Sums of Powers of Primes in Arithmetic Progression

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Abstract - Gerard and Washington proved that, for $k > -1$, the number of primes less than x^{k+1} can be well approximated by summing the k-th powers of all primes up to x. We extend this result to primes in arithmetic progressions: we prove that the number of primes $p \equiv n \pmod{m}$ less than x^{k+1} is asymptotic to the sum of k-th powers of all primes $p \equiv n \pmod{m}$ up to x. We prove that the prime power sum approximation tends to be an underestimate for positive k and an overestimate for negative k, and quantify for different values of k how well the approximation works for x between 10^4 and 10^8.

Keywords : prime number theorem; arithmetic progression

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1 Introduction

1.1 Historical Context

Approximating the distribution of prime numbers is a major motivating question in number theory. This task can be broken down into counting primes up to a given number and estimating where prime numbers appear within the set of natural numbers. The latter is a much more difficult task and falls outside of the scope and context of our results. We instead extend a recent approximation for the prime counting function to primes in arithmetic progressions. We use analogous methods to the work of Gerard and Washington [5], incorporating the findings from Page [10] and Siegel [13] and applying the prime number theorem for arithmetic progressions where Gerard and Washington used the prime number theorem. While our methods are similar, extending them to primes in arithmetic progressions allows to investigate finer questions on the distributions of prime numbers, for example, by studying Chebyshev bias.

Definition 1.1 *For $k \in \mathbb{R}$, define*

$$\pi(x) \ := \sum_{p \le x} 1 \qquad \text{and} \qquad \pi_k(x) \ := \sum_{p \le x} p^k,$$

where p denotes a prime number here and throughout; $\pi(x)$ is the prime-counting function*.*

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Definition 1.2 *For real $x \geq 0$, define the* logarithmic integral

$$\mathrm{li}(x) \ := \ \int_0^x \frac{dt}{\log t}.$$

Definition 1.3 *Let f and g be two functions defined on $\mathbb{R}_{>0}$, and $g(x)$ be strictly positive for all large enough values of x. We say*

$$f(x) \ = \ O(g(x)) \qquad as \qquad x \to \infty$$

if there exist constants M and x_0 such that

$$|f(x)| \ \leq \ M|g(x)| \qquad for\ all \qquad x \ \geq \ x_0.$$

We also say

$$f(x) \ll g(x) \qquad is\ equivalent\ to \qquad f(x) = O(g(x)).$$

Definition 1.4 *Let f and g be two functions defined on $\mathbb{R}_{>0}$, and $g(x)$ be strictly positive for all large enough values of x. We say*

$$f(x) \ = \ o(g(x)) \qquad as \qquad x \to \infty$$

if for every positive constant ε, there exists a constant x_0 such that

$$|f(x)| \ \leq \ \varepsilon g(x) \qquad for\ all \qquad x \ \geq \ x_0$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \ = \ 0.$$

We also say

$$f(x) \lll g(x) \qquad is\ equivalent\ to \qquad f(x) = o(g(x)).$$

Definition 1.5 *Let f and g be two functions defined on $\mathbb{R}_{>0}$, and $g(x)$ be strictly positive for all large enough values of x. We say*

$$f(x) \sim g(x) \qquad as \qquad x \to \infty,$$

read $f(x)$ is asymptotic to *$g(x)$, if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \ = \ 1.$$

In 1896, Vallée-Poussin [16] proved that

$$\pi(x) \sim \mathrm{li}(x) \qquad \text{as} \qquad x \to \infty,$$

which is known as the *prime number theorem*, using techniques which relied on complex analysis and the Riemann ζ-function. Vallée-Poussin [16] also estimated the error term in the prime number theorem by proving that

$$\pi(x) \;=\; \mathrm{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right) \quad (x \geq 2) \tag{1}$$

where c is a positive constant. Further improvements on the error term began to emerge. In 1922, Littlewood (outlined in [7]) proved that

$$\pi(x) - \mathrm{li}(x) \ll x \exp\left(-c\sqrt{\log x \log\log x}\right) \quad (x \geq 2).$$

In 1958, Korobov [6] and Vinogradov [15] proved that

$$\pi(x) - \mathrm{li}(x) \ll x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right) \quad (x \geq 2).$$

Since then, various alternative proofs have been given which do not use complex analysis, with perhaps the most renowned being the proof by Erdős and Selberg [12].

A natural immediate extension of approximating the number of primes up to a certain real number is estimating the *sum of primes* up to a certain real number. Using Vallée-Poussin's result [Equation (1)], Szalay [14, Lemma 1] found the following approximation for the sum of primes less than a real number $x \geq 0$:

$$\pi_1(x) \;=\; \mathrm{li}\left(x^2\right) + O\left(x^2 \exp\left(-c\sqrt{\log x}\right)\right). \tag{2}$$

In 1996, Massias and Robin [9, p. 217] approximated the sum of prime numbers less than a real number $x \geq 0$ as

$$\pi_1(x) \;=\; \mathrm{li}\left(\mathrm{li}^{-1}(x)^2\right) + O\left(x^2 \exp\left(-\gamma\sqrt{\log x}\right)\right)$$

where γ is a positive constant and $\mathrm{li}^{-1}(x)$ denotes the inverse of $\mathrm{li}(x)$.

Massias and Robin [9, Théorème D(v)] further bounded $\pi_1(x)$ above for every $x \geq 24281$ by

$$\pi_1(x) \;\leq\; \frac{x^2}{2\log x} + \frac{3x^2}{10\log^2 x}. \tag{3}$$

Around three decades following the publication of Szalay's result [Equation (2)], Axler [2] applied it to show that

$$\pi_1(x) \;=\; \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{3x^2}{8\log^4 x} + O\left(\frac{x^2}{\log^5 x}\right).$$

Axler [2, Theorem 1.1] also improved on the upper bound given by Massias and Robin [Line (3)] to give the following inequality

$$\pi_1(x) \;<\; \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{5.3x^2}{8\log^4 x},$$

which holds for every $x \geq 110118925$. For $x \geq 905238547$, Axler [2, Theorem 1.2] also gives a lower bound of $\pi_1(x)$, which differs from the upper bound only by a constant in the fourth term from the upper bound.

$$\pi_1(x) \;>\; \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{1.2x^2}{8\log^4 x}.$$

In 2016, shortly following Axler's preprint, Gerard and Washington [5, Theorem 1] generalized these results to sums of fixed powers k of primes. They proved that $\pi_k(x)$ is asymptotic to $\pi(x^{k+1})$:

$$\begin{aligned}\pi_k(x) - \pi(x^{k+1})& \\ &= \begin{cases} O\left(x^{k+1}\exp\left(-A(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) & \text{if } k > 0, \\ O\left(x^{k+1}\exp\left(-(k+1)^{3/5}A(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) & \text{if } -1 < k < 0,\end{cases}\end{aligned}$$

where $A = .2098$. Their result is significant both from the perspective of studying the prime counting function and from the perspective of studying sums of powers of primes, as it relates the two functions with small relative error for large x. From the perspective of studying the prime counting function, their result allows for approximating $\pi(x)$ by $\pi_k\left(x^{1/(k+1)}\right)$, which can significantly reduce the number of primes used in the approximation.

1.2 Statement of Main Results

Throughout this paper, we take $m, n \in \mathbb{Z}_{>0}$ to be two coprime positive integers, specifically with n a unit in $\mathbb{Z}/m\mathbb{Z}$. As mentioned previously, we use p to denote prime numbers throughout.

To extend previous results, we first define the sum of k-powers of primes for arithmetic progressions.

Definition 1.6 *For a fixed $k \in \mathbb{R}$, define*

$$\pi(x; m, n) \;=\; \sum_{\substack{p \leq x \\ p \equiv n \pmod m}} 1 \qquad \text{and} \qquad \pi_k(x; m, n) \;=\; \sum_{\substack{p \leq x \\ p \equiv n \pmod m}} p^k.$$

Also, recall the definition of Euler's totient function:

$$\varphi(m) \;=\; \#\{x \in \mathbb{N} \;:\; x \leq m \text{ and } \gcd(x, m) = 1\},$$

which counts the number of positive integers up to m which are coprime to m.

We extend the work of Gerard and Washington [5], studying sums of powers of primes *in arithmetic progressions*. Dirichlet's theorem on arithmetic progressions states that there are infinitely many primes p in the form $p \equiv n \pmod{m}$. Thus, $\pi(x; m, n)$ grows to ∞ as x approaches ∞. Also, Vallée-Poussin [16] proved the *prime number theorem on arithmetic progressions*, concluding that

$$\pi(x; m, n) \sim \frac{\pi(x)}{\varphi(m)} \sim \frac{x}{\varphi(m) \log x} \qquad \text{as} \qquad x \to \infty, \tag{4}$$

where $\gcd(m, n) = 1$. A proof of (4) is outlined in [1, p. 154-155]. In 1935, Page [10] proved that there exists a positive constant δ such that

$$\pi(x; m, n) = \frac{\text{li}(x)}{\varphi(m)} + O\left(x \exp\left(-(\log x)^{\delta}\right)\right) \quad (x \geq 2),$$

whenever $1 \leq m \leq (\log x)^{2-\delta}$. In the same year, Siegel [13] proved that, for every fixed $\eta > 0$, there exists a positive constant ξ_η depending on η such that

$$\pi(x; m, n) = \frac{\text{li}(x)}{\varphi(m)} + O\left(x \exp\left(-\frac{1}{2}\xi_\eta \sqrt{\log x}\right)\right) \quad (x \geq 2),$$

whenever $1 \leq m \leq (\log x)^{\eta}$. We use the above version of the prime number theorem on arithmetic progressions to obtain our main result.

Following the methods of Gerard and Washington in [5] we obtain the following theorem.

Theorem 1.7 *Fix a real number $k > -1$ and positive integers $m, n \in \mathbb{Z}_{>0}$ such that* $\gcd(m, n) = 1$. *Then we can approximate the number of primes $p \equiv n \pmod{m}$ less than x^{k+1} by the sum of k-powers of primes $p \equiv n \pmod{m}$ less than a real number x:*

$$\begin{aligned} &\pi_k(x; m, n) - \pi(x^{k+1}; m, n) \\ &\quad = \begin{cases} O\left(x^{k+1} \exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) & \text{if } k > 0, \\ O\left(x^{k+1} \exp\left(-\frac{1}{2}\alpha\sqrt{(k+1)\log x}\right)\right) & \text{if } -1 < k < 0, \end{cases} \end{aligned}$$

where α is a positive constant.

Additionally, generalizing the proof of [17, Theorem 2], we prove the following.

Theorem 1.8 *Fix a real number $k > -1$ and positive integers $m, n \in \mathbb{Z}_{>0}$ such that* $\gcd(m, n) = 1$. *Then*

$$\int_1^\infty \frac{\pi_k(t; m, n) - \pi(t^{k+1}; m, n)}{t^{k+2}}\, dt = -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$

Further,

$$\begin{aligned} -\frac{\log(k+1)}{(k+1)\varphi(m)} &< 0 \qquad (k > 0) \\ -\frac{\log(k+1)}{(k+1)\varphi(m)} &> 0 \qquad (-1 < k < 0). \end{aligned}$$

Remark 1.9 In 1853, Chebyshev [3] noticed that there are more primes of the form $p \equiv 3 \pmod 4$ than $p \equiv 1 \pmod 4$. Over the years the synonymous expression "prime number races" has emerged to describe problems originated from Chebyshev. Rubinstein and Sarnak [11] provided a framework for the quantification of Chebyshev's bias in prime number races in arithmetic progressions. They considered the sets $P(m; n_1, n_2) = \{x \geq 2 : \pi(x; m, n_1) > \pi(x; m, n_2)\}$. With an appropriate notion of size, under standard assumptions, they proved that the set $P(4; 3, 1)$ is "larger" than the set $P(4; 1, 3)$.

In Appendix 3.1, we provide calculations of various examples for different values of k, m, and n. The integral computation in Theorem 1.8 confirms the observations from our data that

$$\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$$

tends to be negative when $k > 0$, and tends to be positive when $-1 < k < 0$ for large values of x. Note that this does not definitively specify the sign of the error for given large values of x. Since our result integrates over all reals greater than one, we can only conclude that the "net" sign will either be negative or positive for specific k. A more refined approach to evaluating $\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$ when x is sufficiently large necessitates the utilization of the Riemann ζ-function and the Riemann Hypothesis to analyze the bias between $\pi(x; m, n)$ and $\mathrm{li}(x)$, which allows for a more precise determination of the error's behavior in such cases.

2 Proofs of Theorems

One useful technique used to prove Theorem 1.7 and Theorem 1.8 is Riemann-Stieltjes Integration. The detailed procedure is included in Appendix 3.2.

2.1 Proof of Theorem 1.7

We divide the proof in three parts.

2.1.1 Start of Proof

Proof. By the prime number theorem for arithmetic progressions [8, Theorem 7-25],

$$\pi(x; m, n) \quad = \quad \frac{1}{\varphi(m)} \int_2^x \frac{dt}{\log t} \; + \; O\left(x \exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) \; (x \geq 2) \tag{5}$$

where α is a positive constant. In the rest of the proof, denote

$$E(x) \; = \; O\left(x \exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) \tag{6}$$

as the error term in (5).

By applying (13) on Riemann-Stieltjes integration, for $x \geq 2$ we have

$$\pi_k(x;m,n) \quad = \sum_{\substack{p \leq x \\ p \,\equiv\, n \pmod m}} p^k \quad = \quad \int_2^x t^k \, d\pi(t;m,n). \tag{7}$$

Taking derivatives for (5) yields

$$\begin{aligned} d\pi(x;m,n) &= d\left(\frac{1}{\varphi(m)}\int_2^x \frac{dt}{\log t}\right) + dE(x) \\ &= \frac{1}{\varphi(m)}\frac{dx}{\log x} + dE(x). \end{aligned}$$

Substituting $d\pi(t;m,n)$ into (7), we obtain

$$\begin{aligned} \pi_k(x;m,n) &= \int_2^x t^k \left(\frac{1}{\varphi(m)}\frac{dt}{\log t} + dE(t)\right) \\ &= \frac{1}{\varphi(m)}\int_2^x \frac{t^k}{\log t}\, dt + \int_2^x t^k \, dE(t). \end{aligned}$$

Let $u = t^{k+1}$. Then

$$\begin{aligned} \frac{1}{\varphi(m)}\int_2^x \frac{t^k}{\log t}\, dt &= \frac{1}{\varphi(m)}\int_{2^{k+1}}^{x^{k+1}} \frac{du}{\log u} \\ &= \pi(x^{k+1};m,n) - E(x^{k+1}) - C \end{aligned}$$

where $C = \mathrm{li}(2^{k+1})/\varphi(m)$.

Thus,

$$\frac{1}{\varphi(m)}\int_2^x \frac{t^k}{\log t}\, dt = \pi(x^{k+1};m,n) - E(x^{k+1}) - C.$$

Also, integration by parts yields

$$\begin{aligned} \int_2^x t^k \, dE(t) &= t^k E(t)\Big|_2^x - k\int_2^x t^{k-1}E(t)\, dt \\ &= x^k E(x) - k\int_2^x t^{k-1}E(t)\, dt. \end{aligned}$$

Finally, we obtain

$$\pi_k(x;m,n) = \pi(x^{k+1};m,n) - E(x^{k+1}) - C + x^k E(x) - k\int_2^x t^{k-1}E(t)\, dt. \tag{8}$$

2.1.2 Error Term

We now simplify (8).

First, according to (6),

$$E(x) \;=\; O\left(x\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right).$$

Thus,

$$E(x^{k+1}) \;=\; O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{(k+1)\log x}\right)\right).$$

Also, we have

$$\begin{aligned} x^k E(x) \;&=\; x^k\; O\left(x\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) \\ &=\; O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right). \end{aligned}$$

The next step is to bound

$$k\int_2^x t^{k-1}E(t)\,dt \tag{9}$$

by using [5, Lemma 1]. By applying (6), we can bound (9) as

$$\begin{aligned} \left|k\int_2^x t^{k-1}E(t)\,dt\right| \;&\le\; D\int_2^x t^{k-1}\left(t\exp\left(-\frac{1}{2}\alpha\sqrt{\log t}\right)\right)\,dt \\ &=\; D\int_2^x t^k\exp\left(-\frac{1}{2}\alpha\sqrt{\log t}\right)\,dt \end{aligned} \tag{10}$$

where D is a positive constant.

The integrand of (10) is of the form $\quad t^k\exp(-f(t))\quad$ where $\quad f(t)=\dfrac{1}{2}\alpha\sqrt{\log t}$.
To apply [5, Lemma 1], we verify that

i. $\lim_{t\to\infty}[t^{k+1}\exp(-f(t))] \;=\; \infty$

ii. and $\qquad \lim_{t\to\infty}[tf'(t)] \;=\; 0$

for $k>-1$ in separate computations included in Appendix 3.3. Thus, applying [5, Lemma 1], we obtain

$$k\int_2^x t^{k-1}E(t)\,dt \;=\; O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right).$$

Finally, we reduce

$$O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) - C \;=\; O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right).$$

2.1.3 Conclusion

Based on previous sections, we simplify (8) as

$$\pi_k(x;m,n) - \pi(x^{k+1};m,n) = \begin{cases} O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) & \text{if } k>0, \\ O\left(x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{(k+1)\log x}\right)\right) & \text{if } -1<k<0. \end{cases}$$

Therefore, we conclude that

$$\pi_k(x;m,n) \ \sim \ \pi(x^{k+1};m,n).$$

□

2.2 Proof of Theorem 1.8

Before proving Theorem 1.8, we require additional results.

2.2.1 Abel's Summation Formula and an Additional Lemma

Abel's summation formula is an important tool in analytic number theory derived from integrating Riemann-Stieltjes by parts. The statement of the formula and its hypotheses are given below.

First, recall that a function whose domain is the positive integers is called an *arithmetic function.*

Theorem 2.1 (Abel's Summation Formula) *Let a be an arithmetic function. Define*

$$A(t) \ := \ \sum_{1\le n\le t} a(n)$$

for $t \in \mathbb{R}$. Fix $x, y \in \mathbb{R}$ such that $x < y$ and let f be a continuously differentiable function on $[x, y]$. Then

$$\sum_{x<n\le y} a(n)f(n) \ = \ A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)\,dt. \tag{11}$$

The detailed proof of Abel's Summation Formula can be found in *Apostol*'s Introduction to Analytic Number Theory [1].

Lemma 2.2 *Fix positive integers $m, n \in \mathbb{Z}_{>0}$ such that* $\gcd(m,n) = 1$. *Then for $x \ge 2$,*

$$\sum_{\substack{p\le x \\ p\equiv n \pmod m}} \frac{1}{p} \ = \ \frac{\log\log x}{\varphi(m)} + B + O\left(\frac{1}{\log x}\right),$$

where B is a constant.

Lemma 2.2 can be found in *Davenport's* Multiplicative Number Theory [4]. We provide a proof for that.

Proof. Let

$$f(p) \;=\; \frac{1}{\log p}.$$

Note that f is defined on primes and is continuous on $[2, x]$. Let

$$a(p) \;=\; \begin{cases} \frac{\log p}{p} & \text{if} \quad p \;\equiv\; n \pmod{m} \quad \text{is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

If $x \geq 2$,

$$f'(x) \;=\; -\frac{1}{x \log^2 x}.$$

Define

$$A(x) \;:=\; \sum_{\substack{p \leq x \\ p \equiv n \pmod{m}}} a(p) \;=\; \sum_{\substack{p \leq x \\ p \equiv n \pmod{m}}} \frac{\log p}{p}.$$

According to [1, p.148, Theorem 7.3]

$$\sum_{\substack{p \leq x \\ p \equiv n \pmod{m}}} \frac{\log p}{p} \;=\; \frac{\log x}{\varphi(m)} + R(x), \text{ where } R(x) \;=\; O(1).$$

Applying Abel's Summation Formula (Theorem 2.1), we have

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv n \pmod{m}}} \frac{1}{p} \;&=\; \sum_{\substack{1 < p \leq x \\ p \equiv n \pmod{m}}} a(p) f(p) \\ &=\; \frac{1}{\log x}\left(\frac{\log x}{\varphi(m)} + O(1)\right) + \int_2^x \frac{1}{t \log^2 t}\left(\frac{\log t}{\varphi(m)} + R(t)\right) dt \\ &=\; \frac{1}{\varphi(m)} + O\left(\frac{1}{\log x}\right) + \frac{1}{\varphi(m)} \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{R(t)}{t \log^2 t}\, dt. \qquad (12) \end{aligned}$$

Rewrite

$$\int_2^x \frac{R(t)}{t \log^2 t} dt \;=\; \int_2^\infty \frac{R(t)}{t \log^2 t}\, dt - \int_x^\infty \frac{R(t)}{t \log^2 t}\, dt.$$

Notice that $\int_2^\infty \frac{R(t)}{t \log^2 t}\, dt$ exists because $R(t) = O(1)$.

Further, since $R(t) = O(1)$, we have

$$\int_x^\infty \frac{R(t)}{t \log^2 t}\, dt \;=\; O\left(\int_x^\infty \frac{1}{t \log^2 t}\, dt\right) \;=\; O\left(\frac{1}{\log x}\right).$$

Continuing from (12):

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv n \pmod m}} \frac{1}{p} &= \frac{1}{\varphi(m)} + O\left(\frac{1}{\log x}\right) + \frac{\log\log x}{\varphi(m)} - \frac{\log\log 2}{\varphi(m)} \\ &\quad + \int_2^\infty \frac{R(t)}{t\log^2 t}\,dt - \int_x^\infty \frac{R(t)}{t\log^2 t}\,dt \\ &= \frac{\log\log x}{\varphi(m)} + B + O\left(\frac{1}{\log x}\right),\end{aligned}$$

where

$$B = \frac{1}{\varphi(m)} - \frac{\log\log 2}{\varphi(m)} + \int_2^\infty \frac{R(t)}{t\log^2 t}\,dt.$$

Therefore, we conclude that

$$\sum_{\substack{p\le x\\ p\equiv n \pmod m}} \frac{1}{p} = \frac{\log\log x}{\varphi(m)} + B + O\left(\frac{1}{\log x}\right).$$

□

2.2.2 Proof of Theorem 1.8 by Lemma 2.2

Recall the statement of Theorem 1.8.

Theorem 1.8 Fix a real number $k > -1$ and positive integers $m, n \in \mathbb{Z}_{>0}$ such that $\gcd(m, n) = 1$. Then

$$\int_1^\infty \frac{\pi_k(t; m, n) - \pi(t^{k+1}; m, n)}{t^{k+2}}\,dt = -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$

Further,

$$\begin{aligned}-\frac{\log(k+1)}{(k+1)\varphi(m)} &< 0 \qquad (k > 0)\\ -\frac{\log(k+1)}{(k+1)\varphi(m)} &> 0 \qquad (-1 < k < 0).\end{aligned}$$

Proof. By Lemma 2.2, if $x \ge 2$, we have

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv n \pmod m}} \frac{1}{p} &= \frac{\log\log x}{\varphi(m)} + B + O\left(\frac{1}{\log x}\right)\\ &= \frac{\log\log x}{\varphi(m)} + B + o(1),\end{aligned}$$

where B is a constant.

Let $u = t^{k+1}$. Then

$$\begin{aligned}\int_1^x \frac{\pi(t^{k+1};m,n)}{t^{k+2}}\,dt &= \frac{1}{k+1}\int_1^{x^{k+1}} \frac{\pi(u;m,n)}{u^2}\,du \\ &= \frac{-1}{k+1}\frac{\pi(x^{k+1};m,n)}{x^{k+1}} + \frac{1}{k+1}\int_1^{x^{k+1}} \frac{1}{u}\,d\pi(u;m,n) \\ &= \frac{-1}{k+1}\frac{\pi(x^{k+1};m,n)}{x^{k+1}} + \frac{1}{k+1}\sum_{\substack{p\le x^{k+1} \\ p\equiv n \pmod m}} \frac{1}{p} \\ &= \frac{\log\log(x^{k+1})}{(k+1)\varphi(m)} + \frac{B}{k+1} + o(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_1^x \frac{\pi_k(t,m,n)}{t^{k+2}}\,dt &= \frac{-1}{k+1}\frac{\pi_k(x,m,n)}{x^{k+1}} + \frac{1}{k+1}\int_1^{x} \frac{1}{t^{k+1}}\,d\pi_k(t,m,n) \\ &= \frac{-1}{k+1}\frac{\pi_k(x,m,n)}{x^{k+1}} + \frac{1}{k+1}\sum_{\substack{p\le x \\ p\equiv n \pmod m}} \frac{1}{p^{k+1}}p^k \\ &= \frac{-1}{k+1}\frac{\pi_k(x,m,n)}{x^{k+1}} + \frac{1}{k+1}\sum_{\substack{p\le x \\ p\equiv n \pmod m}} \frac{1}{p} \\ &= \frac{\log\log x}{(k+1)\varphi(m)} + \frac{B}{k+1} + o(1).\end{aligned}$$

Thus,

$$\int_1^x \frac{\pi_k(t,m,n)-\pi(t^{k+1},m,n)}{t^{k+2}}\,dt = -\frac{\log(k+1)}{(k+1)\varphi(m)} + o(1),$$

which implies the theorem. □

3 Appendices

3.1 Appendix: Some Examples of Main Result

In this section, we present tables containing numerical examples that measure the accuracy of the approximations included in our main results. In particular, we are interested in demonstratig how closely $\pi_k(x^{1/(k+1)};m,n)$ approximates $\pi(x;m,n)$.

Utilizing High Performance Computing, [1] we examine the approximation for primes of the form $p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$, $p \equiv 1 \pmod 5$, and $p \equiv 3 \pmod 5$. For these modular cases, we select 9 positive integers between 10^4 and 10^8, and calculate

$$\text{Error } (x, k; m, n) \; := \; \frac{\pi(x; m, n) - \pi_k(x^{1/(k+1)}; m, n)}{\pi(x; m, n)}$$

for $k = 1, \; 1/2, \; -1/10, \; -1/12$.

3.1.1 Error for $k = 1$

We observe that of the 36 errors measured in Tables 1-4, 72.22% are positive.

x	$\pi(x; 4, 1)$	$\pi_1(x^{1/2}; 4, 1)$	Error %
1×10^4	609	515	15.43514%
5×10^4	2549	2025	20.55708%
1×10^5	4783	4418	7.63119%
5×10^5	20731	19668	5.12759%
1×10^6	39175	36628	6.50160%
5×10^6	174193	165373	5.06335%
1×10^7	332180	323048	2.74911%
5×10^7	1500452	1475230	1.68096%
1×10^8	2880504	2863281	0.59792%

Table 1: $\pi(x; 4, 1)$ and $\pi_1(x^{1/2}; 4, 1)$.

x	$\pi(x; 4, 3)$	$\pi_1(x^{1/2}; 4, 3)$	Error %
1×10^4	619	543	12.27787%
5×10^4	2583	2411	6.65892%
1×10^5	4808	4786	0.45757%
5×10^5	20806	20643	0.78343%
1×10^6	39322	39497	−0.44504%
5×10^6	174319	170667	2.09501%
1×10^7	332398	319819	3.78432%
5×10^7	1500681	1516507	−1.05459%
1×10^8	2880950	2873113	0.27203%

Table 2: $\pi(x; 4, 3)$ and $\pi_1(x^{1/2}; 4, 3)$.

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x	$\pi(x;5,1)$	$\pi_1(x^{1/2};5,1)$	Error %
1×10^4	306	215	29.73856%
5×10^4	1274	1181	7.29984%
1×10^5	2387	2536	-6.24214%
5×10^5	10386	10631	-2.35894%
1×10^6	19617	18470	5.84697%
5×10^6	87062	81830	6.00951%
1×10^7	166104	156148	5.99384%
5×10^7	750340	763457	-1.74814%
1×10^8	1440298	1448386	-0.56155%

Table 3: $\pi(x;5,1)$ and $\pi_1(x^{1/2};5,1)$.

x	$\pi(x;5,3)$	$\pi_1(x^{1/2};5,3)$	Error %
1×10^4	310	291	6.12903%
5×10^4	1290	1036	19.68992%
1×10^5	2402	2644	-10.07494%
5×10^5	10382	10539	-1.51223%
1×10^6	19665	18146	7.72438%
5×10^6	87216	90461	-3.72065%
1×10^7	166230	156456	5.87981%
5×10^7	750395	753820	-0.45643%
1×10^8	1440474	1436510	0.27519%

Table 4: $\pi(x;5,3)$ and $\pi_1(x^{1/2};5,3)$.

3.1.2 Error for $k=1/2$

We observe that of the 36 errors measured in Tables 5-8, 72.22% are positive.

x	$\pi(x;4,1)$	$\pi_{1/2}(x^{2/3};4,1)$	Error %
1×10^4	609	617.62512	-1.41628%
5×10^4	2549	2477.64505	2.79933%
1×10^5	4783	4659.83812	2.57499%
5×10^5	20731	20125.89212	2.91886%
1×10^6	39175	38904.00140	0.69176%
5×10^6	174193	173246.23939	0.54351%
1×10^7	332180	329252.45078	0.88131%
5×10^7	1500452	1492885.30185	0.50429%
1×10^8	2880504	2873027.62482	0.25955%

Table 5: $\pi(x;4,1)$ and $\pi_{1/2}(x^{2/3};4,1)$.

x	$\pi(x;4,3)$	$\pi_{1/2}(x^{2/3};4,3)$	Error %
1×10^4	619	591.60159	4.42624%
5×10^4	2583	2502.18366	3.12878%
1×10^5	4808	4833.35209	−0.52729%
5×10^5	20806	20951.89316	−0.70121%
1×10^6	39322	39178.87051	0.36399%
5×10^6	174319	173924.73741	0.22617%
1×10^7	332398	331806.98445	0.17780%
5×10^7	1500681	1502046.79913	−0.09101%
1×10^8	2880950	2879155.53993	0.06229%

Table 6: $\pi(x;4,3)$ and $\pi_{1/2}(x^{2/3};4,3)$.

x	$\pi(x;5,1)$	$\pi_{1/2}(x^{2/3};5,1)$	Error %
1×10^4	306	290.30286	5.12978%
5×10^4	1274	1243.85408	2.36624%
1×10^5	2387	2288.69057	4.11853%
5×10^5	10386	10309.63049	0.73531%
1×10^6	19617	19616.30635	0.00354%
5×10^6	87062	87036.61969	0.02915%
1×10^7	166104	164310.69864	1.07963%
5×10^7	750340	752249.09877	−0.25443%
1×10^8	1440298	1430300.15946	0.69415%

Table 7: $\pi(x;5,1)$ and $\pi_{1/2}(x^{2/3};5,1)$.

x	$\pi(x;5,3)$	$\pi_{1/2}(x^{2/3};5,3)$	Error %
1×10^4	310	321.30898	−3.64806%
5×10^4	1290	1244.44539	3.53136%
1×10^5	2402	2501.69252	−4.15040%
5×10^5	10382	10504.71338	−1.18198%
1×10^6	19665	19604.34768	0.30843%
5×10^6	87216	86272.37280	1.08194%
1×10^7	166230	167998.78804	−1.06406%
5×10^7	750395	748916.40906	0.19704%
1×10^8	1440474	1444351.68992	−0.26920%

Table 8: $\pi(x;5,3)$ and $\pi_{1/2}(x^{2/3};5,3)$.

3.1.3 Error for $k = -1/10$

We observe that of the 36 errors measured in Tables 9-12, 72.22% are negative.

x	$\pi(x;4,1)$	$\pi_{-1/10}(x^{10/9};4,1)$	Error %
1×10^4	609	613.50169	−0.73919%
5×10^4	2549	2562.89963	−0.54530%
1×10^5	4783	4788.03485	−0.10527%
5×10^5	20731	20771.18437	−0.19384%
1×10^6	39175	39266.51644	−0.23361%
5×10^6	174193	174232.64634	−0.02276%
1×10^7	332180	332314.25320	−0.04042%
5×10^7	1500452	1500545.39963	−0.00622%
1×10^8	2880504	2880813.47274	−0.01074%

Table 9: $\pi(x;4,1)$ and $\pi_{-1/10}(x^{10/9};4,1)$.

x	$\pi(x;4,3)$	$\pi_{-1/10}(x^{10/9};4,3)$	Error %
1×10^4	619	618.68563	0.05079%
5×10^4	2583	2579.32406	0.14231%
1×10^5	4808	4821.56790	−0.28219%
5×10^5	20806	20796.90807	0.04370%
1×10^6	39322	39305.82276	0.04114%
5×10^6	174319	174323.13838	−0.00237%
1×10^7	332398	332477.68215	−0.02397%
5×10^7	1500681	1500779.09018	−0.00654%
1×10^8	2880950	2881063.76609	−0.00395%

Table 10: $\pi(x;4,3)$ and $\pi_{-1/10}(x^{10/9};4,3)$.

x	$\pi(x;5,1)$	$\pi_{-1/10}(x^{10/9};5,1)$	Error %
1×10^4	306	306.84917	−0.27750%
5×10^4	1274	1284.72538	−0.84187%
1×10^5	2387	2397.35493	−0.43380%
5×10^5	10386	10381.05165	−0.04764%
1×10^6	19617	19624.30360	−0.03723%
5×10^6	87062	87119.41013	−0.06594%
1×10^7	166104	166161.85950	−0.03483%
5×10^7	750340	750274.35740	0.00875%
1×10^8	1440298	1440380.00374	−0.00569%

Table 11: $\pi(x;5,1)$ and $\pi_{-1/10}(x^{10/9};5,1)$.

x	$\pi(x;5,3)$	$\pi_{-1/10}(x^{10/9};5,3)$	Error %
1×10^4	310	309.65259	0.11207%
5×10^4	1290	1288.73787	0.09784%
1×10^5	2402	2406.10885	−0.17106%
5×10^5	10382	10404.21103	−0.21394%
1×10^6	19665	19658.64066	0.03234%
5×10^6	87216	87152.03182	0.07334%
1×10^7	166230	166229.30179	0.00042%
5×10^7	750395	750428.14712	−0.00442%
1×10^8	1440474	1440534.90175	−0.00423%

Table 12: $\pi(x;5,3)$ and $\pi_{-1/10}(x^{10/9};5,3)$.

3.1.4 Error for $k=-1/12$

We observe that of the 36 errors measured in Tables 13-16, 77.78% are negative.

x	$\pi(x;4,1)$	$\pi_{-1/12}(x^{12/11};4,1)$	Error %
1×10^4	609	611.17719	−0.35750%
5×10^4	2549	2558.04851	−0.35498%
1×10^5	4783	4787.40169	−0.09203%
5×10^5	20731	20762.97056	−0.15422%
1×10^6	39175	39253.59228	−0.20062%
5×10^6	174193	174211.69780	−0.01073%
1×10^7	332180	332319.22077	−0.04191%
5×10^7	1500452	1500497.92446	−0.00306%
1×10^8	2880504	2880766.12283	−0.00910%

Table 13: $\pi(x;4,1)$ and $\pi_{-1/12}(x^{12/11};4,1)$.

x	$\pi(x;4,3)$	$\pi_{-1/12}(x^{12/11};4,3)$	Error %
1×10^4	619	622.36367	−0.54340%
5×10^4	2583	2584.93696	−0.07499%
1×10^5	4808	4816.96821	−0.18653%
5×10^5	20806	20788.86077	0.08238%
1×10^6	39322	39306.76238	0.03875%
5×10^6	174319	174325.21822	−0.00357%
1×10^7	332398	332443.67182	−0.01374%
5×10^7	1500681	1500833.67573	−0.01017%
1×10^8	2880950	2881059.64847	−0.00381%

Table 14: $\pi(x;4,3)$ and $\pi_{-1/12}(x^{12/11};4,3)$.

x	$\pi(x;5,1)$	$\pi_{-1/12}(x^{12/11};5,1)$	Error %
1×10^4	306	308.32261	-0.75902%
5×10^4	1274	1286.24176	-0.96089%
1×10^5	2387	2392.89814	-0.24709%
5×10^5	10386	10367.25069	0.18052%
1×10^6	19617	19631.27471	-0.07277%
5×10^6	87062	87117.75683	-0.06404%
1×10^7	166104	166145.81965	-0.02518%
5×10^7	750340	750274.07185	0.00879%
1×10^8	1440298	1440333.42281	-0.00246%

Table 15: $\pi(x;5,1)$ and $\pi_{-1/12}(x^{12/11};5,1)$.

x	$\pi(x;5,3)$	$\pi_{-1/12}(x^{12/11};5,3)$	Error %
1×10^4	310	310.43590	-0.14061%
5×10^4	1290	1290.55374	-0.04293%
1×10^5	2402	2406.74513	-0.19755%
5×10^5	10382	10393.81954	-0.11385%
1×10^6	19665	19662.29214	0.01377%
5×10^6	87216	87159.52414	0.06475%
1×10^7	166230	166209.29782	0.01245%
5×10^7	750395	750355.12871	0.00531%
1×10^8	1440474	1440579.59405	-0.00733%

Table 16: $\pi(x;5,3)$ and $\pi_{-1/12}(x^{12/11};5,3)$.

3.1.5 Conclusion

Note that when $k > 0$, the error exhibits a positive trend, indicating that $\pi_(x^{k+1};m,n)$ tends to be larger $\pi_k(x;m,n)$. The opposite trend holds when $k < 0$. The explanation for this phenomenon is provided by Theorem 1.8 and Remark 1.9.

3.2 Appendix: Riemann–Stieltjes Integration

3.2.1 Riemann-Stieltjes Integral

Let f be a real-valued function of a real variable on $[a,b]$ where $a,b\in\mathbb{R}$, and let g also be a real-to-real function. Let $P=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a partition of $[a,b]$. Consider

$$S(f,g,P) := \sum_{i=1}^{n} f(x_i^*)[g(x_i)-g(x_{i-1})]$$

where $x_i^*\in[x_{i-1},x_i]$.

If $S(f,g,P)$ converges to a constant L when $\max_{1\le i\le n}\{x_i - x_{i-1}\}$ approaches 0, then we define the Riemann-Stieltjes Integral as follows:

$$\int_a^b f\,dg \;:=\; L.$$

The Riemann-Steltjes integral exists if g is of bounded variation and right-semicontinuous.

3.2.2 Applying Riemann-Stieltjes Integration

Consider

$$\pi(x;m,n) - \pi(x-1;m,n) \;=\; \begin{cases} 1, & \text{if } x \text{ is prime and } x \equiv n \pmod{m}, \\ \\ 0, & \begin{array}{c}\text{if } x \text{ is composite or} \\ x \text{ is prime where } x \not\equiv n \pmod{m}.\end{array} \end{cases}$$

Let f be a real-valued function of a real variable. Then

$$\sum_{\substack{p\le x \\ p \equiv n \pmod{m}}} f(p) \quad = \quad \sum_{t=2}^{x} f(t)[\pi(t;m,n) - \pi(t-1;m,n)].$$

By applying the Riemann-Stieltjes Integral, we have

$$\begin{aligned} \sum_{\substack{p\le x \\ p \equiv n \pmod{m}}} f(p) \quad &= \quad \sum_{t=2}^{x} f(t)[\pi(t;m,n) - \pi(t-1;m,n)] \\ &= \quad \int_2^x f(t)\,d\pi(t;m,n). \end{aligned} \tag{13}$$

This technique is crucial to evaluate the Riemann-Stieltjes Integral in the proofs of Theorem 1.7 and Theorem 1.8.

3.3 Appendix: Limit Computation

From Gerard-Washington [5, Lemma 1], we have the following result.

Let $k > -1$ and let f be a differentiable function on $[2,\infty)$ such that

i. $x^{k+1}\exp(-f(x)) \to \infty$ and

ii. $xf'(x) \to 0$

as $x \to \infty$. Then $\int_2^x t^k \exp(-f(t))\,dt \;=\; O\left(x^{k+1}\exp(-f(x))\right)$.

We apply this result with $f(x) = \frac{1}{2}\alpha\sqrt{\log x}$, where α is a positive constant.

We verified the hypotheses in our case by computing the relevant limits for applying the lemma.

i. We will find

$$L := \lim_{x\to\infty}\left[x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right].$$

Let

$$x = \exp\left(\frac{4y^2}{\alpha^2}\right).$$

When $y \to \infty$, $x \to \infty$. Thus,

$$\begin{aligned} L &= \lim_{y\to\infty}\left[\left[\exp\left(\frac{4y^2}{\alpha^2}\right)\right]^{k+1}\exp\left(-\frac{1}{2}\alpha\cdot\frac{2y}{\alpha}\right)\right] \\ &= \lim_{y\to\infty}\left[\exp\left((k+1)\cdot\frac{4y^2}{\alpha^2}\right)\exp(-y)\right] \\ &= \lim_{y\to\infty}\left[\exp\left(\frac{4k+4}{\alpha^2}\cdot y^2\right)\exp(-y)\right]. \end{aligned}$$

Since $k > -1$, we have $k + 1 > 0 \Longrightarrow 4k + 4 > 0$. Hence,

$$\frac{4k+4}{\alpha^2} > 0,$$

which indicates that

$$y \lll \frac{4k+4}{\alpha^2}\cdot y^2$$

when y approaches ∞.

Thus, $\exp\left(\frac{4k+4}{\alpha^2}\cdot y^2\right)$ grows much faster than $\exp(-y)$ decays, so we conclude that

$$L = \lim_{y\to\infty}\left[\exp\left(\frac{4k+4}{\alpha^2}\cdot y^2\right)\exp(-y)\right] = \infty.$$

Therefore,

$$\lim_{x\to\infty}\left[x^{k+1}\exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right] = \infty.$$

ii. Recall that $f(x) = \frac{1}{2}\alpha\sqrt{\log x}$. Thus

$$\begin{aligned} f'(x) &= \frac{\alpha}{4x\sqrt{\log x}} \\ \Longrightarrow xf'(x) &= \frac{\alpha}{4\sqrt{\log x}} \end{aligned}$$

Since $\lim_{x\to\infty}\sqrt{\log x} = \infty$, it follows that

$$\lim_{x\to\infty}[xf'(x)] = 0.$$

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References

[1] T. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] C. Axler, On the sum of the first n prime numbers, *J. Théor. Nombres Bordeaux*, **31**, (2019), 293–311. Available online at the URL: https://jtnb.centre-mersenne.org/item/?id=JTNB_2019__31_2_293_0

[3] P.L. Chebyshev, Lettre de M. le professeur Tchébyshev à M. Fuss, sur un nouveau théoreme rélatif aux nombres premiers contenus dans la formes $4n \pm 1$ et $4n \pm 3$, *Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg*, **11** (1853), 208.

[4] H. Davenport, *Multiplicative Number Theory*, Second Edition, Graduate Texts in Mathematics, Springer, New York, 1980.

[5] J. Gerard, L.C. Washington, Sums of powers of primes, *Ramanujan J.* **45**, (2018), 171–180. Available online at the URL: https://link.springer.com/article/10.1007/s11139-016-9847-4

[6] N.M. Korobov, Estimates of trigonometric sums and their applications, *Uspehi Mat. Nauk*, **13**, 1958, 185–192. Available online at the URL: https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=7458&option_lang=eng

[7] A. Kumchev, *Distribution of Prime Numbers*, (2005). Available online at the URL: http://tigerweb.towson.edu/akumchev/distributionofprimesnotes.pdf.

[8] W.J. LeVeque, *Topıcs in Number Theory II*, Addison-Wesley Publishing Company, 1961.

[9] J. Massias, G. Robin, Bornes effectives pour certaines fonctions concernant les nombres premiers, *J. Théor. Nombres Bordeaux*, **8**, (1996), 213–238. Available online at the URL: https://jtnb.centre-mersenne.org/item/?id=JTNB_1996__8_1_215_0

[10] A. Page, On the number of primes in an arithmetic progression, *Proc. London Math. Soc. (2)*, **39**, (1935), 116–141. Available online at the URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s2-39.1.116

[11] M. Rubinstein, P. Sarnak, Chebyshev's bias, *Experiment. Math.*, **3**, (1994), 173–197. Available online at the URL: https://projecteuclid.org/journals/experimental-mathematics/volume-3/issue-3/Chebyshevs-bias/em/1048515870.full

[12] A. Selberg, An elementary proof of the prime number theorem, *Ann. Math. (2)*, **50**, (1949), 305–313. Available online at the URL: https://www.jstor.org/stable/1969455?origin=crossref

[13] C.L. Siegel, Über die Classenzahl quadratischer Zahlkörper, *Acta arith.*, **1**, (1935), 83–86. Available online at the URL: https://www.impan.pl/pl/wydawnictwa/czasopisma-i-serie-wydawnicze/acta-arithmetica/all/1/1/93175/uber-die-classenzahl-quadratischer-zahlkorper

[14] M. Szalay, On the maximal order in S_n and S_n^*, *Acta Arith.*, **37**, (1980), 321–331. Available online at the URL: https://www.impan.pl/pl/wydawnictwa/czasopisma-i-serie-wydawnicze/acta-arithmetica/all/37/0/102639/on-the-maximal-order-in-s-n-and-s-n

[15] I.M. Vinogradov, A new estimate for the function $\zeta(1+it)$, *Izv. Akad. Nauk SSSR Ser. Mat.*, **22**, (1958), 161–164. Available online at the URL: https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=im&paperid=3962&option_lang=eng

[16] C.J. de la Vallée-Poussin, Recherches analytiques la théorie des nombres premiers, *Annales de la Société scientifique de Bruxelles*, **20**, (1896), 183–265.

[17] L.C. Washington, Sums of powers of primes II, available online at the URL: `https://arxiv.org/abs/2209.12845v1`.

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