Straightening Identities in the Universal Enveloping Algebra of some Twisted Multiloop Algebras of $\mathfrak{sl}_4$

S. Chamberlin and J. Niraula

Abstract - We have formulated and proven some straightening identities in the universal enveloping algebras of the twisted multiloop algebras of $\mathfrak{sl}_4$ with Chevalley involution twist. These formulas could be useful in investigating the representation theory of this algebra.

Keywords : multiloop algebras; Lie algebras; straightening identities; universal enveloping algebras

Mathematics Subject Classification (2020) **:** 16S30; 17B05; 17B35; 17B65

1 Introduction

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{N}$ denote the positive integers, $\mathbb{Z}$ denote the integers, and $\mathbb{Z}_+$ denote the non-negative integers.

Twisted multiloop algebras associated to complex simple finite-dimensional Lie algebras and their representations have been extensively studied, [1, 7, 10]. Nonetheless, straightening identities in the universal enveloping algebras of these algebras are still unknown.

Straightening identities allow products in non-commutative algebras to be reordered so that the basis elements appear in a selected order. They can be useful in studying the representation theory of the algebra. The famous Poincaré-Birkhoff-Witt Theorem tells us that straightening is always possible, but because it is an existence theorem, it does not tell us how to straighten any particular product.

Straightening identities in the universal enveloping algebra of a simple complex finite-dimensional Lie algebra, $\mathfrak{g}$, were done by B. Kostant in [6]. Some straightening identities in the universal enveloping algebras of the untwisted loop algebras $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ were done by H. Garland in [5]. This was then extended to the untwisted map algebras $\mathfrak{g}\otimes A$, where A is any commutative associative algebra with unity by the first author in [4].

The first twisted straightening identities were done by D. Mitzman for the twisted Kac-Moody algebras in [8]. The next step for twisted straightening identities was done by A. Bianchi and the first author in [2] where they gave straightening identities in the universal enveloping algebra of the Onsager algebra. This work is the first step in the direction of straightening identities in the universal enveloping algebras of the twisted multiloop algebras. In this case, $\mathfrak{g}=\mathfrak{sl}_4$ and the twisting automorphism is the Chevalley involution.

2 Preliminaries

We recall the definition of a complex Lie algebra given in [3].

Definition 2.1 *A vector space, L, over $\mathbb{C}$, with an operation $[.,.]$ (called the Lie bracket), is a complex Lie Algebra if the following axioms are satisfied for all $x, y, z \in L$ and all $c \in \mathbb{C}$:*

$$\begin{aligned} [cx+y,z] &= c[x,z]+[y,z] \\ [x,cy+z] &= c[x,y]+[x,z] \\ [x,x] &= 0 \\ [x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= 0 \end{aligned}$$

The first two axioms imply that the bracket is bilinear and the last is called the Jacobi identity.

Let $\mathbb{C}\langle m\rangle := \mathbb{C}[t_1, t_1^{-1}, \ldots t_m, t_m^{-1}]$ be the set of complex Laurent polynomials in m variables.

Definition 2.2 *Given a complex Lie algebra L, the multiloop algebra of L is the Lie algebra $L\langle m\rangle := L \otimes \mathbb{C}\langle m\rangle$ with Lie bracket given by bilinearly extending the bracket*

$$[x \otimes f, y \otimes g] = [x,y] \otimes fg$$

where $x, y \in L$ and $f, g \in \mathbb{C}\langle m\rangle$.

We now define the Chevalley involution as in Example 3.9 in [9].

Definition 2.3 *Let $\mathfrak{g}$ be a simple complex finite dimensional Lie algebra and $\{e_j, f_j, h_j\}$ be a set of Chevalley generators for $\mathfrak{g}$. Define a Chevalley involution σ on $\mathfrak{g}$ by*

$$\sigma(e_j) = -f_j, \quad \sigma(f_j) = -e_j \quad \sigma(h_j) = -h_j.$$

σ also acts on $\mathbb{C}\langle m\rangle$ by $\sigma(f(t_1,\ldots,t_m)) = f(-t_1, t_2, \ldots, t_m)$. Then we can extend the action of σ to the multiloop algebra $\mathfrak{g}\otimes\mathbb{C}\langle m\rangle$ by defining $\sigma(x\otimes g) = \sigma(x)\otimes\sigma(g)$ for all $x \in \mathfrak{g}$ and $g \in \mathbb{C}\langle m\rangle$. We define the twisted multiloop algebra of $\mathfrak{g}$, $\mathcal{T}_m(\mathfrak{g})$, to be the subalgebra of the multiloop algebra fixed by this action of σ.

2.1 The Twisted Multiloop Algebra of $\mathfrak{sl}_4$

Let $\mathfrak{sl}_4$ be the Lie algebra of the 4×4 matrices with complex entries where the sum of the diagonal elements is 0 with the commutator Lie bracket ($[A,B] = AB - BA$). Let $e_{j,k}$ be the 4×4 matrix with 1 in the jth row and kth column and zeros elsewhere. The set $\{e_{j,k}, e_{l,l} - e_{l+1,l+1} \mid 1 \le j \ne k \le 4,\ l \in \{1,2,3\}\}$ is a basis for $\mathfrak{sl}_4$. The Chevalley involution acts on this basis as $\sigma(e_{j,k}) = -e_{k,j}$ and $\sigma(e_{l,l} - e_{l+1,l+1}) = -(e_{l,l} - e_{l+1,l+1})$ for $1 \le j \ne k \le 4$ and $l \in \{1,2,3\}$. The action on this basis is not sufficiently nice for our purposes. So we change the basis to one on which the Chevalley involution acts by the scalars 1 and -1.

For $j \in \{1,2\}$, define $x_j^{\sigma,\pm}, h_j^{\sigma}$ as follows, where $i = \sqrt{-1} \in \mathbb{C}$.

$$
\begin{aligned}
x_1^{\sigma,+} &:= \frac{1}{2}\left(\left(e_{1,2}-e_{2,1}\right)-\left(e_{3,4}-e_{4,3}\right)-i\left(e_{1,3}-e_{3,1}\right)-i\left(e_{2,4}-e_{4,2}\right)\right)\\
h_1^{\sigma} &:= i\left(\left(e_{2,3}-e_{3,2}\right)-\left(e_{1,4}-e_{4,1}\right)\right)\\
x_1^{\sigma,-} &:= \frac{1}{2}\left(-\left(e_{1,2}-e_{2,1}\right)+\left(e_{3,4}-e_{4,3}\right)-i\left(e_{1,3}-e_{3,1}\right)-i\left(e_{2,4}-e_{4,2}\right)\right)\\
x_2^{\sigma,+} &:= \frac{1}{2}\left(\left(e_{1,2}-e_{2,1}\right)+\left(e_{3,4}-e_{4,3}\right)+i\left(e_{1,3}-e_{3,1}\right)-i\left(e_{2,4}-e_{4,2}\right)\right)\\
h_2^{\sigma} &:= -i\left(\left(e_{2,3}-e_{3,2}\right)+\left(e_{1,4}-e_{4,1}\right)\right)\\
x_2^{\sigma,-} &:= \frac{1}{2}\left(-\left(e_{1,2}-e_{2,1}\right)-\left(e_{3,4}-e_{4,3}\right)+i\left(e_{1,3}-e_{3,1}\right)-i\left(e_{2,4}-e_{4,2}\right)\right)
\end{aligned}
$$

Notice that σ fixes all of the elements above, because $\sigma\left(e_{j,k}-e_{k,j}\right) = -e_{k,j}+e_{j,k}$. In fact, the elements above serve as a basis for the subalgebra of $\mathfrak{sl}_4$ which is fixed by σ (denoted $\mathfrak{sl}_4^{\sigma}$), because they are six linearly independent vectors in a space of dimension 6. So we can write $\mathfrak{sl}_4^{\sigma} \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

For the rest of the basis, define the following elements of $\mathfrak{sl}_4$:

$$
\begin{aligned}
y_1^{\sigma,+} &:= \frac{1}{2}\left(\left(e_{1,2}+e_{2,1}\right)+\left(e_{3,4}+e_{4,3}\right)-i\left(e_{1,3}+e_{3,1}\right)+i\left(e_{2,4}+e_{4,2}\right)\right)\\
y_2^{\sigma,+} &:= \frac{1}{2}\left(\left(e_{1,2}+e_{2,1}\right)-\left(e_{3,4}+e_{4,3}\right)+i\left(e_{1,3}+e_{3,1}\right)+i\left(e_{2,4}+e_{4,2}\right)\right)\\
y_1^{\sigma,-} &:= \frac{1}{2}\left(\left(e_{1,2}+e_{2,1}\right)+\left(e_{3,4}+e_{4,3}\right)+i\left(e_{1,3}+e_{3,1}\right)-i\left(e_{2,4}+e_{4,2}\right)\right)\\
y_2^{\sigma,-} &:= \frac{1}{2}\left(\left(e_{1,2}+e_{2,1}\right)-\left(e_{3,4}+e_{4,3}\right)-i\left(e_{1,3}+e_{3,1}\right)-i\left(e_{2,4}+e_{4,2}\right)\right)\\
y_3^{\sigma,+} &:= \frac{1}{2}\left(i\left(e_{1,4}+e_{4,1}\right)+\left(e_{1,1}-e_{2,2}\right)+\left(e_{2,2}-e_{3,3}\right)+\left(e_{3,3}-e_{4,4}\right)\right)\\
y_3^{\sigma,-} &:= \frac{1}{2}\left(-i\left(e_{1,4}+e_{4,1}\right)+\left(e_{1,1}-e_{2,2}\right)+\left(e_{2,2}-e_{3,3}\right)+\left(e_{3,3}-e_{4,4}\right)\right)\\
Y_1^{\sigma,+} &:= i\left(e_{2,3}+e_{3,2}\right)-\left(e_{2,2}-e_{3,3}\right)\\
Y_2^{\sigma,+} &:= i\left(e_{2,3}+e_{3,2}\right)+\left(e_{2,2}-e_{3,3}\right)\\
z^{\sigma,+} &:= \left(e_{1,1}-e_{2,2}\right)-\left(e_{3,3}-e_{4,4}\right)
\end{aligned}
$$

Notice that for each of the elements, x, above $\sigma(x) = -x$, because $\sigma\left(e_{i,j}+e_{j,i}\right) = -e_{j,i}-e_{i,j}$ and $\sigma\left(e_{k,k}-e_{k+1,k+1}\right) = -\left(e_{k,k}-e_{k+1,k+1}\right)$, for $1 \leq i \neq j \leq 4$ and $k \in \{1,2,3\}$. Notice also that it is straightforward to show that the elements listed are linearly independent. Therefore, they form a basis for $\mathfrak{sl}_4$, because there are 15 of them in a 15-dimensional space.

Notice that, span $\left\{x_j^{\sigma,\pm}, h_j^{\sigma}\right\} \cong$ span $\left\{y_j^{\sigma,\pm}, h_j^{\sigma}\right\} \cong \mathfrak{sl}_2$, for $j \in \{1,2\}$.

We can now formulate a basis for $\mathcal{T}_m(\mathfrak{sl}_4)$ as follows. Given $\mathbf{r} \in \mathbb{Z}^m$ let r_j be the jth coordinate of $\mathbf{r}$ and define $t^{\mathbf{r}} := t_1^{r_1} \dots t_m^{r_m} \in \mathbb{C}\langle m\rangle$. Given $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^m$ with r_1 even and s_1 odd, $j \in \{1,2\}$ and $k \in \{1,2,3\}$, define $x_{j,\mathbf{r}}^{\pm}, y_{k,\mathbf{s}}^{\pm}, h_{j,\mathbf{r}}, Y_{j,\mathbf{s}}^{+}, z_{\mathbf{s}}^{+} \in \mathcal{T}_m(\mathfrak{sl}_4)$ by $x_{j,\mathbf{r}}^{\pm} := x_j^{\sigma,\pm} \otimes t^{\mathbf{r}}$, $y_{k,\mathbf{s}}^{\pm} := y_k^{\sigma,\pm} \otimes t^{\mathbf{s}}$, $h_{j,\mathbf{r}} := h_j^{\sigma} \otimes t^{\mathbf{r}}$, $Y_{j,\mathbf{s}}^{+} := Y_j^{\sigma,+} \otimes t^{\mathbf{s}}$, and $z_{\mathbf{s}}^{+} := z^{\sigma,+} \otimes t^{\mathbf{s}}$.

It is straightforward to see that the set

$$\left\{x_{j,\mathbf{r}}^{\pm}, y_{k,\mathbf{s}}^{\pm}, h_{j,\mathbf{r}}, Y_{j,\mathbf{s}}^{+}, z_{\mathbf{s}}^{+} \mid j \in \{1,2\},\ k \in \{1,2,3\},\ \mathbf{r}, \mathbf{s} \in \mathbb{Z}^m,\ 2 \mid r_1,\ 2 \nmid s_1\right\}$$

is a basis for $\mathcal{T}_m(\mathfrak{sl}_4)$.

For all $j \in \{1,2\}$,

$$\operatorname{span}\left\{x_{j,\mathbf{r}}^{\pm}, h_{j,\mathbf{r}} \mid \mathbf{r} \in \mathbb{Z}^m, 2 \mid r_1\right\} \cong \operatorname{span}\left\{y_{j,\mathbf{s}}^{\pm}, h_{j,\mathbf{r}} \mid \mathbf{r}, \mathbf{s} \in \mathbb{Z}^m, 2 \mid r_1, 2 \nmid s_1\right\} \cong \mathfrak{sl}_2\langle m\rangle$$

and

$$\operatorname{span}\left\{y_{3,\mathbf{s}}^{\pm}, 2\left(h_{1,\mathbf{r}} + h_{2,\mathbf{r}}\right) \mid \mathbf{r}, \mathbf{s} \in \mathbb{Z}^m, 2 \mid r_1, 2 \nmid s_1\right\} \cong \mathfrak{sl}_2\langle m\rangle$$

where the isomorphism is given by $y_3^{\sigma,\pm} \mapsto \frac{1}{2} y_3^{\sigma,\pm}$. Also,

$$\operatorname{span}\left\{Y_{1,\mathbf{s}}^{+}, Y_{2,\mathbf{s}}^{+}, -2\left(h_{1,\mathbf{r}} - h_{2,\mathbf{r}}\right) \mid \mathbf{r}, \mathbf{s} \in \mathbb{Z}^m, 2 \mid r_1, 2 \nmid s_1\right\} \cong \mathfrak{sl}_2\langle m\rangle$$

where the isomorphism is given by $Y_1^{\sigma,+} \mapsto -\frac{1}{2} Y_1^{\sigma,+}$, and $Y_2^{\sigma,+} \mapsto \frac{1}{2} Y_2^{\sigma,+}$ So the straightening identities for these elements have been done in [4].

Given a Lie algebra, L, let $U(L)$ be its universal enveloping algebra.

Definition 2.4 *Given $u \in U(L)$ and $k \in \mathbb{Z}$, define the kth divided power by $u^{(k)} = 0$ if $k < 0$ and*

$$u^{(k)} := \frac{u^k}{k!} \in U(L)$$

if $k \geq 0$.

In order to formulate straightening identities in the universal enveloping algebra we need to order the basis. The order we chose is given by

$$\left(y_1^{\sigma,-}\right)\left(y_2^{\sigma,-}\right)\left(y_3^{\sigma,-}\right)\left(x_1^{\sigma,-}\right)\left(x_2^{\sigma,-}\right) h_1^{\sigma} h_2^{\sigma} z^{\sigma,+}\left(y_1^{\sigma,+}\right)\left(y_2^{\sigma,+}\right)\left(y_3^{\sigma,+}\right)\left(Y_1^{\sigma,+}\right)\left(Y_2^{\sigma,+}\right)\left(x_1^{\sigma,+}\right)\left(x_2^{\sigma,+}\right)$$

3 Straightening Identities

Our straightening identities have two different types. We will first state all of the identities of the first type and then prove one in detail. The remaining identities of this type have similar proofs and so the proofs are omitted.

3.1 Type 1 Identities

Theorem 3.1 *Let $r,s \in \mathbb{N}$, $j,l \in \{1,2\}$ distinct, and $\mathbf{r},\mathbf{s} \in \mathbb{Z}^m$ with $2 \mid r_1$ and $2 \nmid s_1$, then*

$$\left(x_{j,\mathbf{r}}^{+}\right)^{(r)}\left(y_{l,\mathbf{s}}^{+}\right)^{(s)} = \sum_{k=0}^{\min\{r,s\}} \left(y_{l,\mathbf{s}}^{+}\right)^{(s-k)}\left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)^{(k)}\left(x_{j,\mathbf{r}}^{+}\right)^{(r-k)} \tag{1}$$

$$\left(x_{j,\mathbf{r}}^{-}\right)^{(r)}\left(y_{l,\mathbf{s}}^{-}\right)^{(s)} = \sum_{k=0}^{\min\{r,s\}} (-1)^{k}\left(y_{l,\mathbf{s}}^{-}\right)^{(s-k)}\left(y_{3,\mathbf{r}+\mathbf{s}}^{-}\right)^{(k)}\left(x_{j,\mathbf{r}}^{-}\right)^{(r-k)} \tag{2}$$

$$\left(x_{j,\mathbf{r}}^{+}\right)^{(r)}\left(y_{l,\mathbf{s}}^{-}\right)^{(s)} = \sum_{k=0}^{\min\{r,s\}} (-1)^{kl}\left(y_{l,\mathbf{s}}^{-}\right)^{(s-k)}\left(Y_{j,\mathbf{r}+\mathbf{s}}^{+}\right)^{(k)}\left(x_{j,\mathbf{r}}^{+}\right)^{(r-k)} \tag{3}$$

$$\left(y_{j,\mathbf{s}}^{+}\right)^{(s)}\left(x_{l,\mathbf{r}}^{-}\right)^{(r)} = \sum_{k=0}^{\min\{r,s\}} (-1)^{kl}\left(x_{l,\mathbf{r}}^{-}\right)^{(r-k)}\left(y_{j,\mathbf{s}}^{+}\right)^{(s-k)}\left(Y_{j,\mathbf{r}+\mathbf{s}}^{+}\right)^{(k)} \tag{4}$$

Proof. We prove (1) by double induction on r and s. The other proofs are similar. In the case that $s=1$ (1) becomes

$$\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{2,\mathbf{s}}^{+}\right) = \left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} + \left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} \tag{5}$$

We will first prove (5) by induction on r:

Base Case: For $r=s=1$,

$$\begin{aligned}\left(x_{1,\mathbf{r}}^{+}\right)\left(y_{2,\mathbf{s}}^{+}\right) &= \left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right) + \left[x_{1,\mathbf{r}}^{+}, y_{2,\mathbf{s}}^{+}\right]\\ &= \left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right) + \left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\end{aligned}$$

Induction Step for (5): Assume (5) for some $r \in \mathbb{N}$. Observe that,

$$\begin{aligned}\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)}\left(y_{2,\mathbf{s}}^{+}\right) &= \frac{1}{(r+1)!}\left(x_{1,\mathbf{r}}^{+}\right)^{1+r}\left(y_{2,\mathbf{s}}^{+}\right) \text{ by Definition 2.4}\\ &= \frac{r!}{(r+1)!}\left(x_{1,\mathbf{r}}^{+}\right)\left(\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{2,\mathbf{s}}^{+}\right)\right)\\ &= \frac{1}{r+1}\left(x_{1,\mathbf{r}}^{+}\right)\left(\left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} + \left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}\right) \text{ by (5)}\\ &= \frac{1}{r+1}\left(\left(x_{1,\mathbf{r}}^{+}\right)\left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} + \left(x_{1,\mathbf{r}}^{+}\right)\left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}\right)\\ &= \frac{1}{r+1}\left(\left(\left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right) + \left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} + r\left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\right)\\ &= \frac{\left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{r+1}}{(r+1)(r!)} + \frac{(r+1)\left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}}{(r+1)}\\ &= \left(y_{2,\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)} + \left(y_{3,\mathbf{r}+\mathbf{s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\end{aligned}$$

Therefore, (5) holds for all $r \in \mathbb{N}$ by induction. We will now prove (1) for any $r \in \mathbb{N}$ by induction on s. The base case $s = 1$ is (5).

Induction Step for (1): Assume for all $r \in \mathbb{N}$ and some $s \in \mathbb{N}$ that (1) holds. Observe that,

$$\begin{aligned}
(s+1)\left(x^+_{1,\mathbf{r}}\right)^{(r)}\left(y^+_{2,\mathbf{s}}\right)^{(s+1)} =& \frac{(s+1)}{(s+1)!}\left(x^+_{1,\mathbf{r}}\right)^{(r)}\left(y^+_{2,\mathbf{s}}\right)^{1+s} \text{ by Definition 2.4}\\
=& \frac{1}{s!}\left(x^+_{1,\mathbf{r}}\right)^{(r)}\left(y^+_{2,\mathbf{s}}\right)\left(y^+_{2,\mathbf{s}}\right)^{s}\\
=& \left(\left(y^+_{2,\mathbf{s}}\right)\left(x^+_{1,\mathbf{r}}\right)^{(r)} + \left(y^+_{3,\mathbf{r+s}}\right)\left(x^+_{1,\mathbf{r}}\right)^{(r-1)}\right)\left(y^+_{2,\mathbf{s}}\right)^{(s)} \text{ by (5)}\\
=& \left(y^+_{2,\mathbf{s}}\right)\left(\left(x^+_{1,\mathbf{r}}\right)^{(r)}\left(y^+_{2,\mathbf{s}}\right)^{(s)}\right) + \left(y^+_{3,\mathbf{r+s}}\right)\left(\left(x^+_{1,\mathbf{r}}\right)^{(r-1)}\left(y^+_{2,\mathbf{s}}\right)^{(s)}\right)\\
=& \left(y^+_{2,\mathbf{s}}\right)\sum_{k=0}^{\min\{r,s\}}\left(y^+_{2,\mathbf{s}}\right)^{(s-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}\\
&+ \left(y^+_{3,\mathbf{r+s}}\right)\sum_{k=0}^{\min\{r-1,s\}}\left(y^+_{2,\mathbf{s}}\right)^{(s-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-1-k)}\\
&\text{by the induction hypothesis}\\
=& \sum_{k=0}^{\min\{r,s\}}(s+1-k)\left(y^+_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}\\
&+ \sum_{k=0}^{\min\{r-1,s\}}(k+1)\left(y^+_{2,\mathbf{s}}\right)^{(s-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k+1)}\left(x^+_{1,\mathbf{r}}\right)^{(r-1-k)}\\
=& \sum_{k=0}^{\min\{r,s\}}(s+1-k)\left(y^+_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}\\
&+ \sum_{k=1}^{\min\{r,s+1\}}(k)\left(y^+_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}\\
=&(s+1)\left(y^+_{2,\mathbf{s}}\right)^{(s+1)}\left(x^+_{1,\mathbf{r}}\right)^{(r)}\\
&+ \sum_{k=1}^{\min\{r,s\}}(s+1-k)\left(y^+_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}\\
&+ \sum_{k=1}^{\min\{r,s+1\}}(k)\left(y^+_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^+_{3,\mathbf{r+s}}\right)^{(k)}\left(x^+_{1,\mathbf{r}}\right)^{(r-k)}
\end{aligned}$$

$$
\begin{aligned}
=&(s+1)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r)}+\sum_{k=1}^{\min\{r,s+1\}}(s+1-k)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}\\
&+\sum_{k=1}^{\min\{r,s+1\}}(k)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}\\
=&(s+1)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r)}\\
&+\sum_{k=1}^{\min\{r,s+1\}}(s+1-k+k)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}\\
=&(s+1)\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r)}+(s+1)\sum_{k=1}^{\min\{r,s+1\}}\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}\\
&=(s+1)\left(\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r)}+\sum_{k=1}^{\min\{r,s+1\}}\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}\right)\\
&=(s+1)\sum_{k=0}^{\min\{r,s+1\}}\left(y^{+}_{2,\mathbf{s}}\right)^{(s+1-k)}\left(y^{+}_{3,\mathbf{r+s}}\right)^{(k)}\left(x^{+}_{1,\mathbf{r}}\right)^{(r-k)}
\end{aligned}
$$

□

3.2 Type 2 Identities

Theorem 3.2 *Let $r, s \in \mathbb{N}$ and $\mathbf{a}, \mathbf{r}, \mathbf{s} \in \mathbb{Z}^m$ with $2 \mid r_1$, $2 \nmid a_1$ and $2 \nmid s_1$.*

$$\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} = \sum\left(-2y_{2,\mathbf{r+s}}^{-}\right)^{(k_1)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s-k_1-k_2)}\left(-Y_{1,2\mathbf{r+s}}^{+}\right)^{(k_2)}\left(x_{1,\mathbf{r}}^{+}\right)^{(r-k_1-2k_2)} \tag{6}$$

$$\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(Y_{2,\mathbf{s}}^{+}\right)^{(s)} = \sum 2^{k_1}\left(y_{2,\mathbf{r+s}}^{+}\right)^{(k_1)}\left(y_{3,2\mathbf{r+s}}^{+}\right)^{(k_2)}\left(Y_{2,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(x_{1,\mathbf{r}}^{+}\right)^{(r-k_1-2k_2)} \tag{7}$$

$$\left(y_{3,\mathbf{s}}^{+}\right)^{(s)}\left(x_{1,\mathbf{r}}^{-}\right)^{(r)} = \sum(-2)^{k_1}\left(x_{1,\mathbf{r}}^{-}\right)^{(r-k_1-2k_2)}\left(y_{2,\mathbf{r+s}}^{+}\right)^{(k_1)}\left(y_{3,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(Y_{2,2\mathbf{r+s}}^{+}\right)^{(k_2)} \tag{8}$$

$$\left(Y_{1,\mathbf{s}}^{+}\right)^{(s)}\left(x_{1,\mathbf{r}}^{-}\right)^{(r)} = \sum(-2)^{k_1}\left(y_{2,\mathbf{r+s}}^{-}\right)^{(k_1)}\left(y_{3,2\mathbf{r+s}}^{-}\right)^{(k_2)}\left(x_{1,\mathbf{r}}^{-}\right)^{(r-k_1-2k_2)}\left(Y_{1,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)} \tag{9}$$

$$\left(x_{2,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} = \sum(-2)^{k_1}\left(y_{1,\mathbf{r+s}}^{-}\right)^{(k_1)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s-k_1-k_2)}\left(Y_{2,2\mathbf{r+s}}^{+}\right)^{(k_2)}\left(x_{2,\mathbf{r}}^{+}\right)^{(r-k_1-2k_2)} \tag{10}$$

$$\left(x_{2,\mathbf{r}}^{+}\right)^{(r)}\left(Y_{1,\mathbf{s}}^{+}\right)^{(s)} = \sum\left(-2y_{1,\mathbf{r+s}}^{+}\right)^{(k_1)}\left(-y_{3,2\mathbf{r+s}}^{+}\right)^{(k_2)}\left(Y_{1,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(x_{2,\mathbf{r}}^{+}\right)^{(r-k_1-2k_2)} \tag{11}$$

$$\left(y_{1,\mathbf{a}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} = \sum 2^{k_1}\left(y_{3,\mathbf{s}}^{-}\right)^{(s-k_1-k_2)}\left(x_{2,\mathbf{a+s}}^{-}\right)^{(k_1)}\left(y_{1,\mathbf{a}}^{+}\right)^{(r-k_1-2k_2)}\left(Y_{1,2\mathbf{a+s}}^{+}\right)^{(k_2)} \tag{12}$$

$$\left(Y_{2,\mathbf{s}}^{+}\right)^{(s)}\left(y_{1,\mathbf{a}}^{+}\right)^{(r)} = \sum\left(y_{1,\mathbf{a}}^{+}\right)^{(r-k_1-2k_2)}\left(-y_{3,2\mathbf{a+s}}^{+}\right)^{(k_2)}\left(Y_{2,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(-2x_{2,\mathbf{a+s}}^{+}\right)^{(k_1)} \tag{13}$$

$$\left(y_{3,\mathbf{s}}^{+}\right)^{(s)}\left(y_{1,\mathbf{a}}^{-}\right)^{(r)} = \sum\left(y_{1,\mathbf{a}}^{-}\right)^{(r-k_1-2k_2)}\left(y_{3,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(-Y_{2,2\mathbf{a+s}}^{+}\right)^{(k_2)}\left(2x_{2,\mathbf{a+s}}^{+}\right)^{(k_1)} \tag{14}$$

$$\left(Y_{1,\mathbf{s}}^{+}\right)^{(s)}\left(y_{1,\mathbf{a}}^{-}\right)^{(r)} = \sum(-2)^{k_1}\left(y_{1,\mathbf{a}}^{-}\right)^{(r-k_1-2k_2)}\left(y_{3,2\mathbf{a+s}}^{-}\right)^{(k_2)}\left(x_{2,\mathbf{a+s}}^{-}\right)^{(k_1)}\left(Y_{1,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)} \tag{15}$$

$$\left(y_{2,\mathbf{a}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} = \sum\left(y_{3,\mathbf{s}}^{-}\right)^{(s-k_1-k_2)}\left(2x_{1,\mathbf{a+s}}^{-}\right)^{(k_1)}\left(y_{2,\mathbf{a}}^{+}\right)^{(r-k_1-2k_2)}\left(-Y_{2,2\mathbf{a+s}}^{+}\right)^{(k_2)} \tag{16}$$

$$\left(Y_{1,\mathbf{s}}^{+}\right)^{(s)}\left(y_{2,\mathbf{a}}^{+}\right)^{(r)} = \sum 2^{k_1}\left(y_{2,\mathbf{a}}^{+}\right)^{(r-k_1-2k_2)}\left(y_{3,2\mathbf{a+s}}^{+}\right)^{(k_2)}\left(Y_{1,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(x_{1,\mathbf{a+s}}^{+}\right)^{(k_1)} \tag{17}$$

$$\left(Y_{2,\mathbf{s}}^{+}\right)^{(s)}\left(y_{2,\mathbf{a}}^{-}\right)^{(r)} = \sum\left(y_{2,\mathbf{a}}^{-}\right)^{(r-k_1-2k_2)}\left(-y_{3,2\mathbf{a+s}}^{-}\right)^{(k_2)}\left(2x_{1,\mathbf{a+s}}^{-}\right)^{(k_1)}\left(Y_{2,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)} \tag{18}$$

$$\left(y_{3,\mathbf{s}}^{+}\right)^{(s)}\left(x_{2,\mathbf{r}}^{-}\right)^{(r)} = \sum\left(x_{2,\mathbf{r}}^{-}\right)^{(r-k_1-2k_2)}\left(-2y_{1,\mathbf{r+s}}^{+}\right)^{(k_1)}\left(y_{3,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)}\left(-Y_{1,2\mathbf{r+s}}^{+}\right)^{(k_2)} \tag{19}$$

$$\left(Y_{2,\mathbf{s}}^{+}\right)^{(s)}\left(x_{2,\mathbf{r}}^{-}\right)^{(r)} = \sum\left(2y_{1,\mathbf{r+s}}^{-}\right)^{(k_1)}\left(-y_{3,2\mathbf{r+s}}^{-}\right)^{(k_2)}\left(x_{2,\mathbf{r}}^{-}\right)^{(r-k_1-2k_2)}\left(Y_{2,\mathbf{s}}^{+}\right)^{(s-k_1-k_2)} \tag{20}$$

Where all of the sums are over all $k_1, k_2 \in \mathbb{Z}_+$ such that $k_1 + 2k_2 \leq r$ and $k_1 + k_2 \leq s$.

Proof. We will prove (6) in detail by double induction on r and s. The other proofs are similar. In the case $s = 1$, (6) becomes

$$\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right) = \left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} - 2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} - \left(Y_{1,2\mathbf{r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-2)} \tag{21}$$

We will prove (21) by induction on r.

Base Case: For $(r, s) = (1, 1)$, we have

$$\begin{aligned}\left(x_{1,\mathbf{r}}^{+}\right)\left(y_{3,\mathbf{s}}^{-}\right) &= \left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right) + \left[x_{1,\mathbf{r}}^{+}, y_{3,\mathbf{s}}^{-}\right] \\ &= \left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right) - 2\left(y_{2,\mathbf{r+s}}^{-}\right)\end{aligned}$$

Induction Hypothesis: Assume (21) for some $r \in \mathbb{N}$. Observe that,

$$\begin{aligned}
\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)}\left(y_{3,\mathbf{s}}^{-}\right) =& \frac{1}{r+1}\left(x_{1,\mathbf{r}}^{+}\right)\left(\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)\right) \text{ by Definition 2.4} \\
=& \frac{1}{r+1}\left(x_{1,\mathbf{r}}^{+}\right)\Bigg(\left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}-2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} \\
&-\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-2)}\Bigg) \text{ by the induction hypothesis} \\
=& \frac{1}{r+1}\Bigg(\left(x_{1,\mathbf{r}}^{+}\right)\left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}-2\left(x_{1,\mathbf{r}}^{+}\right)\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} \\
&-\left(x_{1,\mathbf{r}}^{+}\right)\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-2)}\Bigg) \\
=& \frac{1}{r+1}\Bigg((r+1)\left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)}-2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} \\
&-2r\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}-2\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} \\
&-(r-1)\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}\Bigg) \\
=& \frac{1}{r+1}\Bigg((r+1)\left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)}-2(r+1)\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)} \\
&-(r+1)\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}\Bigg) \\
=& \left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r+1)}-2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}-\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}
\end{aligned}$$

Therefore, (21) holds for all $r \in \mathbb{N}$.

We will proceed by induction on s to prove (6). Assume the formula for all $r \in \mathbb{N}$ and some $s \in \mathbb{N}$. Then

$$\begin{aligned}
(s+1)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s+1)} &= \left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} \\
&= \Bigg(\left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}-2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)} \\
&\quad -\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-2)}\Bigg)\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} \text{ by (21)} \\
&= \left(y_{3,\mathbf{s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)}-2\left(y_{2,\mathbf{r+s}}^{-}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-1)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)} \\
&\quad -\left(Y_{1,\mathbf{2r+s}}^{+}\right)\left(x_{1,\mathbf{r}}^{+}\right)^{(r-2)}\left(y_{3,\mathbf{s}}^{-}\right)^{(s)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(y^-_{3,\mathbf{s}}\right) \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s}} \left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&- 2\left(y^-_{2,\mathbf{r}+\mathbf{s}}\right) \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r-1\\ k_1+k_2\le s}} \left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-1-k_1-2k_2)}\\
&- \left(Y^+_{1,2\mathbf{r}+\mathbf{s}}\right) \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r-2\\ k_1+k_2\le s}} \left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-2-k_1-2k_2)}\\
&= \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s}} (s+1-k_1-k_2)\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&- 2 \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r-1\\ k_1+k_2\le s}} (-2)^{k_1}(k_1+1)\left(y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1+1)} \left(y^-_{3,\mathbf{s}}\right)^{(s-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-1-k_1-2k_2)}\\
&- \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r-2\\ k_1+k_2\le s}} (-1)^{k_2}(k_2+1)\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s-k_1-k_2)} \left(Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2+1)} \left(x^+_{1,\mathbf{r}}\right)^{(r-2-k_1-2k_2)}\\
&= \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} (s+1-k_1-k_2)\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&- 2 \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} (-2)^{k_1-1}k_1\left(y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&- \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} (-1)^{k_2-1}k_2\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&= \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} (s+1-k_1-k_2)\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&+ \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} k_1\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}\\
&+ \sum_{\substack{k_1,k_2\in\mathbb{Z}_+\\ k_1+2k_2\le r\\ k_1+k_2\le s+1}} k_2\left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}
\end{aligned}
$$

$$= (s+1) \sum_{\substack{k_1,k_2\in\mathbb{Z}_+ \\ k_1+2k_2\le r \\ k_1+k_2\le s+1}} \left(-2y^-_{2,\mathbf{r}+\mathbf{s}}\right)^{(k_1)} \left(y^-_{3,\mathbf{s}}\right)^{(s+1-k_1-k_2)} \left(-Y^+_{1,2\mathbf{r}+\mathbf{s}}\right)^{(k_2)} \left(x^+_{1,\mathbf{r}}\right)^{(r-k_1-2k_2)}$$

Therefore (6) is proven by double induction on r and s. $\square$

4 Further Directions

The next steps are to formulate and prove straightening identities for the following products: $\left(x^+_{1,\mathbf{r}}\right)^{(r)} \left(y^-_{1,\mathbf{s}}\right)^{(s)}$, $\left(x^+_{2,\mathbf{r}}\right)^{(r)} \left(y^-_{2,\mathbf{s}}\right)^{(s)}$, $\left(y^+_{1,\mathbf{s}}\right)^{(s)} \left(x^-_{1,\mathbf{r}}\right)^{(r)}$, and $\left(y^+_{2,\mathbf{s}}\right)^{(s)} \left(x^-_{2,\mathbf{r}}\right)^{(r)}$.

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Samuel Chamberlin
Park University
8700 River Park Drive
Parkville, MO 64152
E-mail: `schamberlin@park.edu`

Jagrit Niraula
Park University
8700 River Park Drive
Parkville, MO 64152
E-mail: `1613771@park.edu`

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