Long-Term Opinion Distributions of an Opinion Formation Model with Averaging Behavior

M. Perez Cazarez and S. Reed*

Abstract - Sociophysics utilizes mathematical tools from physics to study social phenomena. In particular, interactions between individuals can impact emerging trends in the opinions of a population as a whole. This research focuses on the long-term opinion distributions of a population of individuals who can influence one another through pairwise interactions, where one individual modifies their opinion to be more in line with their neighbor. Interactions are governed by a social network, where friends on the network interact while strangers do not. We introduce a new model for opinion formation with averaging behavior where the opinion of an individual at any time t is an integer between $-k$ and k. For example, an opinion of k could indicate a heavily republican opinion and $-k$ a heavily democratic opinion. As the process evolves in time, interactions between neighbors x and y in the social network result in person x updating her opinion to be one step closer to the opinion of person y. We first consider the scenario where everyone in the social network is friends with everyone else. For $k = 1$ we compute explicitly the long-term opinion distribution via differential equations. For values of $k > 1$, we solve for the long-term distribution numerically using Runge-Kutta. We then compute the long-term opinion distribution via simulation, in the more general social network scenario.

Keywords : sociophysics ; Sznajd model; opinion dynamics

Mathematics Subject Classification (2020) **:** 91D25, 91D30, 60J20, 60K35

1 Introduction

Sociophysics is a research area that employs mathematical tools to study the behavior of human populations and how microscopic interactions between individual agents impacts the population on a macroscopic level. The name sociophyics arises from the notion that interacting individuals are analogous to colliding particles on different social networks. Several models have been developed to predict and understand different types interactions between individuals in the real world, including the work of [2], [5], [1].

The Sznajd Model (SM) was initially proposed by Katarzyna Sznajd-Weron and Jozef Sznajd, and was referred to as the "United we stand, divided we fall" (USDF) model. This model proposes that multiple individuals with a shared opinion can convince others to share the same opinion as the group. The USDF model was applied to a one-dimensional

*This work was supported by the Mathematics Summer 2022 Research Program (MSRP) at California State University, Fullerton

lattice, where two neighbors who shared the same opinion would convince their closest neighbors to share the same opinion. Since the publication of the USDF model, researchers such as [1], [3], [7], [8], [10] have been inspired to explore different modifications to the original model. These variations range from adding noise and considering the model in different network structures, to modifying the interaction process.

In her review article [6], Katarzyna Sznajd-Weron provides an overview of the SM and some of the work that has been inspired by the model over the past 20 years since it was developed. Dietrich Stauffer was the first to name the model as the SM and generalized the original one-dimensional USDF model into two dimensions. In [1], he compares the SM with Ising models by representing an opinion with either -1 or $+1$. Stauffer observed the model on a square lattice, where two neighbors convinced their six (eight) nearest neighbors to share the same opinion. Stauffer used this model to explain the distribution of votes in local elections in Brazil.

In their work, [2] argued that compromise should be considered when studying opinion dynamics. This motivated them to investigate a modification of the SM where an individual can change their opinion in a random, diffusive process. This modification captures the spontaneous changes in opinions an individual may have due to other factors, such as the media. [9] explored another modification of the SM that models interactions where every individual has an assigned level of persuasiveness and stubbornness. This model can be used to describe people in a social structure with a higher status that could result in a bias or individuals who never change their minds.

In this paper, we were inspired by [4] who explored a slight modification of the SM, where instead of there being only two opinions ($+1$ or -1), their model introduces a third opinion that accounts for an indecisive opinion $(+1, 0, -1)$. With an algorithm, they performed simulations to observe the interacting opinion dynamical system in the long run, using the "two against one" SM. On the complete graph, they used a system of differential equations that represented their model to study the equilibrium. After simulations, they discovered that when there are three opinions with equally distributed initial conditions, the system is at equilibrium and no change occurs. As soon as there is a change in the initial distribution of the opinions, the effects coincide with other equilibria from their system of differential equations.

In the real world, the sentiment of an individual on a particular topic (say political party preference) often falls on a spectrum. Furthermore, an individual often changes their opinion on a topic gradually over time. For example, it is highly unlikely that through a single interaction, someone who strongly leans republican would spontaneously convert their view to strongly lean democrat. The model introduced and studied in this paper takes these ideas in mind in order to better represent the real world. We describe our model in more detail in the next section.

2 Model Description

The model studied in this paper involves N interacting agents on a social network. The social network can be represented by a finite, connected graph $G = (V, E)$, where the vertex set V is the set of individuals, and the edge set E is the set of social connections. We assume that for all time t, G remains fixed in that no new individuals are added or removed and no social connections are lost or created throughout the life of the process. At each time $t \geq 0$, each individual $x \in V$ has an opinion taken from the set

$$K = \{-k, -k+1, \ldots, k-1, k\}.$$

We interpret the values of K to be representative of a range of opinions on a specific topic. Taking political preference as an example, we may interpret values very close to k to indicate that an individual leans strongly Republican and values close to $-k$ to indicate a strong Democrat preference. The larger the value of k, the more granular the set of opinions becomes.

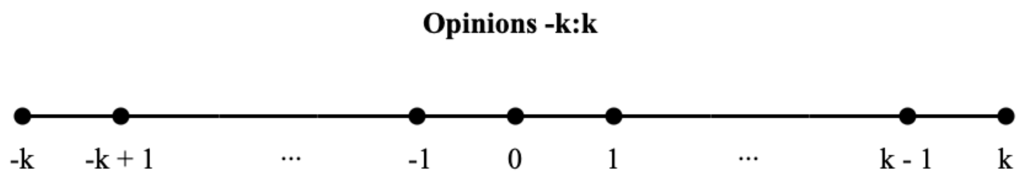

Figure 1: Set of possible opinions: $K = \{-k, \ldots, k\}$

The model under consideration in this paper can be studied from both a deterministic and stochastic point of view, each of which will be given treatment in the present paper. To describe the dynamics of the model, we first consider the idea that in general, most people are unlikely to change their opinion on any given topic drastically at a single point in time. In fact, it seems that most individuals that go through a transformation of sentiment on any given topic go through that transformation over an extended period of time often resulting from multiple exposures to ideologies different from their own. It is with this idea in mind we describe the dynamics that follow.

At the time that an interaction occurs, we choose an edge $(x, y) \in E$ at random. Individuals x and y are to be the two individuals that interact. With equal likelihood, one individual let's say x is to become the "persuader" and the other y is to become the "persuaded." Letting $o(x)$ and $o(y)$ be the pre-interaction opinions and $o(x)'$, $o(y)'$ be the post-interaction opinions of x and y respectively, the opinions of x and y are then updated accordingly:

$$o(x)' = o(x) \text{ and } o(y)' = o(y) + sign(o(x) - o(y))$$

In words, the dynamics above can be described as the "persuaded" updates their opinion to be one step closer to that of the "persuader," while the "persuader" remains fixed in their views.

Now that the procedure for a single interaction has been described, we consider next a few cases of our model:

(i) Process is deterministic, G is the complete graph.

(ii) Process is stochastic, G is the complete graph, at each time step every individual interacts with someone else by randomly pairing individuals.

(iii) Process is stochastic, G need not be complete, at each time step a single interaction occurs by selecting at random an edge $(x, y) \in E$.

2.1 Deterministic Case

The simplest version of the model to analyze is the one in which the evolution is completely deterministic. In this case we utilize ordinary differential equations as the mathematical vehicle for analysis. To make the analysis tenable, we make the assumption that the social network is the complete graph with N vertices as shown in Figure 2.1. This allows all individuals to interact with all other individuals.

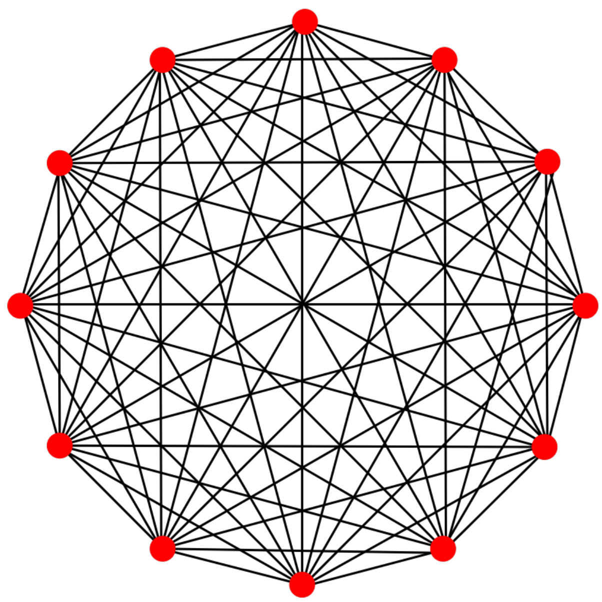

Figure 2: The complete graph with N vertices. Vertices represent individuals and edges between vertices represent social connections.

We define the function $P_i(t)$, $i \in K$ to be the fraction of individuals with opinion i at time t. Note that for all time $t \geq 0$, we must have that

$$0 \leq P_i(t) \leq 1 \quad \text{for all } i \in K \tag{1}$$

and

$$\sum_{i=-k}^{k} P_i(t) = 1. \tag{2}$$

Assuming that neighbors on the graph interact at rate 1, we can derive the system of $2k + 1$ differential equations given in Equation (3).

$$
\begin{cases}
\dfrac{dP_{-k}}{dt} = P_{-k}(P_{-k+1}) - P_{-k}(1 - P_{-k}) \\
\dfrac{dP_i}{dt} = P_{i-1}\left(\displaystyle\sum_{j=i}^{k} P_j\right) + P_{i+1}\left(\displaystyle\sum_{j=-k}^{i} P_j\right) - P_i(1 - P_i) \quad \text{for} - k < i < k \\
\dfrac{dP_k}{dt} = P_k(P_{k-1}) - P_k(1 - P_k)
\end{cases}
\tag{3}
$$

We are particularly interested in finding the long-term opinion distribution, namely we wish to find $\lim_{t\to\infty} P_i(t)$ for each $i \in K$. It will be shown in Section 3 that in the case where $k = 1$, the explicit solution to the system of ODEs in Equation (3) can be found and for $k > 1$, the equilibrium can be found.

2.2 Stochastic Case

We now consider a stochastic version of the model by utilizing discrete-time Markov chains as the analytical tool. In this case, we let $(X_t)_{t\geq 0}$ be a discrete-time Markov with state space S containing all functions of the form:

$$
f : V \to K, \tag{4}
$$

and use the notation $X_t(x) = i$ to indicate that at time t, individual $x \in V$ has opinion $i \in K$.

Under this stochastic scenario, we will analyze two different schemes for how interactions occur:

- Scheme 1: At each time step, every individual interacts with someone else. In this case, G must be the complete graph.
- Scheme 2: At each time step, one pair of neighbors are chosen at random to interact. In this case G need only to be finite and connected.

The evolution of scheme 1 is described below by Algorithm 1.

Algorithm 1

Set initial conditions for each individual.
for $t = 1, 2, \ldots$ **do**
  Randomly form $\frac{N}{2}$ pairs from the N individuals such that each individual is paired with someone else where $x_i \neq y_i$, $(i = 1, 2, \ldots, N)$ and
  for Each pair (x, y) **do**
    if $X_{t-1}(x) < X_{t-1}(y)$ **then**
      Set $X_t(y) = X_t(y) - 1$
    else if $X_{t-1}(x) > X_{t-1}(y)$ **then**
      Set $X_t(y) = X_t(y) + 1$
    else
      Leave opinions unchanged
    end if
  end for
end for

When randomly forming pairs for Algorithm 1, person x is the persuader, and person y is the persuaded therefore, $(x, y) \neq (y, x)$. Also note each individual has equal likelihood to be chosen as either person x or y with $P(x) = 0.5, P(y) = 0.5$.

The evolution of scheme 2 is described below by Algorithm 2. In particular, we will consider when G is the 1 and 2-dimensional lattice with reflective boundary conditions as shown in Figures 3 and 4.

Algorithm 2

Set initial conditions for each individual.
for $t = 1, 2, \ldots$ **do**
  Randomly select an edge $(x, y) \in E$
  if $X_{t-1}(x) < X_{t-1}(y)$ **then**
    Set $X_t(y) = X_t(y) - 1$
  else if $X_{t-1}(x) > X_{t-1}(y)$ **then**
    Set $X_t(y) = X_t(y) + 1$
  else
    Leave opinions unchanged
  end if
end for

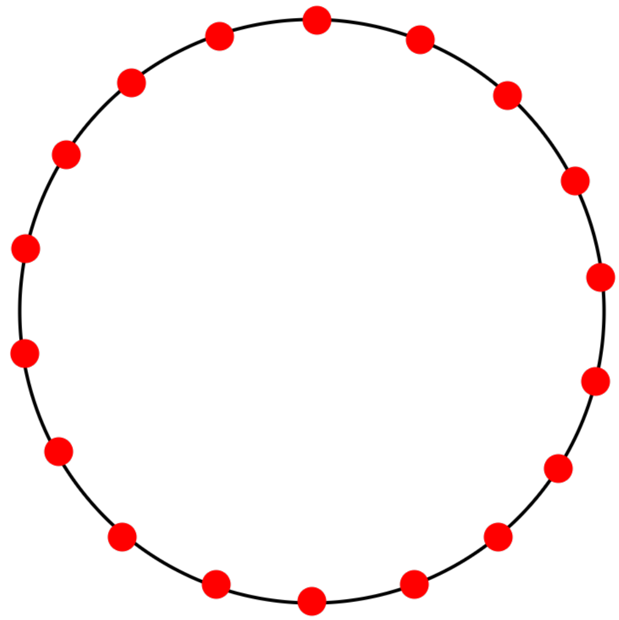

Figure 3: Each vertex represents a person, the lines represent the interactions that an individual may have with other individuals on a 1-Dimensional Lattice.

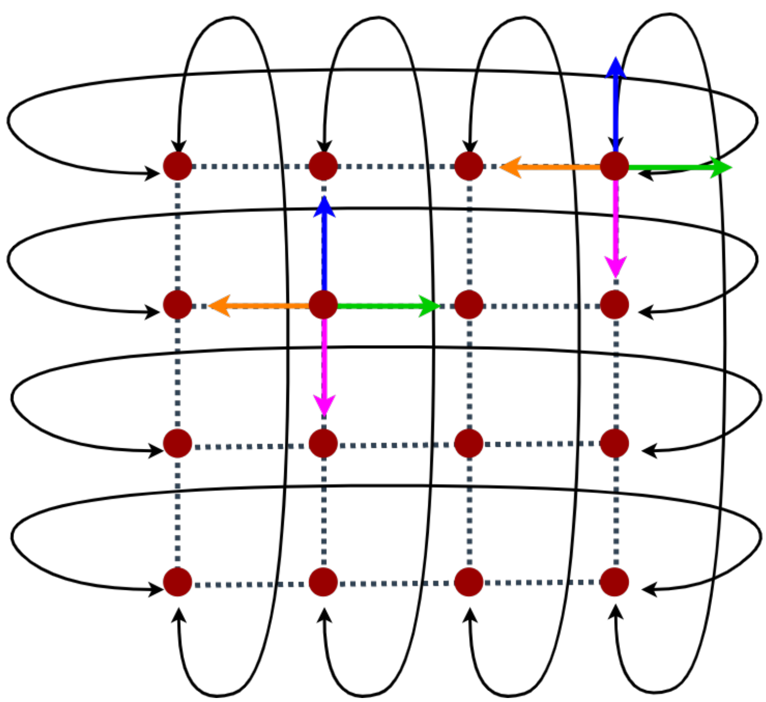

Figure 4: Each vertex represents a person, the lines represent the interactions that an individual may have with other individuals on a 2-Dimensional Lattice.

3 Main Results - Deterministic Case

As stated in Section 2, we are interested in the long-term opinion distribution $\lim_{t\to\infty} P_i(t)$ for each $i \in K$. We consider first the case where $k = 1$ and $K = \{-1, 0, 1\}$. When $k = 1$, Equation (3) reduces to

$$\begin{cases} \dfrac{dP_{-1}}{dt} = -P_{-1}P_1 \\ \dfrac{dP_0}{dt} = 2P_{-1}P_1 \\ \dfrac{dP_1}{dt} = -P_{-1}P_1 \end{cases} \tag{5}$$

which has solution given by

$$P_{-1}(t) = C_0 + \frac{C_0C_1}{e^{tC_0} - C_1}, \qquad P_0(t) = \frac{-2C_0C_1}{e^{tC_0} - C_1} + C_2, \qquad P_1(t) = \frac{C_0C_1}{e^{tC_0} - C_1} \tag{6}$$

where,

$$C_0 = P_{-1}(0) - P_1(0), \qquad C_1 = \frac{P_1(0)}{P_1(0) + C_0}, \qquad C_2 = P_0(0) + 2P_1(0) \tag{7}$$

if $P_{-1}(0) \neq P_1(0)$ and solution

$$P_{-1}(t) = \frac{1}{t + C_3}, \qquad P_0(t) = \frac{-2}{t + C_3} + 1, \qquad P_1(t) = \frac{1}{t + C_3} \tag{8}$$

where,

$$C_3 = \frac{1}{P_1(0)}, \tag{9}$$

when $P_{-1}(0) = P_1(0)$. In this case, we can easily find the long-term opinion distribution.

$$\lim_{t\to\infty} P_{-1}(t) = P_{-1}(0) - P_1(0), \quad \lim_{t\to\infty} P_0(t) = P_0(0) + 2P_1(0), \quad \lim_{t\to\infty} P_1(t) = 0$$

if $C_3 > 0$, and

$$\lim_{t\to\infty} P_{-1}(t) = 0, \quad \lim_{t\to\infty} P_0(t) = P_0(0) + 2P_{-1}(0), \quad \lim_{t\to\infty} P_1(t) = P_1(0) - P_{-1}(0)$$

if $C_3 < 0$,and

$$\lim_{t\to\infty} P_{-1}(t) = 0, \quad \lim_{t\to\infty} P_0(t) = 1, \quad \lim_{t\to\infty} P_1(t) = 0$$

if $P_{-1}(0) = P_1(0)$.

In words, the long-term behavior of the model is such that $P_0(t)$ continues to grow until $P_{-1}(t)$ or $P_1(t)$ or both $P_{-1}(t)$ and $P_1(t)$ go to zero, in which case an equilibrium is reached. In the specific case where $P_{-1}(0) = P_1(0)$, opinion 0 always becomes sole opinion at equilibrium.

When $k > 1$, it becomes very difficult to find an explicit solution to Equation (3) and so we use Runge-Kutta to find a numerical solution. Although we were not able to find the analytical solution to Equation (3), we were able to find the equilibrium values for the system.

In order to understand the long-term behavior of the model for $k > 1$, we use Runge-Kutta to numerically solve Equation (3) for several values of k with varying initial conditions as shown in Figure 5.

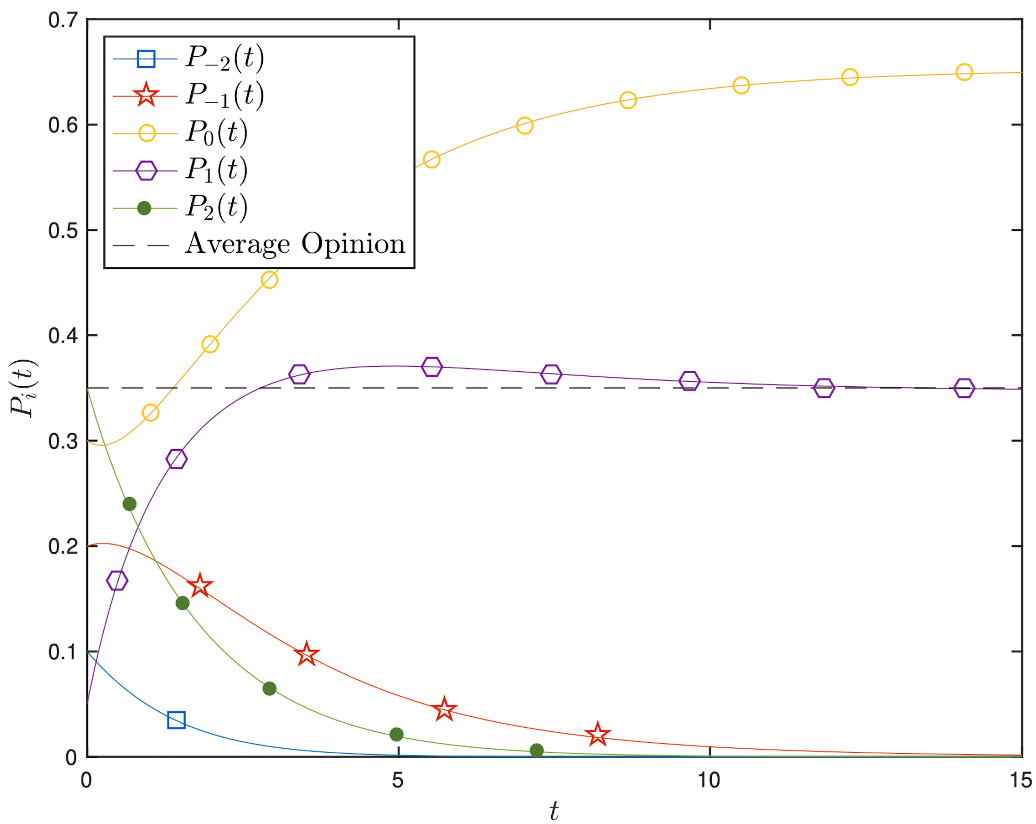

Figure 5: Numerical Runge-Kutta solution to Equation (3) with $k = 2$ and initial opinion distribution given by $[.1,\ .2,\ .3,\ .05,\ .35]$.

Upon visually inspecting the numerical solution for Equation (3), it appears as though the "average opinion" given by $\sum_{i=-k}^{k} iP_i(t)$ remains constant for all $t \geq 0$ regardless of k

or the initial conditions. This observation leads us to the following results.

Theorem 3.1 *There is some C, such that*

$$\sum_{i=-k}^{k} iP_i(t) = C$$

for all $t \geq 0$.

Proof. To prove the desired result, we will show that, $\frac{d}{dt}\left(\sum_{i=-k}^{k} iP_i(t)\right) = 0$. We have that

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{i=-k}^{k} iP_i(t)\right) &= \sum_{i=-k}^{k} i\frac{dP_i}{dt} \\
&= -k\left[P_{-k+1}P_{-k} - P_{-k}(1-P_{-k})\right] + k\left[P_{k-1}P_k - P_k(1-P_k)\right] \\
&\qquad + \sum_{i=-k+1}^{k-1} i\left[P_{i-1}\left(\sum_{j=i}^{k} P_j\right) + P_{i+1}\left(\sum_{j=-k}^{i} P_j\right) - P_i(1-P_i)\right] \\
&= -kP_{-k+1}P_{-k} + kP_{k-1}P_k + \sum_{i=-k+1}^{k-1} i\left[P_{i-1}\left(\sum_{j=i}^{k} P_j\right) + P_{i+1}\left(\sum_{j=-k}^{i} P_j\right)\right] \\
&\qquad - \sum_{i=-k}^{k} iP_i(1-P_i) \\
&= -kP_{-k+1}P_{-k} + \sum_{i=-k+1}^{k-1} iP_{i+1}\left(\sum_{j=-k}^{i} P_j\right) + kP_{k-1}P_k + \sum_{i=-k+1}^{k-1} iP_{i-1}\left(\sum_{j=i}^{k} P_j\right) \\
&\qquad - \sum_{i=-k}^{k} iP_i(1-P_i) \\
&= \sum_{i=-k}^{k-1}\left[(i+1)P_i\sum_{j=i+1}^{k} P_j\right] + \sum_{i=-k+1}^{k}\left[(i-1)P_i\sum_{j=-k}^{i-1} P_j\right] - \sum_{i=-k}^{k} iP_i(1-P_i) \\
&= \sum_{i=-k+1}^{k-1}\left[(i+1)P_i\sum_{j=i+1}^{k} P_j + (i-1)P_i\sum_{j=-k}^{i-1} P_j\right] + (-k+1)P_{-k}\sum_{j=-k+1}^{k} P_j \\
&\qquad + (k-1)P_k\sum_{j=-k}^{k-1} P_j - \sum_{i=-k}^{k} iP_i(1-P_i) \\
&= \sum_{i=-k+1}^{k-1} iP_i(1-P_i) + \sum_{i=-k+1}^{k-1} P_i\left(\sum_{j=i+1}^{k} P_j - \sum_{j=-k}^{i-1} P_j\right) + (-k)P_{-k}(1-P_{-k})
\end{aligned}$$

$$+ kP_k(1-P_k) + P_{-k}(1-P_{-k}) + P_k(1-P_k) - \sum_{i=-k}^{k} iP_i(1-P_i)$$

$$= \sum_{i=-k+1}^{k-1} P_i \left(\sum_{j=i+1}^{k} P_j - \sum_{j=-k}^{i-1} P_j \right) + P_{-k}(1-P_{-k}) + P_k(1-P_k)$$

$$= \sum_{i=-k+1}^{k-1} P_i \sum_{j=i+1}^{k} P_j - \sum_{i=-k+1}^{k-1} P_i \sum_{j=-k}^{i-1} P_j + P_{-k}(1-P_{-k}) + P_k(1-P_k)$$

$$= \sum_{j=-k+2}^{k} P_j \sum_{i=-k+1}^{j-1} P_i - \sum_{j=-k}^{k-2} P_j \sum_{i=j+1}^{k-1} P_i + P_{-k}(1-P_{-k}) + P_k(1-P_k)$$

$$= P_k \sum_{i=-k+1}^{k-1} P_i + \sum_{j=-k+2}^{k-1} P_j \sum_{i=-k+1}^{j-1} P_i - P_{-k} \sum_{i=-k+1}^{k-1} P_i$$

$$- \sum_{j=-k+1}^{k-2} P_j \sum_{i=j+1}^{k-1} P_i + P_{-k}(1-P_{-k}) - P_k(1-P_k)$$

$$= P_k(1-P_k-P_{-k}) + \sum_{j=-k+2}^{k-1} P_j \sum_{i=-k+1}^{j-1} P_i - P_{-k}(1-P_k-P_{-k})$$

$$- \sum_{j=-k+1}^{k-2} P_j \sum_{i=j+1}^{k-1} P_i + P_{-k}(1-P_{-k}) - P_k(1-P_k)$$

$$= P_k(1-P_k) - P_kP_{-k} + \sum_{j=-k+2}^{k-1} P_j \sum_{i=-k+1}^{j-1} P_i - P_{-k}(1-P_{-k}) + P_kP_{-k}$$

$$- \sum_{j=-k+1}^{k-2} P_j \sum_{i=j+1}^{k-1} P_i + P_{-k}(1-P_{-k}) - P_k(1-P_k)$$

$$= \sum_{j=-k+2}^{k-1} P_j \sum_{i=-k+1}^{j-1} P_i - \sum_{j=-k+1}^{k-2} P_j \sum_{i=j+1}^{k-1} P_i$$

$$= \sum_{j=-k+2}^{k-1} P_j \sum_{i=-k+1}^{j-1} P_i - \sum_{i=-k+2}^{k-1} P_i \sum_{j=-k+1}^{i-1} P_j = 0,$$

and the result is shown. □

Theorem 3.1 establishes that the average opinion $\sum_{i=-k}^{k} iP_i(t)$ remains constant for all $t \geq 0$. The next result given in Theorem 3.3 gives that at equilibrium, there are at most two neighboring opinions j, $j+1$ that become dominant in the long run. For ease of writing let us first introduce some notation that will be used throughout the rest of the paper. Define P_i^* as

$$P_i^* = \lim_{t\to\infty} P_i(t), \ i \in K.$$

The remainder of Section 3 shall be limited to the case where Conjecture 3.2 holds.

Conjecture 3.2 For all $k \geq 1$, and all $i \in K$, we have that P_i^* exists.

Proof. In support of Conjecture 3.2, we give the following inductive heuristic argument.

Base Case: As we have explicit solutions when $k = 1$ for $P_i(t)$, it is easily seen that Conjecture 3.2 holds for $k = 1$ by investigating Equations (8) and (9).

Inductive Step: Assume that 3.2 holds for $k = r$ where $r \geq 1$. Let $k = r + 1$. From Equation (3), $\frac{dP_k}{dt}$ can be written in the form

$$\frac{dP_k}{dt} = P_k(P_k + P_{k-1} - 1). \tag{10}$$

Equations (1) and (2) give that P_k is bounded and monotone, and so we must have that P_k^* exists. Specifically, at equilibrium, it must be the case that either

$$P_k^* + P_{k-1}^* = 1 \text{ or } P_k^* = 0.$$

Similarly, we must also have that

$$P_{-k}^* + P_{-k+1}^* = 1 \text{ or } P_{-k}^* = 0.$$

Case 1: ($P_k^* + P_{k-1}^* = 1$ or $P_{-k}^* + P_{-k+1}^* = 1$) In this case, Equations (1) and (2) imply that $P_i^* = 0$ for all $i \neq k-1, k$ or $i \neq -k+1, -k$ respectively and thus P_i^* exists.

Case 2: ($P_k^* = P_{-k}^* = 0$) In this case, when t is very large, we end up with a system of $2r + 1$ differential equations satisfying Equation (3). Our inductive hypothesis suggests that P_i^* exists for all $-r < i < r$. □

Theorem 3.3 *Let* $j = \min(i \in K | P_i^* > 0)$. *Then,* $P_j^* + P_{j+1}^* = 1$ *and* $P_i^* = 0$ *for all* $i \neq j, j+1$.

Proof. Let $j = \min(i \in K | P_i^* > 0)$. By definition, P_i^* is the value of $P_i(t)$ at equilibrium and so we must have that

$$\frac{dP_i^*}{dt} = 0 \quad \text{for all } i \in K.$$

Combining this with Equation (3) gives that

$$\begin{aligned}\frac{dP_j^*}{dt} &= P_{j-1}^* \left(\sum_{i=j}^{k} P_i^* \right) + P_{j+1}^* \left(\sum_{i=-k}^{j} P_i^* \right) - P_j^*(1 - P_j^*) \\ &= P_{j+1}^* \left(\sum_{i=-k}^{j} P_i^* \right) - P_j^*(1 - P_j^*)\end{aligned}$$

$$
\begin{aligned}
&= P_{j+1}^* \left(P_j^* + \sum_{i=-k}^{j-1} P_i^* \right) - P_j^*(1 - P_j^*) \\
&= P_{j+1}^* P_j^* - P_j^*(1 - P_j^*) \\
&= P_j^*(P_j^* + P_{j+1}^* - 1) \\
&= 0
\end{aligned}
$$

The above equation implies that $P_j^* = 0$ or $P_j^* + P_{j+1}^* = 1$. We have already assumed that $P_j^* > 0$ and so it must be the case that $P_j^* + P_{j+1}^* = 1$ and the result is shown. □

Our next result makes use of Theorems 3.1 and 3.3 to find explicit values for P_i^* for all $i \in K$.

Theorem 3.4 *Let* $C = \sum_{i=-k}^{k} iP_i(0)$ *and let* $j = \min(i \in K | P_i^* > 0)$ *then,*

$$j \leq C \leq j + 1,$$

and

$$P_j^* = 1 - C + j, \;\; P_{j+1}^* = C - j \quad \text{and} \quad P_i^* = 0 \quad \text{for all} \quad i \neq j, j+1.$$

Proof. Let $C = \sum_{i=-k}^{k} iP_i(0)$ and let $j = \min(i \in K | P_i^* > 0)$ Then Theorem 3.3 gives that $P_j^* + P_{j+1}^* = 1$ and $P_i^* = 0$ for all $i \neq j, j+1$. Combining this with Theorem 3.1 gives that

$$
\begin{aligned}
C &= \sum_{i=-k}^{k} iP_i(0) \\
&= \sum_{i=-k}^{k} iP_i(t) \;\; \text{for all} \;\; t \geq 0 \\
&= \sum_{i=-k}^{k} iP_i^* \\
&= jP_j^* + (j+1)P_{j+1} \\
&= j + P_{j+1},
\end{aligned}
$$

and so we must have that $P_j^* = 1 - C + j$ and $P_{j+1}^* = C - j$. Furthermore, we must have that for any δ satifying $0 \leq \delta \leq 1$,

$$j \leq j\delta + (j+1)(1-\delta) \leq j + 1.$$

And so we also have that $j \leq C \leq j + 1$, and the result is shown. □

Theorem 3.4 establishes not only an expression for the equilibria but also that if $P_j^* + P_{j+1}^* = 1$, then $j \leq C \leq j + 1$ because the average opinion remains constant for all time.

4 Main Results - Stochastic Case

In the present section we present the main results in the case where the model is stochastic as described in Subsection 2.2. Recalling the definition of $(X_t)_{t\geq 0}$ as a discrete-time Markov chain with finite state space, we can conclude that there is not a unique stationary distribution because the process $(X_t)_{t\geq 0}$ is not irreducible. In fact, there are many absorbing states for the process of the form:

$$f_i : V \to \{i\} \text{ where } i \in K.$$

Reaching state f_i at some time T would indicate that $X_t(x) = i$ for all $x \in V$ and $t \geq T$. We can therefore conclude that regardless of X_0, for each realization of the process $(X_t)_{t\geq 0}$, there will be some $i \in K$ and $T < \infty$ such that $X_t = f_i$ for all $t \geq T$. In words, it is guaranteed that a consensus will be reached among the population in a finite amount of time. This behavior differs greatly from what we see in the deterministic version of the model where it is possible for two adjacent opinions to dominate in the long run.

Knowing that in the stochastic case, one opinion will eventually take over, we are interested three things:

1. Given X_0, what is $\lim_{t\to\infty} P(X_t = f_i)$ for all $i \in K$?
2. Does the structure of the social network have an effect on the above probability?
3. Does the number of interactions per time-step have an impact on the above probability?

Because of the stochastic nature of the model and the difficulty of studying this model analytically, we will use Monte-Carlo simulation to glean insight into the behavior of the stochastic model.

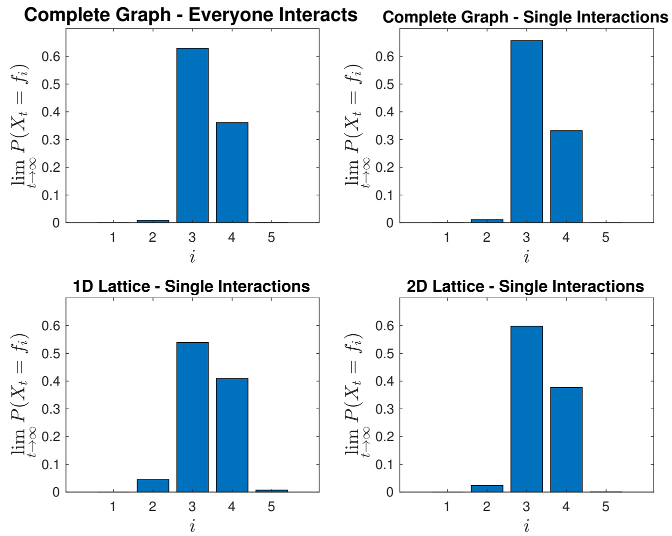

Figure 6: In each of the four subplots, $N = 100$, $K = \{-2,\ -1,\ 0,\ 1,\ 2\}$, initial distribution of opinions is [.1 .2 .3 .05 .35] and the number of simulations is 1000. The simulations resulted in a variance of .2489, .2402, .3395, and .2793 respectively.

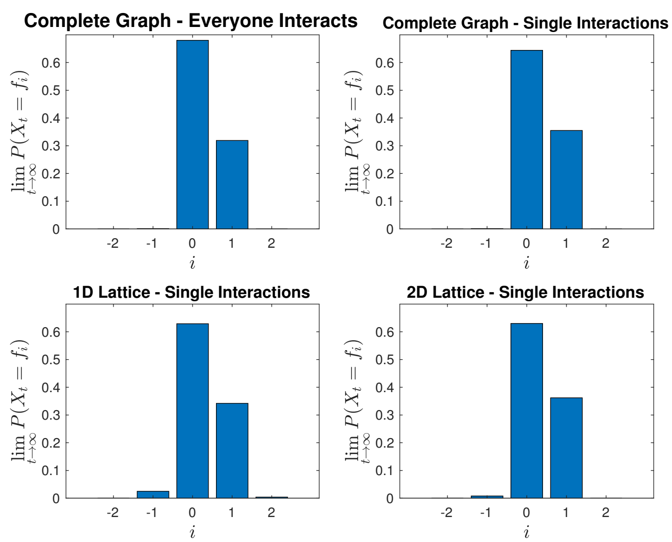

Figure 7: In each of the four subplots, $N = 400$, $K = \{-2,\ -1,\ 0,\ 1,\ 2\}$, initial distribution of opinions is [.1 .2 .3 .05 .35] and the number of simulations is 1000. The simulations resulted in a variance of .2191, .2309, .2777, and .2449 respectively.

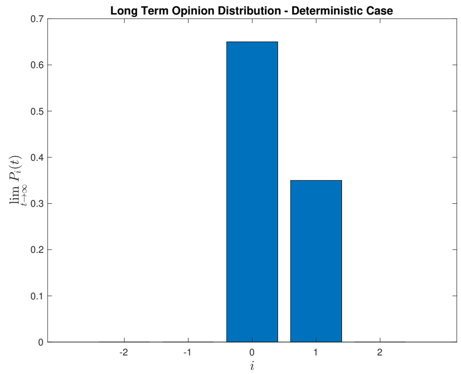

Figure 8: Long term opinion distribution $\lim_{t\to\infty} P_i(t)$ in the deterministic case for $i \in \{-2,\ -1,\ 0,\ 1,\ 2\}$ with initial distribution of opinions given by [.1 .2 .3 .05 .35]. Thinking of $\lim_{t\to\infty} P_i(t)$ as the weights of a probably mass function for random variable X, then $\mathrm{Var}(X) = .2275$.

Comparing Figures 6 and 7 with Figure 8, we can see that though there are slight differences in the long term behavior of the stochastic model for the process on the complete graph vs. 1D and 2D lattice, there does not appear to be much of a difference when everyone interacts at each time-step vs. when only one interaction occurs at a time or with different values of N. Interestingly, the long-term behavior in each of the four cases studied seems to fit well with the long term opinion distribution in the deterministic version of the model.

5 Conclusion

In conclusion, sociophysics offers a unique and valuable perspective for understanding the behavior of human populations. Through the use of mathematical models, researchers have been able to analyze and predict how interactions between individuals can lead to macroscopic phenomena. The Sznajd Model has gained significant popularity among researchers in the field, and in this paper, we proposed a modification to better capture the gradual evolution of opinions in the real world, building upon previous work in the area.

By introducing a range of opinions, our modified model allows for a more nuanced and accurate representation of real-world situations where beliefs and opinions are not binary, but rather fall on a spectrum. Moreover, our model allows for individuals to be influenced by others' opinions without entirely replacing their own. Through simulations on various graph structures(1D, 2D Lattice), we observed that the impact of the interaction process on the distribution of opinions in the long-run yield similar results to the deterministic model.

Our work contributes to the expanding field of sociophysics and highlights the potential of mathematical models in understanding human behavior. Future research can continue to build upon our ideas, deepening our understanding of how microscopic interactions between individuals can influence macroscopic phenomena. In our future work, we aim to conduct a more comprehensive exploration of our model, which may help us gain deeper insights into the factors that contribute to the different outcomes observed in the one-dimensional graph.

References

[1] D. Stauffer, Sociophysics: the Sznajd model and its applications, *Comput. Phys. Commun.*, **146** (2002), 93–98.

[2] E. Ben-Naim, Opinion dynamics: Rise and fall of political parties, *Europhys. Lett.*, **69** (2005), 671.

[3] F. Slanina, H. Lavicka, Analytical results for the Sznajd model of opinion formation, *Eur. Phys. J. B Condens. Matter Phys.*, **35** (2003), 279–288.

[4] M. Fraia, A.Tosin, The Boltzmann legacy revisited: kinetic models of social interactions, *Matematica nella Societa e nella Cultura*, **5** (2002), 93–98.

[5] K. Sznajd-Weron, J. Sznajd, Opinion Evolution in closed community, *Internat. J. Modern Phys. C*, **11** (2000), 1157–1165.

[6] K. Sznajd-Weron, J. Sznajd, T. Weron, A review on the Sznajd model — 20 years after, *Phys. A*, **565** (2021), 125537.

[7] G. Kondrat, How to introduce temperature to the 1D Sznajd model, *Phys. A*, **390** (2011), 2087–2095.

[8] F.W.S. Lima, Transition under noise in the Sznajd model on square lattice, *Internat. J. Modern Phys. C* , **27**, (2016), 1650026.

[9] M. Pérez-Llanos, J.Pinasco, N. Saintier, A. Silva , Opinion Formation Models with Heterogeneous Persuasion and Zealotry, *SIAM J. Math. Anal.*, **50** (2018), 4812–4837.

[10] A.T. Bernardes, U.M.S. Costa, A.D Araujo, D. Stauffer, Damage spreading, coarsening dynamics and distribution of political votes in Sznajd model on square lattice, *Internat. J. Modern Phys. C*, **12** (2001), 159–167.

Milka Perez Cazarez
California State University, Fullerton
800 North State College Blvd
Fullerton, California
E-mail: milkaperez@csu.fullerton.edu

Stephanie Reed
California State University, Fullerton
800 North State College Blvd
Fullerton, California
E-mail: sjreed@fullerton.edu

Received: June 6, 2023 **Accepted**: December 1, 2023
Communicated by Malgorzata Peszynska