigma^m \in \Sigma_k(\mathcal{A})$ so that $\sigma^i \subseteq \sigma$ for every $i \in \{1, 2 \ldots, m\}$. There exists a σ^0 such that

$$\tilde{\theta}(\pi^{-1}(\sigma^i)) \leq \tilde{\theta}(\pi^{-1}(\sigma^0)),$$

for all $i \in I$. We claim that σ^0 is the unique face contained in σ which is not positive. As done previously, suppose there are two distinct k-faces σ^0 and σ^1 with non-positive boundaries in H_0 for contradiction. Then σ must have two distinct positive boundaries in H_0 which is a contradiction by the above.

Thus, for any k-face σ in the set $\{\sigma \in \Sigma_k(\mathcal{A}') \colon \sigma \cap H_0 \neq \emptyset\}$ can be written as

$$\sigma = \bigsqcup_{i \in I} \sigma^i,$$

where $\sigma^i \in \Sigma_k(\mathcal{A})$ and σ^i is not positive for exactly one index $i \in I$. The faces σ^i with a positive boundary are in correspondence with the $(k-1)$-faces of $\mathcal{A}''$ which are contained in σ, while the σ^i without a positive boundary are in correspondence with σ. We conclude that

$$(2) = (4) \sqcup \Sigma_{k-1}(\mathcal{A}'').$$

Therefore the k-faces of $\mathcal{A}$ can be written as

$$\Sigma_k(\mathcal{A}) \cong \Sigma_k(\mathcal{A}') \sqcup \Sigma_k(\mathcal{A}'') \sqcup \Sigma_{k-1}(\mathcal{A}''),$$

and enumerating the sets yields

$$f_k(\mathcal{A}) = f_k(\mathcal{A}') + f_k(\mathcal{A}'') + f_{k-1}(\mathcal{A}'').$$

□

We conjecture the generalised deletion-restriction applies to arrangements containing non-simply connected regions as well, which we demonstrate in the following simple example.

Example 4.7 Suppose we have the standard arrangement $\mathcal{I}$ on $\mathbb{T}^2$ and we want to count the number of edges using the generalised deletion-restriction. Then $f_1(\mathcal{I}') = 0$ because there are no simply-connected faces in $\mathcal{I}'$. However, $\mathcal{I}''$ is essentially an arrangement on $\mathbb{T}^1$ of one hyperplane hence $f_1(\mathcal{I}'') = f_0(\mathcal{I}'') = 1$, so the relation rightly yields $f_1(\mathcal{I}) = 2$.

We make use of Lemma 4.6 to prove the main result of the paper.

Theorem 4.8 (Symmetry of f-vectors) *Suppose we have a hyperplane arrangement $(\mathcal{A}, \mathbb{T}^n)$ such that $|\mathcal{A}| \geq n$. Then the* $\mathbf{f}$*-vector associated to $\mathcal{A}$ is symmetric, that is, $f_k(\mathcal{A}) = f_{n-k}(\mathcal{A})$.*

Proof. We proceed by induction on the dimension of the ambient space first. For the base case, consider any arrangement on $\mathbb{T}^0$ for which the statement holds trivially.

We do a second induction on the number of hyperplanes in $\mathcal{A}$. Using the same notation as the proof of Lemma 4.2, we have the case of n hyperplanes on $\mathbb{T}^n$,

$$f_{n-k} = |\det(\Theta')| \binom{n}{k} = |\det(\Theta)| \binom{n}{k} = f_k(\mathcal{A}),$$

because $\Theta' : \mathbb{T}^n \to \mathbb{T}^n$ is a $|\det(\Theta')|$-fold cover. We can thus assume that the statement holds for arrangements $(\mathcal{A}, \mathbb{T}^n)$ with $|\mathcal{A}| = n$.

Now suppose the induction hypothesis holds for all arrangements $|\mathcal{A}| < m$ on $\mathbb{T}^l$ with $l \leq n$. Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement with $|\mathcal{A}| = m + 1$. We investigate what happens when we remove a hyperplane H_{m+1} from $\mathcal{A}$, so that $|\mathcal{A}'| = m$. By Lemma 4.6, we write

$$f_k(\mathcal{A}) = f_k(\mathcal{A}') + f_k(\mathcal{A}'') + f_{k-1}(\mathcal{A}''),$$
$$f_{n-k}(\mathcal{A}) = f_{n-k}(\mathcal{A}') + f_{n-k}(\mathcal{A}'') + f_{n-k-1}(\mathcal{A}'').$$

Clearly, $f_k(\mathcal{A}') = f_{n-k}(\mathcal{A}')$ because $|\mathcal{A}'| = m$. Moreover, $\mathcal{A}''$ is an arrangement of not necessarily simple hyperplanes on $\mathbb{T}^{m-1}$ with $|\mathcal{A}''| \leq m - 1$. Suppose that the induction hypothesis also holds for all arrangements for which $|\mathcal{A}| < m$ on $\mathbb{T}^n$ and all arrangements $\mathcal{A}$ on $\mathbb{T}^l$ with $l < n$. If we make use of Remark 2.6, by induction, we also have

$$f_k(\mathcal{A}'') = f_{n-k-1}(\mathcal{A}'') \text{ and } f_{k-1}(\mathcal{A}'') = f_{n-k}(\mathcal{A}''),$$

and the relation holds. □

As a quick sanity check, we make sure that we have not broken any topological invariants of $\mathbb{T}^n$, namely the Euler characteristic. Recall the Euler characteristic of an arrangement $(\mathcal{A}, \mathbb{T}^n)$ is defined as

$$\sum_{i=0}^{n} (-1)^i f_i(\mathcal{A}).$$

In particular, we must have $\sum_{i=0}^{n} (-1)^i f_i(\mathcal{A}) = 0$. For odd n, it is easy to see that having $f_k(\mathcal{A}) = f_{n-k}(\mathcal{A})$ respects the alternating sum equalling 0.

5 Regions of Toric Arrangements

5.1 Counting the Regions

Previously, we have defined regions to be the top dimensional objects of an arrangement $\mathcal{A}$. We have also seen that regions can only either be simply connected or have an infinite cyclic fundamental group. In the case regions of an arrangement $(\mathcal{A}, \mathbb{T}^n)$ of n hyperplanes are simply connected, we have a definite answer in Lemma 4.2. We would like to describe similar cases. From this point, we will always assume arrangements $(\mathcal{A}, \mathbb{T}^n)$ are spanning and in general position (unless otherwise stated).

We begin by providing geometric intuition of the idea in proof of Lemma 4.2. We will use the symmetry of f-vectors, namely $f_0(\mathcal{A}) = f_n(\mathcal{A})$. In the toy example of $\mathbb{T}^2$, we used the fact that a normal vector $\langle a_1, a_2\rangle$ of a hyperplane on $\mathbb{T}^2$ can be written as $a_1\langle 1, 0\rangle + a_2\langle 0, 1\rangle$, counted the number of line intercepts and then applied the symmetry. Continuing the example on $\mathbb{T}^2$, let $H_1 = \langle a_{11}, a_{12}\rangle$ and $H_2 = \langle a_{21}, a_{22}\rangle$ be hyperplanes given by their normal vectors, as usual. By the definition of a hyperplane, we have

$$H_1 = \ker\left(\theta_1 : \mathbb{T}^2 \to S^1;\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto a_{11}x_1 + a_{12}x_2\right),$$
$$H_2 = \ker\left(\theta_2 : \mathbb{T}^2 \to S^1;\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto a_{21}x_1 + a_{22}x_2\right),$$

Clearly the number of intersections, or 0-faces, is given by $|H_1 \cap H_2|$ which is the same as the cardinality of the kernel of the direct product of group homomorphisms $\theta_1 \times \theta_2$, that is $\ker(\mathbb{T}^2 \xrightarrow{\theta_1,\theta_2} S^1 \times S^1)$. Furthermore, $|\ker(\theta_1 \times \theta_2)|$ is given by the number of lattice point in $\mathbb{Z}^2$ inside[2] the parallelogram determined by the normal vectors of the hyperplanes. Pick's theorem then says that the number of lattice points is equal to the area, which is precisely the determinant. Therefore, by applying Theorem 4.8, we have that the number of regions of $\mathcal{A}$ is given by

$$\left|\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right|.$$

We want to employ a similar idea for a general case on $\mathbb{T}^n$, in particular, we will use Smith normal form to change bases of hyperplanes. For every $i \leq n$, consider the set of hyperplanes that are defined as kernels of group homomorphisms

$$H_i = \ker\left(\theta_i : \mathbb{T}^n \to S^1;\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto a_{i1}x_1 + \ldots + a_{in}x_n\right).$$

The direct product $\theta_N = \theta_1 \times \ldots \times \theta_n$ is a group homomorphism with $\ker(\theta_N) = \bigcap_{i=1}^{n} \ker(\theta_i)$. In other words, we are looking for all vectors $\tilde{\mathbf{x}} \in \mathbb{R}^n$ such that

[2]Since we are working on $\mathbb{T}^2$, we count lattice points on the normal vectors and inside the parallelogram.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x_1} \\ \vdots \\ \tilde{x_n} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where each $b_i \in \mathbb{Z}$ as such vectors $\mathbf{x}$ will be elements of the kernel. Hence, finding the number of elements in $\ker(\theta_N)$ is equivalent to searching for the number of lattice points inside the parallelepiped given rows of $\Theta = (H_1|\dots|H_n)^T$. Since any matrix over $\mathbb{Z}$ has a Smith normal form, we consider the Smith normal form of Θ which we will call Θ'. By Lemma 4.1, the Smith normal form of Θ now represents a rectangular parallelepiped. The problem is invariant under translation and rotation, so we may identify one vertex of the rectangular parallelepiped to the origin and the row vectors of Θ' with the axes. Note that we are only interested in counting the lattice points inside the half-open parallelepiped. We can thus associate to each unit of volume a point, which gives a bijection between the number of lattice points and the volume of the parallelepiped, i.e. $|\det(\Theta)|$. The result then follows as an application of Theorem 4.8.

Corollary 5.1 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement of m hyperplanes such that $m > n$. To each hyperplane H_i associate its normal vector $\mathbf{a_i}$. Then the number of n-dimensional faces is given by the sum over all n-tuples of $\mathcal{A}$,*

$$f_n(\mathcal{A}) = \sum_{\substack{S \subseteq \{1,2,\dots,m\} \\ |S|=n}} |\det(\mathbf{a_{i_1}} \mid \mathbf{a_{i_2}} \mid \dots \mid \mathbf{a_{i_n}})|.$$

Proof. The arrangement $\mathcal{A}$ being in general position implies that the 0-faces of $\mathcal{A}$ do not appear with multiplicities. If we take two subsets of n hyperplanes such that they differ by at least one, their intersection points are disjoint. Therefore, we can enumerate the number of 0-face of all hyperplane subsets of cardinality n which yields the number of n-faces of $\mathcal{A}$. □

Remark 5.2 The definition of regions $R(\mathcal{A})$ does not exclude arrangements $\mathcal{A}$ which are not spanning. Let $\mathcal{A}$ be an arrangement on $\mathbb{T}^n$ of m hyperplanes such that the hyperplanes are pairwise parallel. Clearly, m points divide $\mathbb{T}^1 = S^1$ into m segments. If we interpret $\mathbb{T}^n = S^1 \times S^1 \times \dots \times S^1$, then the hyperplanes are parallel, and we may regard them as a subset of

$$\underbrace{S^1 \times S^1 \times \dots \times S^1}_{n-1},$$

and they only intersect one of the S^1 in the product. The single n-intersection of each hyperplane then defines one region, therefore $r(\mathcal{A}) = m$. However, since all hyperplanes are parallel, all the regions have a non-trivial fundamental group, so by our definition of faces, $f_i = 0$ for $i \in \{0, 1, \dots, n\}$.

5.2 Analogue of Zaslavky's Theorem

Another possible way to count the regions of an arrangement $(\mathcal{A}, \mathbb{T}^n)$ is to use the characteristic polynomial of $\mathcal{A}$. In the context of $\mathbb{R}^n$, this was proven due to T. Zaslavsky in 1975.

Theorem 5.3 (Zaslavsky) *Let $(\mathcal{A}, \mathbb{R}^n)$ be a real hyperplane arrangement and let $\chi_{\mathcal{A}}$ be the characteristic polynomial (as in Definition 5.7). Then*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1).$$

There exists a toric analogue of Theorem 5.3 that has been proven; see Theorem 3.6 in [2]. However, we provide another type of proof using the fact that on general arrangements $(\mathcal{A}, \mathbb{T}^n)$ we apply $f_0(\mathcal{A}) = f_n(\mathcal{A})$ given by Theorem 4.8. Of course, in our case, we still assume that $\mathcal{A}$ is spanning and in general position.

Definition 5.4 *Let P be a locally finite poset. Define a function $\mu = \mu_P : \mathrm{Int}(P) \to \mathbb{Z}$, where $\mathrm{Int}(P)$ is the set of all closed intervals of P, called the Möbius function of P by the conditions:*

$$\mu(x, x) = 1, \quad \textit{for all } x \in P,$$
$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z), \quad \textit{for all } x < y \textit{ in } P.$$

If P has a minimum element $\hat{0}$, we write $\mu(x) = \mu(\hat{0}, x)$. We will also denote by $\mathrm{Int}(x)$ the set of closed intervals $[\hat{0}, x]$.

Lemma 5.5 *For $x \in L(\mathcal{A})$, the restricted poset $\mathrm{Int}(x) \hookrightarrow L(\mathcal{A})$ is a Boolean lattice.*

Proof. Let $L(\mathcal{A})$ be an intersection poset of an arrangement $(\mathcal{A}, \mathbb{T}^n)$ where $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$. Let $S \subseteq \{1, 2, \ldots, m\} = \underline{m}$. By Proposition 2.3. in [8] the poset $L(\mathcal{A})$ is a meet-semilattice, i.e. for every pair of elements $x, y \in L(\mathcal{A})$ there exists a meet $x \wedge y \in L(\mathcal{A})$. Since μ depends only on $\mathrm{Int}(x)$, we will only want to consider $\mathrm{Int}(x)$ for any $x \in L(\mathcal{A})$. But x is a maximal element for $\mathrm{Int}(x)$, so $\mathrm{Int}(x)$ must be a lattice.

Define

$$x_S = \bigcap_{i \in S} H_i.$$

Then $L(\mathcal{A})$ is a set of (non-empty) x_S for all S ordered by reverse set-inclusion. Let $2^{\underline{m}}$ denote the Boolean poset that arises from the subsets of the set $\underline{m}$ and consider an inclusion map $\mathrm{Int}(x) \hookrightarrow 2^{\underline{m}}$ given by $x_S \mapsto S$. Note that x corresponds to a subset $\{i_1, i_2, \ldots, i_v\} \subseteq \underline{m}$. Then $2^{\underline{m}} \big|_{\mathrm{Int}(x)}$ is also Boolean, since it respects the relations given in $2^{\underline{m}}$. Hence, $\mathrm{Int}(x)$ is a Boolean lattice. □

Proposition 5.6 *For an arrangement $(\mathcal{A}, \mathbb{T}^n)$, we have $\mu(x) = (-1)^{\mathrm{codim}(x)}$.*

Proof. We induct on the rank of an arbitrary element $x \in L(\mathcal{A})$. As a base case, we take the minimal element $\mathbb{T}^n$ for which trivially $\mu(\mathbb{T}^n) = (-1)^{\operatorname{codim}(\mathbb{T}^n)} = 1$. For our induction hypothesis, we will assume that $\mu(x) = (-1)^{\operatorname{codim} x}$ for any $x \in L(\mathcal{A})$ with rank less than k, for $k \leq n$. [3] Suppose now that $\operatorname{codim}(x) = k$. By definition,

$$\mu(x) = -\sum_{y \leq x} \mu(y) = \sum_{j<k} (-1)^{j+1} |\{y \in \operatorname{Int}(x) \colon \operatorname{codim}(y) = j\}|.$$

For each j, we have $|\{y \in \operatorname{Int}(x) \colon \operatorname{codim}(y) = j\}| = \binom{k}{j}$, as the set can be thought of as a set of subsets with cardinality j. Hence

$$\mu(x) = \sum_{j<k} (-1)^{j+1} \binom{k}{j} = -(-1)^{k+1} = (-1)^k.$$

□

Definition 5.7 *The characteristic polynomial $\chi_{\mathcal{A}}(t)$ of the arrangement $(\mathcal{A}, \mathbb{T}^n)$ is defined by*

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot t^{\dim(x)}.$$

It is clear from the definitions above that the coefficient of t^{n-1} in $\chi_{\mathcal{A}}(t)$ is the number of hyperplanes of $\mathcal{A}$. We now state the toric version of Zaslavky's theorem.

Proposition 5.8 *Let $(\mathcal{A}, \mathbb{T}^n)$ be an arrangement of hyperplanes. Then*

$$f_n(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(0).$$

Proof. By Proposition 5.6, for any element $x \in L(\mathcal{A})$ with $\dim(x) = 0$ we have $\mu(x) = (-1)^n$. Therefore, the constant term in the characteristic polynomial $\chi_{\mathcal{A}}(t)$ is $(-1)^n f_0$. The result then follows by Theorem 4.8. □

6 Classification of Line Arrangements

In the last section, we look into classifying hyperplane arrangements following from work in [7]. For the remainder of the paper, we focus on arrangements in $\mathbb{T}^2$. We consider two arrangements to be combinatorially equivalent if one of them can be obtained from the other by continuously translating lines without ever creating a triple intersection point. Note that being combinatorially equivalent is a stronger condition than having the same intersection poset.

Definition 6.1 *Let $(\mathcal{A}_1, \mathbb{T}^2)$ and $(\mathcal{A}_2, \mathbb{T}^2)$ be two arrangements. If it is possible to continuously translate hyperplanes of $\mathcal{A}_1$ without creating a triple intersection to $\mathcal{A}_2$, then $\mathcal{A}_1 \sim \mathcal{A}_2$.*

[3] Note that rank of an element $x \in L(\mathcal{A})$ is equal to $\operatorname{codim}(x)$.

Example 6.2 Let $\mathcal{A}$ be an arrangement on $\mathbb{T}^2$ consisting of lines given by the normal vectors $\langle 1,1\rangle, \langle -1,1\rangle$, and $\langle 1,0\rangle$, as shown in Fig. 7. We choose to fix $\langle 1,1\rangle, \langle -1,1\rangle$ and count by brute force the number of arrangements that arise by adding $\langle 1,0\rangle$. However, there is a restriction: we do not allow $\langle 1,0\rangle$ passing through points A or B. This means that there are two regions where the line can fall into, hence there are at most two distinct arrangements in this case. However, we need to take the symmetry of $\mathbb{T}^2$ into consideration. Once we allow a global translation of the torus, the two arrangements are equivalent.

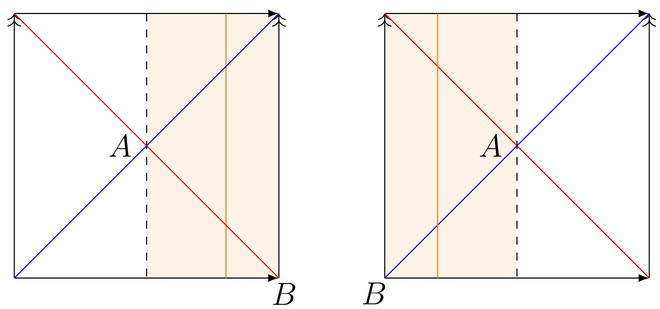

Figure 7: An example of two arrangements consisting of the same hyperplanes which are equivalent under $\sim$.

Lemma 6.3 *Let $(\mathcal{A}, \mathbb{T}^2)$ be an arrangement such that $\mathcal{A} = \{H_1, H_2\}$. Then there is a unique arrangement of hyperplanes H_1 and H_2.*

Proof. It is not possible to have a triple intersection with only two lines. Therefore, we can translate the two lines so that their intersection is at the origin using a global translation of the torus. □

Lemma 6.4 *Let $(\mathcal{A}, \mathbb{T}^2)$ be an arrangement of three hyperplanes H_1, H_2, and H_3. Then there is a unique arrangement of the hyperplanes up to equivalence.*

Proof. Suppose that H_1 and H_2 intersect in j points. Then there is at most j ways to create triple intersections only by translating H_3. As depicted in Example 6.2, H_3 must fall in an interval defined by the intersections of H_1 and H_2, so that $\mathcal{A}$ is in general position. However, accounting for the global symmetry of $\mathbb{T}^2$, all of the considered arrangements are equivalent because we can translate any intersection of H_1 and H_2 to the origin. As in Lemma 6.3, translating any vertex to the origin corresponds to a symmetry which fixes the hyperplanes. □

6.1 An Upper Bound for the Number of Distinct Line Arrangements

In order to establish two arrangements are not equivalent, we make use of the intercept information that appears when defining a hyperplane. Translating a hyperplane is the same as changing the intercept. In consequence, we want to parametrise the intercept and describe a new arrangement, as we demonstrate in the next example.

Example 6.5 We continue Example 6.2. We write the hyperplanes in $\mathcal{A}$ as

$$\begin{aligned} \theta_1 &: \mathbb{T}^2 \to S^1; \quad \theta(\mathbf{x}) = x_1 + x_2 \\ \theta_2 &: \mathbb{T}^2 \to S^1; \quad \theta(\mathbf{x}) = -x_1 + x_2 \\ \theta_3 &: \mathbb{T}^2 \to S^1; \quad \theta(\mathbf{x}) = x_1 \end{aligned}$$

Using the definition of hyperplanes, we describe the arrangement by

$$\Theta : \mathbb{T}^2 \xrightarrow{(\theta_1,\, \theta_2,\, \theta_3)} (S^1)^3$$
$$\left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right),$$

for which we want to parameterise the intercepts b_1, b_2, and b_3 corresponding to triple intersections of the hyperplanes. Therefore, we identify each arrangement with a point in $(S^1)^3$. Since the space of possible arrangements is going to be identified with $\mathbb{T}^3$, we look at each column of the original system of hyperplanes as a hyperplane in $\mathbb{T}^3$. We will call the first column $\mathbf{a_1}$ and the second column $\mathbf{a_2}$. We describe three arrangements in $(S^1)^3$ by taking the cross product of $\mathbf{a_1}$ and $\mathbf{a_2}$,

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

We claim that the hyperplane given by $\mathbf{n}$ corresponds to the intercepts b_1, b_2, and, b_2 giving a triple intersection. Observe that the i-th entry in $\mathbf{n}$ gives the number of equivalence classes of arrangements we get by translating the i-th hyperplane, ignoring the symmetry of $\mathbb{T}^2$ for now. That is to say, taking the cross product gives us a vector whose entries are the solution to the restricted systems of linear equations. The i-th hyperplane is parameterized while the other two are fixed, so solving for the other two gives us the number of possible triple intersections when translating H_i. Since the i-th entry are hyperplanes on S^1 which correspond to line arrangements with a triple intersection, the 1-faces now represent an equivalence class of line arrangements under $\sim$, disregarding the symmetry of $\mathbb{T}^2$. For instance, the last entry suggests there are two distinct hyperplane arrangements under $\sim$, not taking the symmetry of $\mathbb{T}^2$ into consideration, which we have already intuitively verified.

After the example, we go on to set the general construction. Considering an arrangement $(\mathcal{A}, \mathbb{T}^2)$ of m lines, we put the normal vectors of hyperplanes into a system,

$$(\Theta, \mathbf{b}) = \left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ \vdots & \vdots \\ a_{2m-1} & a_{2m} \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \right).$$

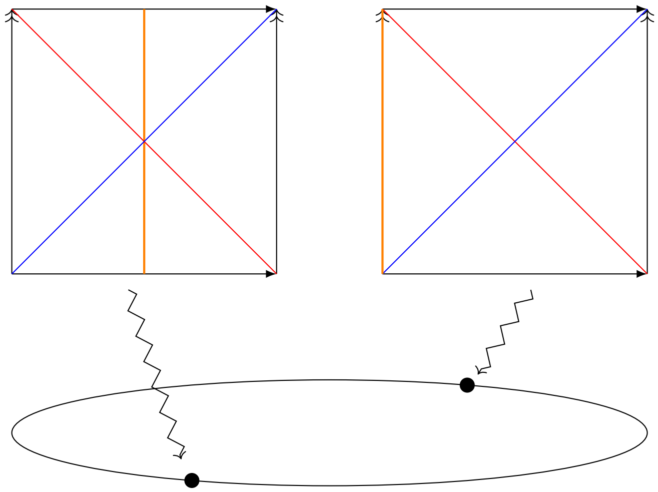

Figure 8: The intercept b_3 parametrised as an arrangement on S^1. The hyperplane $\langle 2 \rangle$ gives two arrangements with a triple intersection obtained by translating H_3.

Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$ consisting of three hyperplanes which are spanning (such a subarrangement always exists because $\mathcal{A}$ is spanning) for which we write

$$\begin{pmatrix} a_i & a_{i+1} \\ a_j & a_{j+1} \\ a_k & a_{k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_i \\ b_j \\ b_k \end{pmatrix}.$$

We want to parametrise the intercepts b_i, b_j, b_k of the hyperplanes and inspect which intercepts correspond to the subarrangement containing triple intersections. The arrangements with a triple intersection must be such that the system above has a solution.

Lemma 6.6 *Given a system of hyperplanes Θ and an intercept vector $\mathbf{b}$, there exists a vector $\mathbf{x}$ satisfying*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

if and only if

$$(\mathbf{a_1} \times \mathbf{a_2}) \cdot \mathbf{b} = 0,$$

where $\mathbf{a_1}$ and $\mathbf{a_2}$ stand for the first and second column of Θ respectively.

Proof. Suppose we have a vector $\mathbf{x}$ which satisfies $\Theta\mathbf{x} = \mathbf{b}$. We rewrite the relation,

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} = \mathbf{0}.$$

Then, since there exists a non-zero vector in the kernel of $(\mathbf{a_1} \mid \mathbf{a_2} \mid \mathbf{b})$, we have $\det(\mathbf{a_1} \mid \mathbf{a_2} \mid \mathbf{b}) = 0$ which is equivalent to $(\mathbf{a_1} \times \mathbf{a_2}) \cdot \mathbf{b} = 0$ by Laplace expansion along the last column.

□

Hence, $\mathbf{n} \cdot \mathbf{b} = (\mathbf{a_1} \times \mathbf{a_2}) \cdot \mathbf{b}$ is the triple intersection locus in $(S^1)^3$ and $\mathbf{n}$ defines a hyperplane parametrised by the intercepts.

Definition 6.7 *Let $(\mathcal{A}, \mathbb{T}^2)$ be an arrangement of m lines. Denote by $\mathbf{n_{ijk}}$ the primitive cross product of column vectors of the system of hyperplanes for a spanning triple of normal vectors $\mathbf{a_i}, \mathbf{a_j}, \mathbf{a_k}$. Let $\overline{\mathbf{n}}_{\mathbf{ijk}}$ be the extended vector of dimension m where the i-th, j-th, and k-th coordinate are taken to be coordinates of $\mathbf{n_{ijk}}$ respectively and the rest of the entries are zero.*

We define a new arrangement on $(S^1)^m$ of $\binom{m}{3}$ hyperplanes defined by normal vectors $\overline{\mathbf{n}}_{\mathbf{ijk}}$ which is the parameter space of arrangements and is given by the system

$$(\overline{\Theta}, \mathbf{0}) = \left(\begin{pmatrix} \leftarrow \overline{\mathbf{n}}_{\mathbf{I_1}} \rightarrow \\ \leftarrow \overline{\mathbf{n}}_{\mathbf{I_2}} \rightarrow \\ \vdots \\ \leftarrow \overline{\mathbf{n}}_{\mathbf{I}_{\binom{m}{3}}} \rightarrow \end{pmatrix}, \mathbf{0} \right),$$

where the normal vectors of hyperplanes are indexed by unordered triples $I_i \subseteq \mathcal{A}$.

We can further refine the method. It follows from Lemma 6.3 that we can always fix the intercept of two lines at the origin in an arrangement $\mathcal{A}$. Therefore, for an arrangement of two hyperplanes $H_1 = \langle a_1, a_2 \rangle$ and $H_2 = \langle a_3, a_4 \rangle$ on $\mathbb{T}^2$, we can fix the intercepts b_1 and b_2 of the two hyperplanes at the origin, that is, two entries of the intercept vector $\mathbf{b}$ can be replaced by zero.

Example 6.8 Suppose we have hyperplanes H_1, H_2, H_3, H_4 on $\mathbb{T}^2$ given by normal vectors

$$\mathbf{a_1} = \langle 1, 0 \rangle, \mathbf{a_2} = \langle 0, 1 \rangle, \mathbf{a_3} = \langle 1, -2 \rangle, \mathbf{a_4} = \langle 1, 2 \rangle$$

respectively. We fix the intercepts of H_3 and H_4 at the origin. Therefore,

$$(\Theta, \overline{\mathbf{b}}) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix} \right)$$

is the system associated to the arrangement. We would like to construct a geometric space, say $\mathcal{M}$, where points are in correspondence with line arrangements and lines are in correspondence with arrangements containing triple intersections. Using the method above, we look at triples of spanning subarrangements of three hyperplanes. We start with the subarrangement with system of hyperplanes given by

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \right),$$

for which we compute the cross product of columns and then extend the normal vector,

$$\mathbf{n}_{123} = \begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \qquad \overline{\mathbf{n}}_{123} = \begin{pmatrix}-1\\2\\1\\0\end{pmatrix}.$$

We repeat the process to obtain the extended system of hyperplanes in terms of $\mathbf{b}$,

$$(\overline{\Theta}, \mathbf{0}) = \left(\begin{pmatrix} -1 & 2 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ 2 & 0 & -1 & -1 \\ 0 & 4 & 1 & -1 \end{pmatrix}, \mathbf{0} \right).$$

Fixing the intercepts $b_3 = b_4 = 0$, we can reduce the arrangement to get

$$(\overline{\Theta}_{\text{red}}, \overline{\mathbf{b}}_{\text{red}}) = \left(\begin{pmatrix} 1 & -2 \\ 1 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{0} \right).$$

In addition, the intercepts of the hyperplanes are given by the fact that there is a unique arrangement where all four hyperplanes intersect, so each hyperplane has to pass through the origin. Since each hyperplane now corresponds to an arrangement containing a triple intersection, the n-faces represent distinct arrangements in general position.

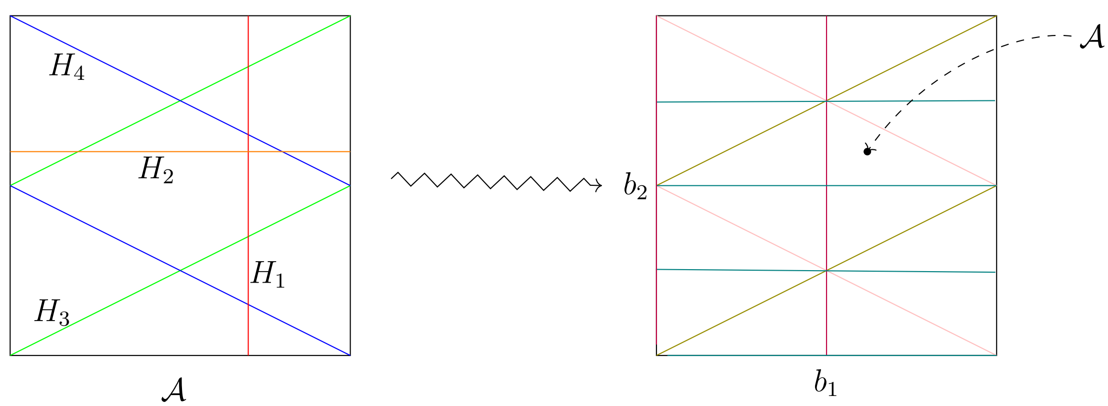

Figure 9: An example of constructing the space of arrangements consisting of hyperplanes $\mathcal{M}$ coming from vectors $\mathbf{n_{123}}$ (olive), $\mathbf{n_{124}}$ (pink), $\mathbf{n_{134}}$ (purple), and $\mathbf{n_{234}}$ (teal).

In fact, we can say more from the space of hyperplane arrangements. Since there are four arrangements with a quadruple intersection, the four points must be the same arrangement. Hence, the Fig. 9 suggests that the space of arrangements has a four-fold symmetry as a result of the additional symmetries of the torus. It suffices to check the fundamental domain of $\mathcal{M}$ to answer how many distinct hyperplane arrangements of H_1, H_2, H_3, and H_4 we have, in this case there are 4 distinct arrangements. Note that if we fixed $b_1 = b_2 = 0$ instead, we would get an arrangement corresponding to the fundamental domain of the parametrized arrangement. This follows from the fact that fixing the hyperplanes with the least intersections gives us the least amount of additional symmetries.

The parameter space of arrangement is often times not in general position, therefore most of the results do not apply nicely to this arrangement. However, we can indicate an upper bound in the next theorem.

Theorem 6.9 *Let $(\mathcal{A}, \mathbb{T}^2)$ be a line arrangement of m lines with $m > 3$ given by normal vectors $\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_m}$. Fix the intercept of two hyperplanes, given by $\mathbf{a_g}$ and $\mathbf{a_h}$ for which the absolute value of the determinant $\det(\mathbf{a_g} \mid \mathbf{a_h})$ is minimised, at the origin. We define an arrangement on $\mathbb{T}^{m-2}$ using the system*

$$\rho_{gh}(\overline{\Theta})\mathbf{b} = \mathbf{0},$$

where we get $\overline{\Theta}_{gh}$ by omitting the g-th and h-th column of $\tilde{\Theta}$. Then the upper bound for the number of distinct arrangements of the m hyperplanes is the number of $(m-2)$-faces of the arrangement described by $\rho_{gh}(\overline{\Theta})$,

$$\frac{1}{|\det(\mathbf{a_g} \mid \mathbf{a_h})|} \sum_{\substack{I_i \subset \{1,2,\dots,m\} \\ |I|=3}} |\det\left(\rho_{gh}(\overline{\mathbf{n}}_{\mathbf{I_1}} \mid \overline{\mathbf{n}}_{\mathbf{I_2}} \mid \dots \mid \overline{\mathbf{n}}_{\mathbf{I_m}})\right)|.$$

Proof. Extending the normal vectors $\mathbf{n_{ijk}}$ to $\overline{\mathbf{n}}_{\mathbf{ijk}}$ gives us a new arrangement on $\mathbb{T}^m$ where each hyperplane is the locus of ijk-intersection locus. By omitting two columns of the system of hyperplanes $\overline{\Theta}$, we remove some additional symmetry of the arrangement, and $\rho_{gh}(\overline{\Theta})$ now represents a spanning arrangement on $\mathbb{T}^{m-2}$. We can also definitely remove the symmetry of the fixed hyperplanes, which is the $|\det(\mathbf{a_g} \mid \mathbf{a_h})|$-fold symmetry. We count the $(m-2)$-faces using Corollary 5.1, but note that the arrangement need not be in general position, hence we only get an upper bound. $\square$

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