An Exploration of Very Triangular Numbers

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Abstract - We present a collection of results concerning the location and distribution of very triangular numbers among triangular numbers, including the twin very triangular number theorem, the existence of arbitrarily long gaps between – and an analog of Bertrand's postulate for – very triangular numbers.

Keywords : triangular numbers; very triangular numbers; Hamming weight

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1 Introduction

Our investigation was prompted by an essay in [6] considering *very triangular numbers*, which are triangular numbers whose Hamming weight[1] is also a triangular number. While the formula $t_n = \frac{n(n+1)}{2}$ for the nth triangular number is known, to the best of our knowledge no explicit formula is known for the nth very triangular number. The lack of such a formula opens the way to investigations such as ours, concerning the distribution of very triangular numbers among the triangular numbers. We point out that the study of very triangular numbers is a special case of a general task: given a (polynomial) function p, characterize all positive integers of the form $p(n)$, whose digit sum in base b belongs to a pre-defined subset of $\mathbb{N}$. Other special cases of this problem include finding all squares whose digit sum is also a square (see [8]), and finding all $n \in \mathbb{N}$ whose digit sum is a square (see [7]).

A recurring theme in the present work is finding appropriate values of n so that the binary representation of t_n can be handled swiftly. Naturally, various (combinations of) powers of 2 surface as 'appropriate', making the theoretical arguments easier but the numerical work more and more memory expensive. Nonetheless, elementary programming in Mathematica™ provided a wealth of suggestive patterns and led to the formulation of many of our results.

After (re-)establishing the infinitude of very triangular numbers, we consider arithmetic progressions and related results including the twin very triangular number theorem, and address the existence of strings of consecutive triangular numbers that either must, or do not contain a very triangular number. The paper concludes with some open problems and potential further avenues of investigation.

[1] The Hamming weight of a positive integer is simply the binary digit-sum of the number.

In what follows we shall use the the following notational conventions: T denotes the set of triangular numbers; VT denotes the set of very triangular numbers, and $|(n)_2^1|$ denotes the Hamming weight of the positive integer n.

2 The Infinitude of Very Triangular Numbers

In priming the reader for further inquiry, Tanton [6, p.217, Theorem] shows that there are infinitely many very triangular numbers, by noting that $|(t_{2^n+3})_2^1| = 6 \in T$ for all $n \in \mathbb{N}$. We generalize this observation in the next theorem.

Theorem 2.1 (Infinitude of VT-II) *Suppose that $2(\ell+1)$ is an even triangular number, and let $n > 2\ell - 1$. Then $t_{2^n+2^\ell-1} \in VT$. In particular, there are infinitely many very triangular numbers.*

Proof. Note that

$$\begin{aligned} \frac{(2^n + \sum_{i=0}^{\ell-1} 2^i)(2^n + 2^\ell)}{2} &= (2^n + \sum_{i=0}^{\ell-1} 2^i)(2^{n-1} + 2^{\ell-1}) \\ &= 2^{2n-1} + 2^{n+\ell-1} + \sum_{i=0}^{\ell-1} 2^{n-1+i} + \sum_{i=0}^{\ell-1} 2^{\ell-1+i}. \end{aligned}$$

The restriction on n ensures that each power of 2 appearing in the last expression is distinct ($2n-1 > n+\ell-1 > n+\ell-2 > \cdots > n-1 > 2\ell-2 > \cdots > \ell-1$). Since there are exactly $2(\ell+1)$-many distinct powers, we conclude that

$$\left|(t_{2^n+2^\ell-1})_2^1\right| = 2(\ell+1) \in T,$$

and consequently, $t_{2^n+2^\ell-1} \in VT$ as claimed. □

Theorem 2.1 shows that given any even triangular number k, there are infinitely many very triangular numbers with Hamming weight equal to k. Our next result shows that given *any* triangular number k, there are very triangular numbers with Hamming weight equal to k.

Theorem 2.2 (Infinitude of VT-III) *Let $k \in T$ be a triangular number. Then $t_{2^k-2^\ell} \in VT$ for all $0 \le \ell \le \lfloor \frac{k}{2} \rfloor$. Moreover, $|(t_{2^k-2^\ell})_2^1| = k$ for all $0 \le \ell \le \left\lfloor \frac{k}{2} \right\rfloor$.*

Proof. Let $k \in T$ be given. The cases $i = 0, 1$ are treated in Theorem 3.4. Suppose now that $2 \leq \ell \leq \left\lfloor \frac{k}{2} \right\rfloor$. We compute

$$\begin{aligned} t_{2^k-2^\ell} &= \frac{(2^k-2^\ell)(2^k-2^\ell+1)}{2} = \frac{1}{2}\left(2^{k+\ell+1}(2^{k-\ell-1}-1)+2^\ell(2^{k-\ell}-1)+2^{2\ell}\right) \\ &= \sum_{j=0}^{k-\ell-2} 2^{k+\ell+j} + \sum_{i=0}^{k-\ell-1} 2^{i+\ell-1} + 2^{2\ell-1} \\ &= \sum_{j=0}^{k-\ell-2} 2^{k+\ell+j} + \sum_{i=0}^{\ell-1} 2^{i+\ell-1} + \underbrace{\sum_{i=\ell}^{k-\ell-1} 2^{i+\ell-1} + 2^{2\ell-1}} \\ &= \sum_{j=0}^{k-\ell-2} 2^{k+\ell+j} + \sum_{i=0}^{\ell-1} 2^{i+\ell-1} + \qquad 2^{k-1}. \end{aligned}$$

Given the restriction $2 \leq \ell \leq \left\lfloor \frac{k}{2} \right\rfloor$ we see that all powers of 2 in the last expression are distinct. Since there are $(k-\ell-1)+\ell+1 = k$ many of these, we conclude that $|(t_{2^k-2^\ell})_2^1| = k$, as claimed. $\square$

One may wonder whether given an odd triangular number m, are there very triangular numbers with Hamming weight equal to m, and if so, how many? Setting $\ell = 0$ in Theorem 2.2 shows that $|(t_{2^k-1})_2^1| = (2k-2)-(k-1)+1 = k$, hence answering the first question in the positive. The answer to the second question – as presented in the following two propositions – is perhaps somewhat surprising. While there are infinitely many very triangular numbers $vt \in VT$ with $|(vt)_2^1| = 15, 21, 45, 55, \ldots$(all odd triangular numbers greater than 3), we are reasonably certain that there are only four very triangular numbers with Hamming weight equal to three.

Proposition 2.3 *Let $\ell > 1$ so that $2\ell + 1$ is a triangular number. There are infinitely many very triangular numbers $vt \in VT$ with $|(vt)_2^1| = 2\ell + 1$.*

Proof. Given ℓ as in the statement, consider the triangular numbers $t_{2^{2\ell}-2^\ell+1}$. We compute

$$\begin{aligned} t_{2^{2\ell}-2^\ell+1} &= \frac{1}{2}\left(2^{2\ell}-2^\ell+1\right)\left(2^{2\ell}-2^\ell+2\right) \\ &= 2^{4\ell-1}-2^{3\ell}+2^{2\ell+1}-2^\ell-2^{\ell-1}+1 \\ &= \sum_{i=0}^{\ell-2} 2^{3\ell+i} + \sum_{i=0}^{\ell} 2^{\ell+i} - 2^{\ell-1} + 1 \\ &= \sum_{i=0}^{\ell-2} 2^{3\ell+i} + \sum_{i=1}^{\ell} 2^{\ell+i} + 2^{\ell-1} + 1. \end{aligned}$$

Given the restriction on ℓ, the the powers of 2 in the above summands are all distinct, and there are $(\ell-1)+\ell+1+1 = 2\ell+1$ many of them. $\square$

We complete this section with a discussion of triangular numbers with Hamming weight equal to 3. Theorem 2.2 ascertains that $t_{2^3-1} = t_7$ and $t_{2^3-2} = t_6$ have Hamming weight equal to three, but Proposition 2.3 (which would imply the infinitude of such) does not apply to the case $2\ell + 1 = 3$. As a curiosity, we mention here that the sequence A212192 (see [9]) in the Online Encyclopedia of Integer Sequences® lists $21, 28, 276, 1540$ as the sequence of triangular numbers with Hamming weight equal to 3, with the comment that there are no more triangular numbers among the first one million positive integers with Hamming weight equal to 3. Whether or not there are infinitely many triangular numbers with Hamming weight equal to three appears to remain an unsolved problem. Our next result shows that $21, 28, 276$ and 1540 are the only triangular numbers with Hamming weight equal to three in a large (infinite) collection of triangular numbers.

Proposition 2.4 *The only very triangular numbers t_n with $|(n)_2^1| \leq 5$ and $|(t_n)_2^1| = 3$ are* $21, 28, 276$ *and* 1540*.*

Proof. The case of $|(n)_2^1| = 1$ trivially yields $|(t_n)_2^1| = 2 \neq 3$. The cases $|(n)_2^1| = 2, 3, 4, 5$ are all handled by case analysis. Lest we fall victim of paralysis by (sub)-case analysis, we only present here the case $|(n)_2^1| = 3$, as this case is 'big enough' to be representative of the others, and to indicate the general method of reasoning; yet small enough not to cause the reader any undue discomfort. Suppose thus that $|(n)_2^1| = 3$, and write $n = 2^{\sigma(1)} + 2^{\sigma(2)} + 2^{\sigma(3)}$ for some increasing function $\sigma : \{1, 2, 3\} \hookrightarrow \mathbb{N} \cup \{0\}$. We will need to distinguish between the cases when $\sigma(1) = 0$ and when $\sigma(1) > 0$.
Consider first the case $\sigma(1) \geq 1$, and the corresponding diagram (see Figure 1) representing the summands in $n(n+1)$, with solid arrows in the diagram pointing towards known larger quantities, and dashed ones between quantities we cannot compare without further assumptions.

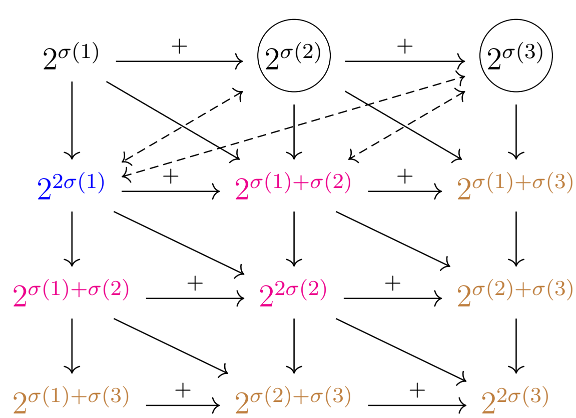

Figure 1: The case $|(n)_2^1| = 3$ and $\sigma(1) \geq 1$.

If $\sigma(2) > 2\sigma(1)$, then the sum looks like $2 + 2^{2\sigma(1)} + 2^{\sigma(2)} +$ terms whose powers are all greater than $\sigma(2)$. As such, we conclude that $|(n(n+1))_2^1| \geq 4$.

If $\sigma(2) = 2\sigma(1)$, then

$$n(n+1) = 2^{\sigma(1)} + 2^{2\sigma(1)+1} + 2^{3\sigma(1)+1} + 2^{4\sigma(1)} + \text{ terms with powers involving } \sigma(3). \quad (1)$$

We now consider the subcases $\sigma(1) = 1$ and $\sigma(1) > 1$. In the former, we obtain

$$n(n+1) = 2 + 2^3 + 2^5 + \text{terms with powers involving } \sigma(3).$$

The smallest of the remaining powers is $\sigma(3) \geq 3$. If $\sigma(3) = 4$ or $\sigma(3) \geq 6$, we trivially obtain that $|(n(n+1))_2^{\frac{1}{2}}| \geq 4$. If $\sigma(3) = 3$, we get $|(n(n+1))_2^{\frac{1}{2}}| = 4$, while if $\sigma(3) = 5$, we obtain $|(n(n+1))_2^{\frac{1}{2}}| = 6$. If on the other hand $\sigma(1) > 1$, then the four displayed powers of 2 in (1) are all distinct. We must therefore have $\sigma(3) = 2\sigma(1) + 1$, or $\sigma(3) = 3\sigma(1) + 1$, as all other cases immediately yield $|(n(n+1))_2^{\frac{1}{2}}| \geq 4$. Suppose first that $\sigma(3) = 2\sigma(1) + 1$. Then

$$n(n+1) = 2^{\sigma(1)} + 2^{2\sigma(1)+2} + 2^{3\sigma(1)+1} + 2^{3\sigma(1)+2} + 2^{4\sigma(1)+3}$$

with all terms distinct, since $\sigma(1) > 1$. Thus $|(n(n+1))_2^{\frac{1}{2}}| = 5$. Finally, if $\sigma(3) = 3\sigma(1)+1$, then

$$n(n+1) = 2^{\sigma(1)} + 2^{2\sigma(1)+1} + 2^{3\sigma(1)+2} + 2^{4\sigma(1)} + 2^{4\sigma(1)+2} + 2^{5\sigma(1)+2} + 2^{6\sigma(1)+2},$$

from which $|(n(n+1))_2^{\frac{1}{2}}| \geq 6$ readily follows.
We now consider the case $\sigma(1) = 0$. The corresponding diagram is given in Figure 2:

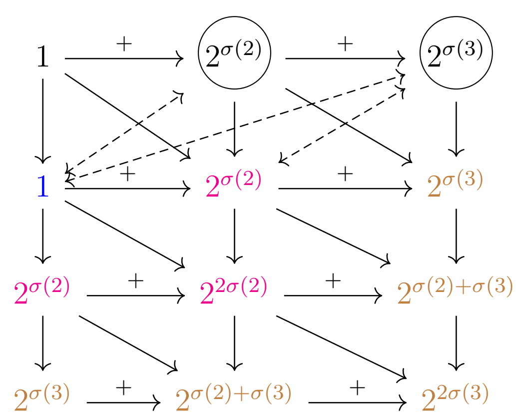

Figure 2: The case $|(n)_2^{\frac{1}{2}}| = 3$ and $\sigma(1) = 0$.

We see that in this case

$$(\dagger) \quad n(n+1) = 2 + 2^{\sigma(2)} + 2^{\sigma(2)+1} + 2^{2\sigma(2)} + \text{ terms with powers involving } \sigma(3).$$

The only way for the first four summands not all to be distinct is if $\sigma(2) + 1 = 2\sigma(2)$, or $\sigma(2) = 1$. We rewrite

$$n(n+1) = 2^2 + 2^3 + \text{ terms with powers involving } \sigma(3).$$

If $\sigma(3) > 3$, we obtain $|(n(n+1))_2^1| \geq 4$. If $\sigma(3) = 3$, we get $n(n+1) = 2^2 + 2^7$, hence $|(n(n+1))_2^1| = 2$. Finally, if $\sigma(3) = 2$, we arrive at $n(n+1) = 2^3 + 2^4 + 2^5$, corresponding to the very triangular number $t_n = \frac{1}{2}n(n+1) = 2^2 + 2^3 + 2^4 = 28$.
If the first four summands in (†) are distinct, we must have $\sigma(3) = \sigma(2) + 1$ if we want to reduce the number of summands. In this case however

$$n(n+1) = 2 + 2^{\sigma(2)} + 2^{\sigma(2)+3} + 2^{2\sigma(2)} + 2^{2\sigma(2)+3},$$

which means that $|(n(n+1))_2^1| \geq 4$ no matter what $\sigma(2)$ is. Since we have exhausted all the cases, we conclude that the only very triangular number t_n with $|(n)_2^1| = 3$ and $|(t_n)_2^1| = 3$ is $t_7 = 28$. For the sake of completeness we present (without proof) the remaining very triangular numbers with three 1s in their binary representation: $|(n)_2^1| = 2$ gives $t_6 = 21$; $|(n)_2^1| = 4$ gives $t_{23} = 276$, and $|(n)_2^1| = 5$ yields $t_{55} = 1540$. □

Based on the extensive numerical, and partial theoretical evidence, we are confident to make the following

Conjecture 2.5 If $n \in \mathbb{N}$ is such that $|(n)_2^1| \geq 6$, then $|(t_n)_2^1| \geq 4$.

3 The Twin Very Triangular Number Theorem and Arithmetic Progressions

We now turn our attention to the location and distribution of very triangular numbers among the triangular numbers. We begin by showing that there are infinitely many pairs of consecutive triangular numbers which are also very triangular. We provide two proofs: the first (Theorem 3.4) is what the reader would be led to when considering (perhaps using a computer algebra system) very triangular numbers which have special binary representations, while the second (Proposition 3.7) establishes – and then uses – the fact that the digit sum function $|(t_n)_2^1|$ is constant over well chosen sequences in its domain. The first proof requires two preliminary results.

Lemma 3.1 *Let $k \in \mathbb{N}$. Then $2^{2k-2} + 2^{2k-3} + \cdots + 2^{k-1}$ is a triangular number.*

Corollary 3.2 *Any number of the form $2^{2k-2}+2^{2k-3}+\cdots+2^{k-1}, k \in T$ is very triangular.*

Remark 3.3 (i) Corollary 3.2 provides an alternative proof of the Theorem on page 217 of [6] establishing the existence of infinitely many very triangular numbers.

(ii) We also see that there is at least one very triangular number with k many 1s in its binary representation for every $k \in T$.

(iii) Not every very triangular number is of this form. For example, $(21)_2 = 10101$.

Theorem 3.4 (The twin very triangular number theorem) *There are infinitely many integers $n \in \mathbb{N}$ such that $t_n, t_{n+1} \in VT$.*

Proof. Let $k \in T$, $k > 1$ be given. By Corollary 3.2

$$\underbrace{11\cdots1}_{k}\underbrace{00\cdots0}_{k-1} = \sum_{i=k-1}^{2k-2} 2^i \in VT.$$

Let $n+1 := \underbrace{11\cdots1}_{k} = \sum_{i=0}^{k-1} 2^i = 2^k - 1$. Then

$$t_{n+1} = \frac{(n+1)(n+2)}{2} = \frac{(2^k-1)2^k}{2} = \left(\sum_{i=0}^{k-1} 2^i\right) 2^{k-1} = \sum_{i=0}^{k-1} 2^{k-1+i} = \sum_{i=k-1}^{2k-2} 2^i.$$

Since $t_{n+1} = t_n + (n+1)$, we see that $t_{n+1} - (n+1) = (\underbrace{11\cdots1}_{k}\underbrace{00\cdots0}_{k-1}) - (\underbrace{11\cdots1}_{k}) \in T$. The reader will note that

$$\begin{aligned}
\underbrace{11\cdots1}_{k}\underbrace{00\cdots0}_{k-1} - \underbrace{11\cdots1}_{k} &= \sum_{i=k-1}^{2k-2} 2^i - \sum_{i=0}^{k-1} 2^i \\
&= \sum_{i=k-1}^{2k-2} 2^i - (2^k - 1) \\
&= \sum_{i=k+1}^{2k-2} 2^i + 2^{k-1} + 1,
\end{aligned}$$

and that

$$\left| \left(\sum_{i=k+1}^{2k-2} 2^i + 2^{k-1} + 1 \right)_2^1 \right| = (k-2) + 2 = k \in T.$$

We conclude that $t_n \in VT$. Since $t_{n+1} \in VT$ as well, the proof is complete. □

Remark 3.5 The following observations are in order:

(i) The proof of Theorem 3.4 shows that if $k \in T$, then $t_{2^k-2}, t_{2^k-1} \in VT$. Not all twin pairs of very triangular numbers are of this form, however, as demonstrated by the pair $(903, 946)$.

(ii) While the theorem exhibits twin very triangular numbers t_n, t_{n+1} with the property that $|(t_n)_2^1| = |(t_{n+1})_2^1| = k$ for some $k \in T$, this property does not hold in general for twin pairs. For instance, $|(t_{175})_2^1| = 10$, while $|(t_{176})_2^1| = 6$.

(ii) These pairs cannot be 'continued' into triples of consecutive very triangular numbers, since $t_{2^k} \notin VT$ for any $k \in \mathbb{N}$.

Theorem 3.4 raises the question of just how long strings of consecutive triangular numbers one can have that are also very triangular. The reader can easily check that $|(t_{581})_2^1| = |(t_{582})_2^1| = |(t_{583})_2^1| = 10$ and that$|(t_{1702})_2^1| = |(t_{1703})_2^1| = |(t_{1704})_2^1| = |(t_{1705})_2^1| = 10$ (there are many more such examples) which would indicate that perhaps one can find arbitrarily long strings of consecutive triangular numbers that are also very triangular.
Open question: Is it true that for any positive integer k, there exists an (or infinitely many) arithmetic progression $AP(k)$ of length k so that $t_j \in VT$ for all $j \in AP(k)$?

A first attempt to answer this open question could start with the following theorem of Szemerédi.

Theorem 3.6 (Szemerédi, [5]) *If $A \subset \mathbb{N}$ is such that*

$$\limsup_{N\to\infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N} > 0,$$

then for all $k \in \mathbb{N}$, A contains infinitely many arithmetic progressions of length k.

If $\pi_{VT}(x)$ and $\pi_T(x)$ denote the counting functions of very triangular and triangular numbers less than or equal to x, then

$$\frac{\pi_{VT}(t_N)}{\pi_T(t_N)} = \frac{|VT \cap \{t_1, t_2, \ldots, t_N\}|}{N} = \frac{|\sigma(\mathbb{N}) \cap \{1, 2, \ldots, N\}|}{N},$$

where $\{t_{\sigma(n)}\}_{n=1}^{\infty}$ is the sequence of very triangular numbers. Consequently, by applying Theorem 3.6 we see that if $\limsup_{N\to\infty} \frac{\pi_{VT}(t_N)}{\pi_T(t_N)} > 0$, then $\sigma(\mathbb{N})$ contains infinitely many arithmetic progressions of length k for every $k \in \mathbb{N}$. Alas, this approach cannot work, since $\limsup_{N\to\infty} \frac{\pi_{VT}(t_N)}{\pi_T(t_N)} = 0$ (see the discussion preceding Proposition 4.5 and Theorem 4.2). Nonetheless, the answer to the open question may still be in the positive, as is the case in the celebrated Green-Tao theorem (see [2]) regarding arithmetic progressions of primes.

While we can not answer the question as posed, following historical precedence[2], we can prove the existence of infinitely many arithmetic progressions $AP(k)$ of length six or less, such that $\{t_j\}_{j\in AP(k)} \subset VT$. In fact, we establish the existence of strings of consecutive very triangular numbers which are also consecutive triangular numbers. The only result we need in order to be able to demonstrate our claim is the following proposition.

Proposition 3.7 *Suppose that $n > 5$. Then for all $m \geq n$ and $0 \leq k < 2^{\lfloor (n-1)/2 \rfloor}$, the following equality holds:*

$$\left|(t_{2^m+3+k})_2^1\right| = \left|(t_{2^n+3+k})_2^1\right|.$$

Proof. Note first that for any n and $0 \leq k$,

$$t_{2^n+3+k} = 2^{2n-1} + 2^{n+1} + (k+1)2^n + 2^{n-1} + 4 + 2 + t_k + 3k.$$

[2]Before proving the full result, Szemerédi first established the existence of infinitely many arithmetic progressions with lengths $k = 4$ (see [4])

Given the restriction $k < \lfloor (n-1)/2 \rfloor$, we see that

$$t_k = \frac{k(k+1)}{2} < \frac{2^{\lfloor (n-1)/2 \rfloor}(2^{\lfloor (n-1)/2 \rfloor}+1)}{2} \leq 2^{n-2} + 2^{\lfloor (n-1)/2 \rfloor - 1} < 2^{n-2} + 2^{n-4},$$

and

$$3k < 3 \cdot 2^{\lfloor (n-1)/2 \rfloor} = 2^{\lfloor (n-1)/2 \rfloor + 1} + 2^{\lfloor (n-1)/2 \rfloor} < 2^{n-2} + 2^{n-3}.$$

It follows that $t_k + 3k < 2^{n-1}$, and hence

$$\begin{aligned}
\left|(t_{2^m+3+k})_2^1\right| &= \left|(2^{2n-1} + 2^{n+1} + (k+1)2^n + 2^{n-1})_2^1\right| + \left|(t_k + 3k + 4 + 2)_2^1\right| \\
&= \left|(2^{2m-1} + 2^{m+1} + (k+1)2^m + 2^{m-1})_2^1\right| + \left|(t_k + 3k + 4 + 2)_2^1\right| \\
&= \left|(t_{2^n+3+k})_2^1\right|.
\end{aligned}$$

The proof is complete. □

Corollary 3.8 *There are infinitely many $j \in \mathbb{N}$ such that $\{t_{j+\ell}\}_{\ell=0}^5 \subset VT$.*

Proof. In light of Proposition 3.7, it suffices to find an $n > 5$ and integers $0 \leq k < k+1 < k+2 < k+3 < k+4 < k+5 < 2^{\lfloor (n-1)/2 \rfloor}$ so that $\{t_{2^n+3+k+\ell}\}_{\ell=0}^5 \subset VT$. Using a CAS, we find that $n = 30$, and $k = 1870 < 1871 < 1872 < 1873 < 1874 < 1875 < 2^{\lfloor (30-1)/2 \rfloor} = 2^{14}$ is such a pair. In fact

$$\begin{aligned}
&\left|(t_{2^{30}+1873})_2^1\right| = \left|(t_{2^{30}+1874})_2^1\right| = \left|(t_{2^{30}+1875})_2^1\right| = \\
&\left|(t_{2^{30}+1876})_2^1\right| = \left|(t_{2^{30}+1877})_2^1\right| = \left|(t_{2^{30}+1878})_2^1\right| = 21.
\end{aligned}$$

□

We remark that the first instance of six consecutive triangular numbers that are all very triangular occurs much sooner: $\{t_{30301}, t_{30302}, t_{30303}, t_{30304}, t_{30305}, t_{30306}\} \subset VT$. Corollary 3.8 generalizes the twin very triangular number theorem, and could be viewed as the sextuplet very triangular number theorem, although we suspect that such a moniker is not likely to catch on.

4 Gaps and Consecutive Strings

The first results of this section finds ranges in the sequence of triangular numbers that must contain at least one very triangular number.

Theorem 4.1 (Bertrand's postulate for very triangular numbers)
For $n = 4, 5, 6$ and $n > 9$, there exists a very triangular number $m \in VT$ with $t_n < m < t_{2n}$.

Proof. Suppose first that $n > 9$. Then there exists $k \in \mathbb{N}$ such that either

(i) $n = 2^{k-1} + 1$,

(ii) $n = 2^{k-1} + 2$, or

(iii) $2^{k-1} + 2 < n \leq 2^k$.

It's easy to check that in cases (i) and (ii), $n < 2^{k-1} + 3 < 2n$, and consequently $t_n < t_{2^{k-1}+3} < t_{2n}$. Since $n > 9$, we see that $k > 4$, and by Theorem 2.1 $t_{2^{k-1}+3} \in VT$. In case (iii), we have $n \leq 2^k < 2^k + 3 < 2^k + 4 < 2n$, and once more we conclude that $t_n < t_{2^k+3} < t_{2n}$, with $t_{2^k+3} \in VT$. Table 1 completes the proof by checking the cases for $n = 4, 5, 6$.

Table 1: Bertrand's postulate for very triangular numbers (small n)

n	t_n	$m \in VT$	t_{2n}
4	10	21,28	36
5	15	21,28	55
6	21	28	78
7	28	N/A	105
8	26	N/A	136
9	45	N/A	171
10	55	190	210
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	210	231, 276, 378, 435, 630	741
$\vdots$	$\vdots$	$\vdots$	$\vdots$

□

The very triangular numbers also possess the property that is a natural dual to the one described in Theorem 4.1:

Theorem 4.2 (The gap theorem for very triangular numbers) *For any $k \in \mathbb{N}$ there exist consecutive very triangular numbers with at least k triangular numbers between them.*

We provide two proofs of Theorem 4.2. The first, non-constructive proof uses the following result of Drmota, Mauduit and Rivat (see [1] and the reference therein to [3] as well as the discussion of applicability to general quadratic $P(n) \in \mathbb{Z}[X]$), in which $s_q(n)$ denotes the sum of the digits of the non-negative integer n in its q-ary digital expansion.

Theorem 4.3 (Theorem E in [1]) *For any integers $q \geq 2$ and $m \geq 2$, and quadratic $P(n) \in \mathbb{Z}[X]$, there exists $\sigma_{q,m} > 0$ such that for any $a \in \mathbb{Z}$,*

$$\mathit{card}\{n \leq x : s_q(P(n)) \equiv a \mod m\} = \frac{x}{m} Q(a, D) + \mathcal{O}_{q,m}\left(x^{1-\sigma_{q,m}}\right),$$

where $D = \gcd(q-1, m)$ and

$$Q(a, D) = \mathit{card}\{0 \leq n < D \ : \ P(n) \equiv a \mod D\}.$$

Proposition 4.4 *Let $\pi_{VT}(x)$ and $\pi_T(x)$ denote the counting functions of very triangular and triangular numbers less than or equal to x. Then*

$$\limsup_{N\to\infty} \frac{\pi_{VT}(t_N)}{\pi_T(t_N)} = 0. \tag{2}$$

Proof. Let $\varepsilon > 0$ be given, and let $m = \prod^k p_j$ be the product of the first k odd primes, with k large enough so that $(2/3)^k < \varepsilon$. Since m is odd, we see that the number of residue classes represented by a triangular number is the same as the number of residue classes mod m represented by $8t_n + 1$, which by virtue of the identity $8t_n + 1 = (2n+1)^2$ is equal to the number of residue classes mod m represented by a square mod m. The reader may recall (or rediscover using an argument involving primitive roots) that for an odd prime p, there are $\frac{\phi(p)}{2} + 1 = \frac{p+1}{2}$ many squares mod p. Consequently, using the simple bound $\frac{p+1}{2} < \frac{2p}{3}$, $(p \geq 3)$ we conclude that there are fewer than

$$\prod_{j=1}^{k} \left(\frac{2p_j}{3}\right) = \left(\frac{2}{3}\right)^k m < \varepsilon m$$

residue classes mod m represented by triangular numbers. We now employ Theorem 4.3 with $q = 2, D = 1$ and $p(n) = t_n$ to conclude that

$$\begin{aligned} \frac{\pi_{VT}(t_N)}{\pi_T(t_N)} &= \frac{\text{card}\left\{n < N \ : \ |(t_n)_2^1| \in T\right\}}{N} \\ &\leq \frac{\text{card}\left\{n < N \ : \ |(t_n)_2^1| \equiv a \mod m \text{ for some } a \in T\right\}}{N} \\ &\leq \varepsilon \frac{m}{N}\frac{N}{m} + \mathcal{O}_{2,m}\left(N^{-\sigma_{2,m}}\right) < 2\varepsilon, \qquad (N \gg 1). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows. □

We are now in position to give the first argument vis-à-vis gaps between consecutive very triangular numbers.

Proof. [Proof 1 of the existence of arbitrarily long gaps between very triangular numbers] We proceed by contradiction. To this end, suppose that there is an upper bound $K \in \mathbb{N}$ on the size of gaps between consecutive very triangular numbers: given any $t_{\sigma(j)} \in VT$, there exists a very triangular number satisfying $t_{\sigma(j)} < t_{\sigma(\ell)} < t_{\sigma(j)+K}$. This would imply that

$$\limsup_{N\to\infty} \frac{\pi_{VT}(t_N)}{\pi_T(t_N)} \geq \lim_{N\to\infty} \frac{\lfloor \frac{N}{K} \rfloor}{N} = \frac{1}{K},$$

presenting a contradiction to Proposition 4.4. □

Our second – constructive – proof of the existence of the gaps is essentially the proof of Proposition 4.5. Since the proof consists of a (long) series of manipulation of sums and is not particularly illuminating, we delegate it to the Appendix, and content ourselves with merely stating the result here.

Proposition 4.5 *Suppose that $k \in T$ and that $4 \mid k$. If $2^k - 2^{\frac{k}{2}} < n \leq 2^k - 2^{\frac{k}{2}} + \frac{k}{4}$, then $t_n \notin VT$.*

5 Future Work

We invite the reader to carry out investigations similar to ours, either to make our understanding of very triangular numbers more complete, or to extend these results to other sets of integers. Some results analogous to those presented in this work are known about very square numbers (i.e., squares whose Hamming weight is also a square) but a treatment of all quadratic $p(n)$ (c.f. Introduction) would be a quite nice result.

Regarding Conjecture 2.5: using techniques similar to those presented in this paper we were able to show that if $|(n)_2^1| = 6$, then $|(t_n)_2^1| \geq 4$. We suspect however that this approach will not suffice in settling the conjecture for all $|(n)_2^1| \geq 7$.

Naturally, the open question posed in Section 3 should be settled, along with the related question of whether or not one can partition the set VT with finitely many arithmetic progressions, and finally, whether it is possible to find a closed formula for the nth very triangular number. Obtaining such a formula would answer many of the posed questions in one fell swoop.

Finally, as a generalization of the present work, we propose the study of subsets of the natural numbers $f : \mathbb{N} \hookrightarrow \mathbb{N}$ and $S \subset f(\mathbb{N})$ so that, using the notation of Theorem 4.3, $s_q(S) \subset f(\mathbb{N})$, and understanding how properties of S depend on the choice of a particular f.

Appendix

We now present the proof of Proposition 4.5

Proof. Let $n, k = 4\ell$ be as in the statement, and write $n = 2^k - 2^{\frac{k}{2}} + m$. Suppose first that $m = 2^p$ for some $0 \leq p \leq \lfloor \log_2 \ell \rfloor$. In this case t_n can be written as

$$\begin{aligned}
t_n &= \frac{1}{2}\left(\sum_{j=2\ell}^{4\ell-1} 2^j + 2^p\right)\left(\sum_{j=2\ell}^{4\ell-1} 2^j + 2^p + 1\right) \\
&= \frac{1}{2}\left(\left(\sum_{j=2\ell}^{4\ell-1} 2^j\right)^2 + \sum_{j=2\ell+p+1}^{4\ell+p} 2^j + \sum_{j=2\ell}^{4\ell-1} 2^j + 2^{2p} + 2^p\right) \\
&= \frac{1}{2}\left(\sum_{j=6\ell+1}^{8\ell-1} 2^j + 2^{4\ell} + \sum_{j=2\ell+p+1}^{4\ell+p} 2^j + \sum_{j=2\ell}^{4\ell-1} 2^j + 2^{2p} + 2^p\right) \\
&= \frac{1}{2}\left(\sum_{j=6\ell+1}^{8\ell-1} 2^j + 2^{4\ell+p+1} + \sum_{j=2\ell+p+1}^{4\ell-1} 2^j + \sum_{j=2\ell}^{4\ell-1} 2^j + 2^{2p} + 2^p\right) \\
&= \frac{1}{2}\left(\sum_{j=6\ell+1}^{8\ell-1} 2^j + 2^{4\ell+p+1} + \sum_{j=2\ell+p+2}^{4\ell} 2^j + \sum_{j=2\ell}^{2\ell+p} 2^j + 2^{2p} + 2^p\right).
\end{aligned}$$

Given that $0 \leq p \leq \lfloor \log_2 \ell \rfloor$, we see that

$$p \leq 2p < 2\ell \leq 2\ell + p < 2\ell + p + 2 < 4\ell < 4\ell + p + 1 < 6\ell + 1$$

as soon as $\ell \geq 2$, and hence

$$|(t_n)_2^1| = \begin{cases} 4\ell+1 = k+1 & \text{if } p=0 \\ 4\ell+2 = k+2 & \text{if } p \geq 1 \end{cases} \notin T \quad \text{for any} \quad k \in T \quad \text{with} \quad k \geq 3.$$

Finally, if $\ell = 1$, then $n = 13$ and $|(t_{13})_2^1| = 5 \notin T$.
Assume now that $n, k = 4\ell$ are as in the statement, and write $n = 2^k - 2^{\frac{k}{2}} + m$, with

$$m = \sum_{j=1}^{r} 2^{\sigma(j)} \quad \text{for some} \quad 2 \leq r \leq \lfloor \log_2 \ell \rfloor,$$

and

$$\sigma : \{1, 2, \dots, r\} \hookrightarrow \{0, 1, 2, \dots, \lfloor \log_2 \ell \rfloor\}.$$

A simple calculation shows that

$$t_n = \frac{1}{2}\left(2^{k/2} + \left(\sum_{j=k/2+1}^{k} 2^j + \sum_{j=1}^{r} 2^{\sigma(j)}\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right) + \sum_{j=3k/2+1}^{2k-1} 2^j\right).$$

The reader will note that

$$2 \leq \sum_{j=1}^{r} 2^{\sigma(j)} + 1 \leq 2^{\sigma(r)+1} \leq 2^{r+1} \leq 2^{\log_2(\ell)+1} \leq \frac{k}{2} < 2^{k/2},$$

and that

$$2\sigma(r) + 1 \leq 2r + 1 \leq 2\log_2(\ell) + 1 < 2\ell = \frac{k}{2}.$$

Consequently, the largest power of 2 in the expression

$$\left(\sum_{j=k/2+1}^{k} 2^j\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right)$$

is less than or equal to $3k/2$, whereas the largest power of 2 in the expression

$$\left(\sum_{j=1}^{r} 2^{\sigma(j)}\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right)$$

is less than $k/2$. Therefore,

$$\begin{aligned} |(t_n)_2^1| &= \left|\left(\left(\sum_{j=k/2+1}^{k} 2^j + \sum_{j=1}^{r} 2^{\sigma(j)}\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right)\right)_2^1\right| + \frac{k}{2} \\ &= \left|\left(\left(\sum_{j=k/2+1}^{k} 2^j\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right)\right)_2^1\right| \\ &+ \left|\left(\left(\sum_{j=1}^{r} 2^{\sigma(j)}\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1\right)\right)_2^1\right| + \frac{k}{2} \end{aligned} \tag{3}$$

We compute the first summand in (3). To this end, note that

$$
\begin{aligned}
&\left(\sum_{j=k/2+1}^{k} 2^j\right)\left(\sum_{j=1}^{r} 2^{\sigma(j)}+1\right) = \sum_{j=k/2+1}^{k} 2^j + \sum_{j=1}^{r}\left(2^{\sigma(j)+k+1} - 2^{\sigma(j)+k/2+1}\right)\\
&= \sum_{j=k/2+1}^{k} 2^j + \sum_{i=1}^{r} \sum_{j=\sigma(i)+k/2+1}^{k+\sigma(i)} 2^j\\
&= \left(\sum_{j=\sigma(r)+k/2+1}^{k+\sigma(r)} 2^j\right) + \sum_{j=k/2+1}^{k} 2^j + \sum_{i=1}^{r-1} \sum_{j=\sigma(i)+k/2+1}^{k+\sigma(i)} 2^j\\
&= 2^{k+1} + \sum_{j=k/2+1}^{\sigma(r)+k/2} 2^j + \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j\right) + \sum_{i=1}^{r-1} \sum_{j=\sigma(i)+k/2+1}^{k+\sigma(i)} 2^j\\
&= 2^{k+1} + 2^{k/2+\sigma(r)+1} + \sum_{j=k/2+1}^{\sigma(r-1)+k/2} 2^j + \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j + \sum_{j=\sigma(r-1)+k/2+2}^{k+\sigma(r-1)} 2^j\right)\\
&+ \sum_{i=1}^{r-2} \sum_{j=\sigma(i)+k/2+1}^{k+\sigma(i)} 2^j\\
&= 2^{k+1} + 2^{k/2+\sigma(r)+1} + 2^{k/2+\sigma(r-1)+1} + \sum_{j=k/2+1}^{\sigma(r-2)+k/2} 2^j\\
&+ \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j + \sum_{j=\sigma(r-1)+k/2+2}^{k+\sigma(r-1)} 2^j + \sum_{j=\sigma(r-2)+k/2+2}^{k+\sigma(r-2)} 2^j\right)\\
&+ \sum_{i=1}^{r-3} \sum_{j=\sigma(i)+k/2+1}^{k+\sigma(i)} 2^j\\
&= 2^{k+1} + 2^{k/2+\sigma(r)+1} + 2^{k/2+\sigma(r-1)+1} + \cdots + 2^{k/2+\sigma(2)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j\\
&+ \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j + \sum_{j=\sigma(r-1)+k/2+2}^{k+\sigma(r-1)} 2^j + \sum_{j=\sigma(r-2)+k/2+2}^{k+\sigma(r-2)} 2^j + \cdots + \sum_{j=\sigma(1)+k/2+2}^{k+\sigma(1)} 2^j\right)\\
&= 2^{k+\sigma(1)+1} + 2^{k/2+\sigma(r)+1} + 2^{k/2+\sigma(r-1)+1} + \cdots + 2^{k/2+\sigma(2)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j\\
&+ \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j + \cdots + \sum_{j=\sigma(3)+k/2+2}^{k+\sigma(3)} 2^j + \sum_{j=\sigma(2)+k/2+2}^{k+\sigma(2)} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k} 2^j\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{k+\sigma(2)+1} + 2^{k/2+\sigma(r)+1} + 2^{k/2+\sigma(r-1)+1} + \cdots + 2^{k/2+\sigma(2)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j \\
&+ \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r)} 2^j + \cdots + \sum_{j=\sigma(3)+k/2+2}^{k+\sigma(3)} 2^j + \sum_{j=\sigma(2)+k/2+2}^{k+\sigma(1)} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k} 2^j \right) \\
&= 2^{k+\sigma(r)+1} + 2^{k/2+\sigma(r)+1} + 2^{k/2+\sigma(r-1)+1} + \cdots + 2^{k/2+\sigma(2)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j \\
&+ \left(\sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r-1)} 2^j + \sum_{j=\sigma(r-1)+k/2+2}^{k+\sigma(r-2)} 2^j + \cdots + \sum_{j=\sigma(2)+k/2+2}^{k+\sigma(1)} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k} 2^j \right) \\
&= 2^{k+\sigma(r)+1} + \sum_{j=2}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j + \sum_{j=2}^{r} \left(\sum_{i=\sigma(j)+k/2+2}^{k+\sigma(j-1)} 2^i \right) + \sum_{j=\sigma(1)+k/2+2}^{k} 2^j \\
&= 2^{k+\sigma(r)+1} + \sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r-1)} 2^j + \left(\sum_{j=3}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=2}^{r-1} \left(\sum_{i=\sigma(j)+k/2+2}^{k+\sigma(j-1)} 2^i \right) \right) \\
&+ 2^{k/2+\sigma(2)+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k} 2^j \\
&= 2^{k+\sigma(r)+1} + \sum_{j=\sigma(r)+k/2+2}^{k+\sigma(r-1)} 2^j + \left(\sum_{j=3}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=2}^{r-1} \left(\sum_{i=\sigma(j)+k/2+2}^{k+\sigma(j-1)} 2^i \right) \right) \\
&+ 2^{k+1} + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k/2+\sigma(2)} 2^j \\
&= 2^{k+\sigma(r)+1} + 2^{k+\sigma(r-1)+1} + \sum_{j=\sigma(r)+k/2+2}^{k} 2^j \\
&+ \left(\sum_{j=3}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=2}^{r-1} \left(\sum_{i=\sigma(j)+k/2+2}^{k+\sigma(j-1)} 2^i \right) \right) + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j + \sum_{j=\sigma(1)+k/2+2}^{k/2+\sigma(2)} 2^j .
\end{aligned}
$$

Now,

$$\begin{aligned}
&\sum_{j=3}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=2}^{r-1} \left(\sum_{i=\sigma(j)+k/2+2}^{k+\sigma(j-1)} 2^i \right) \\
=& \sum_{j=3}^{r} 2^{k/2+\sigma(j)+1} + \sum_{j=3}^{r} \left(\sum_{i=\sigma(j-1)+k/2+2}^{k+\sigma(j-2)} 2^i \right) \\
=& \sum_{j=3}^{r} \left(2^{k+\sigma(j-2)+1} + \sum_{i=\sigma(j-1)+k/2+2}^{k/2+\sigma(j)} 2^i \right),
\end{aligned}$$

hence, we arrive at the expression

$$\sum_{j=1}^{r} 2^{k+\sigma(j)+1} + \sum_{j=\sigma(r)+k/2+2}^{k} 2^j + \sum_{j=2}^{r} \left(\sum_{i=\sigma(j-1)+k/2+2}^{k/2+\sigma(j)} 2^i \right) + \sum_{j=k/2+1}^{\sigma(1)+k/2} 2^j.$$

Therefore,

$$\begin{aligned}
&\left| \left(\left(\sum_{j=k/2+1}^{k} 2^j \right) \left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1 \right) \right)_2^1 \right| \\
=& \ r + (k - (\sigma(r) + k/2 + 2) + 1) + \sum_{j=2}^{r} (k/2 + \sigma(j) - (\sigma(j-1) + k/2 + 2) + 1) \\
+& \ (\sigma(1) + k/2 - (k/2 + 1) + 1 \\
=& \ r + k/2 - \sigma(r) - 1 + \sum_{j=2}^{r} (\sigma(j) - \sigma(j-1)) - (r-1) + \sigma(1) \\
=& \ k/2 - \sigma(r) + (\sigma(r) - \sigma(1)) + \sigma(1) = k/2.
\end{aligned}$$

We now turn our attention to the second summand in (3). Since

$$\sum_{j=1}^{r} 2^{\sigma(j)} + 1 \leq \sum_{j=0}^{\lfloor \log_2 \ell \rfloor} 2^j + 1 \leq 2^{\lfloor \log_2 \ell \rfloor + 1},$$

we see that

$$\left(\sum_{j=1}^{r} 2^{\sigma(j)} \right) \left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1 \right) \leq \sum_{j=1}^{r} 2^{\sigma(j) + \lfloor \log_2 \ell \rfloor + 1} < 2^{2(\lfloor \log_2 \ell \rfloor + 1)}.$$

Consequently,

$$\begin{aligned}
1 \ &\leq \left| \left(\left(\sum_{j=1}^{r} 2^{\sigma(j)} \right) \left(\sum_{j=1}^{r} 2^{\sigma(j)} + 1 \right) \right)_2^1 \right| \\
&\leq 2\lfloor \log_2 \ell \rfloor + 1 \leq 2(\log_2 k - \log_2 4) + 1 = 2\log_2 k - 3.
\end{aligned}$$

Recall that $k \in T$. If we write $k = \frac{s(s+1)}{2}$ for some $s \in \mathbb{N}$, then the smallest triangular number greater than k is $\frac{s(s+1)}{2} + (s+1)$. On the other hand,

$$\begin{aligned} 2\log_2 k - 3 &= 2\log_2\left(\frac{s(s+1)}{2}\right) - 3 = 2(\log_2 s + \log_2(s+1)) - 5 \\ &< 4\log_2(s+1) - 5 \overset{\star}{<} s+1, \end{aligned}$$

where the starred inequality is equivalent to $(s+1)^4 < 2^{s+6}$, which one easily establishes by induction. We conclude that if $t_n \in T$ with $n = 2^k - 2^{k/2} + m$ for $1 \leq m \leq k/4$, then

$$k = \frac{s(s+1)}{2} < |(t_n)_2^1| \leq \frac{k}{2} + \frac{k}{2} + 2\log_2 k - 3 < \frac{s(s+1)}{2} + (s+1),$$

and consequently $t_n \notin VT$. The proof is complete. □

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