The KnotLink Game on Families of Whitehead and Pretzel Links

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Abstract - In this paper, we study the KnotLink Game, a two-player game that can be played on the shadow of a knot or link diagram. This game was introduced by Adams, Henrich, and Stoll, who investigated winning strategies on certain infinite families of rational links. We focus our work on determining winning strategies for two additional infinite families of links: the pretzel link family and a family of Whitehead links.

Keywords : knot; link; game; Whitehead link; pretzel links

Mathematics Subject Classification (2020) **:** 57K10; 91A05

1 Introduction

In 2011, a group of researchers introduced some of the first 2-player games that are played on knot diagrams [4]. Chief among these fascinating new mental toys was a game that came to be known as the Knotting-Unknotting Game. This game was inspired by research that investigated the following question: if we are given a diagram of a knot that is missing some of its over/under crossing information, when can we determine that it is unknotted or, alternatively, is nontrivially knotted? Answering this question for knots (and the analogous question for links) is the key to unlocking winning strategies for knot and link games.

Since the Knotting-Unknotting Game was introduced, researchers have introduced and studied many other 2-player games on knot and link diagrams, including the Link Smoothing Game, the Linking-Unlinking Game, the Region Unknotting Game, the Region Smoothing Swap Game, the Arc Unknotting Game, Untangle, and more. See [3] for a summary of several of these. The focus of this paper is an interesting hybrid of the Knotting-Unknotting Game and the Linking-Unlinking Game, called the KnotLink Game. In [1], Adams, Henrich, and Stoll introduced this game and studied winning strategies for the families of rational knots and links that can be made with two twists. In this paper, we explore winning strategies for the KnotLink Game when the gameboards are from a family of Whitehead-like links or pretzel links.

Before we delve into an investigation of the KnotLink Game, let's define the basic objects and tools we're going to be working with. Later, equipped with these resources, we'll explore the world of game-playing and strategy and share our results.

2 Knot Theory Essentials

2.1 The Basics

A **knot** is a closed curve (i.e., a circle) embedded in 3-dimensional Euclidean space ($\mathbb{R}^3$ or $\mathbb{S}^3$) that doesn't intersect itself. A **link** is a collection of knots, where each knot in the collection is referred to as a **component** of the link. An **oriented** knot or link is one in which we choose a preferred direction we'd like to travel around a knot or link—it is often denoted by arrows. Two (oriented) knots or links are considered to be **equivalent** if you can bend, twist, stretch, pull, expand, or contract one of them to turn it into the other. The idea you might have in mind for knot equivalence moves in 3-space is that of a stretchy, knotted up rubber band. In this process of reforming a knot (or link) into one that is topologically equivalent, we only prohibit *cutting and regluing* it (or magically passing a knot/link through itself).

Knots and links are typically represented by knot and link *diagrams*. A **knot diagram** is a 2-dimensional closed curve with a finite number of transverse self-intersections (meaning that self-intersection points are not points of tangency). Similarly, a **link diagram** is a 2-dimensional representation of a link. In knot and link diagrams, we decorate the transverse intersection points to make it clear which strand passes over and which passes under. These are called **crossings**. If some of these over/under decorations are missing, we call the 2-dimensional picture a **pseudodiagram** and refer to the unknown crossings as **precrossings**. When *all* of the crossing information is missing, we call this pseudodiagram a **shadow**, while a diagram (with all of its crossing information intact) is also considered to be a special type of pseudodiagram [2]. See Figure 1.

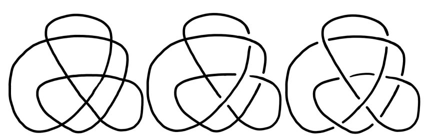

Figure 1: Three pseudodiagrams: a shadow (left), a "mixed" pseudodiagram (center), and a knot diagram (right)

There are a few standard knots and links that occur often enough to warrant descriptions and illustrations, among them are the unknot and the unlink. The **unknot** is the unique knot that can be represented by a diagram with no crossings. All knots that are *not* unknots are classified as **nontrivial knots**. An **unlink**, similarly, is a disjoint (or "unlinked") collection of unknots. All links that are not unlinks are classified as **nontrivial links**. One of the simplest nontrivial links is the **Hopf link**, as it is—diagrammatically speaking—made up of two crossings with opposite overstrand slopes to intertwine the two components. A **trefoil** knot is the simplest nontrivial knot. A **figure-eight knot** is one that can be represented by a diagram with four crossings, but no fewer. Examples of each of these are shown in Figure 3.

While a knot diagram represents just one kind of knot, a knot has infinitely many ways it can be represented diagrammatically. One thorny problem in knot theory is: when you're given two knot diagrams, how can you tell if they represent equivalent knots in 3-space? In the 1920s, Kurt Reidemeister proved [8] that two knot diagrams represent equivalent knots if and only if the diagrams can be related by a sequence of **Reidemeister moves** and planar isotopies (i.e., *wiggling* in such a way that no crossings are introduced or removed). The three types of Reidemeister moves, or R-moves, are shown in Figure 2. Reidemeister I (RI) introduces or removes a small loop. Reidemeister II (RII) slides two portions of the knot so that one lies above the other or, conversely, pulls two overlapping strands apart. Reidemeister III (RIII) moves a strand of the knot that lies entirely over or entirely under two crossing strands past the crossing.

These moves—especially RI and RII—appear frequently in our results, as they help simplify our game board to determine if our result is an unknot/unlink or a nontrivial knot/link. If a crossing in a diagram is one that can be removed with an RI move (possibly after a sequence of other RI moves), then we call it a **reducible crossing**.[1]

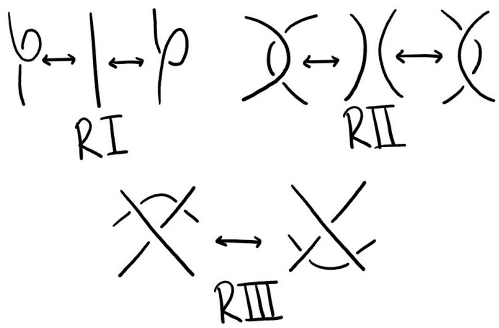

Figure 2: Reidemeister moves: RI, RII, and RIII

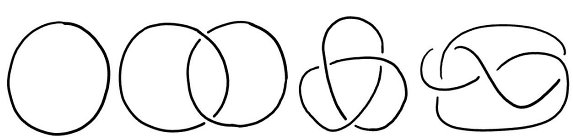

Figure 3: A collection of simple knots and links: unknot (left), the Hopf link (center left), trefoil (center right), and figure-eight knot (right)

2.2 Orientation & Crossing Signs

To describe salient features of a knot or link diagram, it can be helpful to refer to the **sign** of a crossing in the diagram. We are able to assign +1 or −1 to each crossing if we

[1]There are sneakier reducible (aka "nugatory") crossings that don't quite fit this description, but we won't encounter them in this paper.

first give our diagram an **orientation**. To specify a knot or link diagram's orientation, we choose a place on the diagram and begin drawing arrows, following one component in one direction until we have traced the whole component and are back to where we started. We do this for each individual component (choosing a direction we will travel around it) if we have a link. See Figure 4.

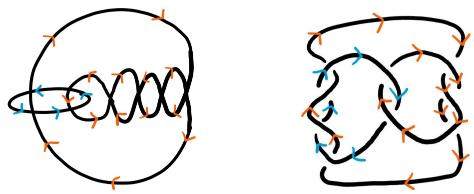

Figure 4: Two oriented links

Then, to determine the **sign** of a crossing, look at a crossing in the diagram from the perspective that both strands involved in the crossing are oriented upwards. (You may have to rotate your head sideways or upside down!) From this perspective, if the strand with the positive slope is the overstrand, we say the crossing is **positive** and associate to it a sign of +1. Otherwise, we say it's **negative** and associate to the crossing a sign of −1. See Figure 5.

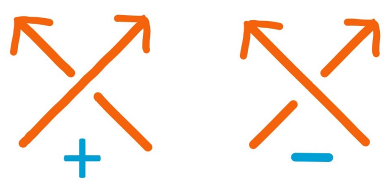

Figure 5: The signs of a crossing

The sum of the signs of all crossings in a knot diagram is called the **writhe** of the diagram. In a link diagram, we're often interested in just the sum of the signs of the crossings where one link component interacts with another, as in Section 2.3. In other situations, we might want to know the **local writhe**, i.e. the sum of the crossing signs of a smaller portion of a knot or link diagram. For both writhe and local writhe on pseudodiagrams, we ignore precrossings and sum the signs of the crossings which have been resolved.

As an exercise, we invite the reader to take a look at the RII move. Choose orientations for both strands of the knot or link that are involved in the move. Show that, regardless of how these orientations are chosen, the two crossings in the move have opposite signs, so the local writhe is zero. This observation will be useful as we study game strategies later!

2.3 Linking Number

The **linking number** of an oriented, 2-component link is an invariant that can be used to identify when a link is nontrivial. In essence, it is a measure of how intertwined two components of a link are with each other.

To compute the linking number, lk, from an oriented, 2-component link diagram, we take half the sum of the signs of those crossings that involve both components. So, if L is our link with components L_1 and L_2, we have:

$$lk(L) = \frac{1}{2} \sum_{c \in L_1 \cap L_2} sign(c).$$

If the linking number of any two components of an oriented link, L, is zero, then L may or may not be an unlink. On the other hand, if the linking number of two oriented link components is nonzero, the resulting diagram is definitively a nontrivial link, and so the linking number is a useful tool for detecting nontriviality of links.

We note that, if you are interesting in studying a link with more than two components, you may compute the linking number of pairs of components in the link to get information. For any two links with the same number of components, any differences in the profiles of pairwise linking numbers for the two links will indicate the links are not equivalent to each other.

2.4 Tricolorability

One of the most interesting nontrivial links is the **Whitehead link**. (See Figure 6.) What makes this link so interesting is that it has a linking number of zero, so the linking number cannot detect its nontriviality. This means that we could use more tools for detecting nontriviality.

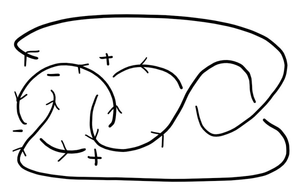

Figure 6: The Whitehead link, a nontrivial link with linking number zero

Tricolorability is an invariant of both knots and links that assigns a color—of three possible colors—to each arc in a knot or link diagram (i.e., to each portion of the diagram between adjacent underpasses). The assignment is made following some simple rules:

- Rule 1: Three different colors or one single color must be present at a crossing, but never just two. (See Figure 7.)
- Rule 2: At least two colors must be used in the diagram as a whole in order for a tricoloring to be valid.

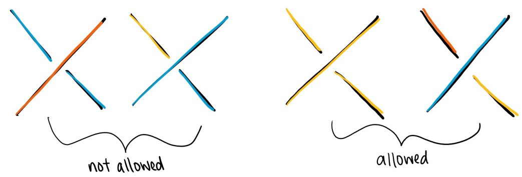

Figure 7: Color combination options for tricolorability

Tricolorability is a knot and link invariant because Reidemeister moves preserve the tricolorability of diagrams. In other words, a diagram will be tricolorable before a Reidemeister move if and only if it is tricolorable after the move.

As an invariant, tricolorability is useful for distinguishing knots and links from one another. Since the unknot is *not* tricolorable, any other knot that *is* tricolorable must be nontrivial. Likewise, since the unlink *is* tricolorable, any link that *fails* to be tricolorable must be nontrivial. See Figure 8 for some examples.

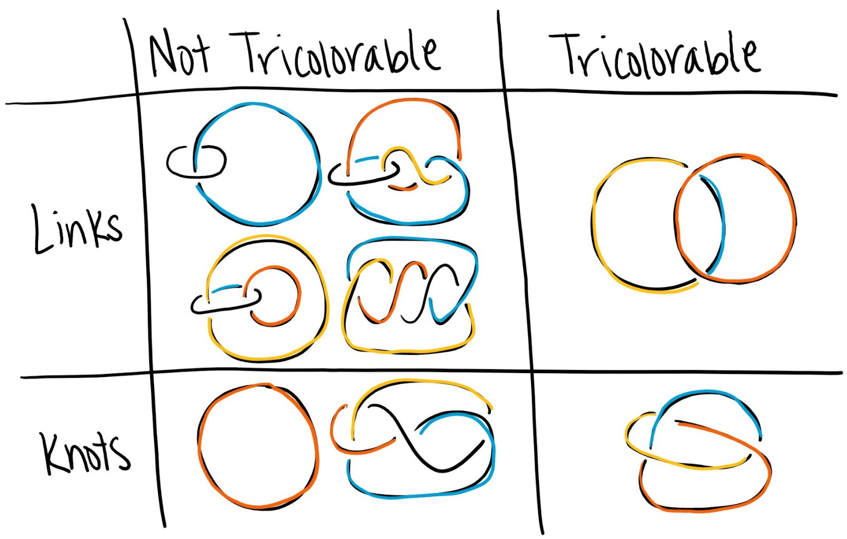

Figure 8: Tricolorability (or not) of several knot and link diagrams

Returning to the example of a Whitehead link, the diagram in Figure 6 fails to be tricolorable. (See for yourself! Try coloring this link and find out what goes wrong.) Therefore, it cannot be the unlink.

3 The KnotLink Game

Now, we come to the heart of our work: the KnotLink Game, a game that can be played on shadows of knots and links. There are two players in this game, the Simplifier and the Complicator, and each of these players has a unique goal. The Simplifier's goal is to turn the game board into an unknot or an unlink. The Complicator wants to turn the game board into a nontrivial knot or link. At the beginning of the game, the players are given a game board: a knot or link shadow. Primarily, players take turns "resolving" crossings in the diagram until every crossing has been resolved. **Resolving** a crossing is when a player decides which strand will cross over and which strand will cross under at a precrossing. While most moves consist of resolving crossings, in addition, each player may choose—no more than one time during the game—to **smooth** rather than resolve a crossing. There are two types of **smoothings** the players can choose from, shown in Figure 9.

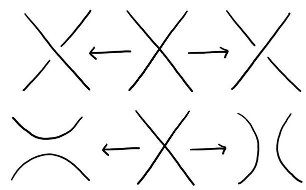

Figure 9: (Top) two possible ways to resolve a precrossing; (bottom) two possible smoothings of a precrossing, a horizontal smoothing shown on the left and a vertical smoothing shown on the right.

Let's see how the game is played by walking through the sample game shown in Figure 10. We begin with a knot shadow. Here, the Simplifier moves first on precrossing 1, choosing to use their one special smoothing move (shown in orange). This turns the knot into a link. The Complicator responds (in blue) by resolving crossing 2. The remaining moves are played (on precrossings labeled in numerical order), with the Simplifier's (orange) moves alternating with the Complicator's (blue) moves. At the end of the game, the linking number of the link is ± 1 (depending on how orientations for the components are chosen), so the link is nontrivial. Hence, the Complicator wins.

The KnotLink Game is a type of **combinatorial game** in which one of the two players can guarantee themselves a win (regardless of how their opponent plays) if they follow a **winning strategy**. So, the most natural question to ask of the KnotLink Game is: given a particular starting shadow, who has a winning strategy?

In the paper that introduced the KnotLink Game [1], the authors determined winning strategies for certain families of knots and links (in the "rational" knot/link family), including $(2, m)$-torus knots and links. Here, we study winning strategies for the KnotLink Game on two other infinite families: a family of Whitehead-like links and a pretzel link family.

4 The KnotLink Game on a Whitehead Link Family

One of the link families we were most excited to explore for the KnotLink Game is a family of linking-number-zero links related to the Whitehead link. The particular diagram shown of the Whitehead link in Figure 6 has a twist with two crossings (on the right) contained

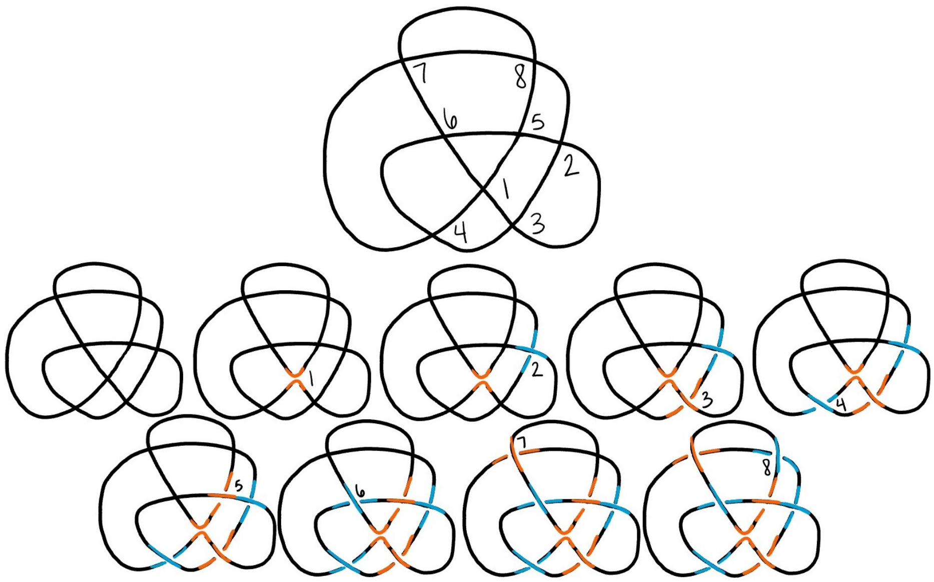

Figure 10: A sample KnotLink Game with Simplifier playing first (shown in orange) and Complicator playing second (shown in blue)

in one of the link's two components. To generate an infinite family of links from this particular link, we replace the 2-crossing twist with a twist having an arbitrary number of crossings. See Figure 11 for an image of a general link in this family.

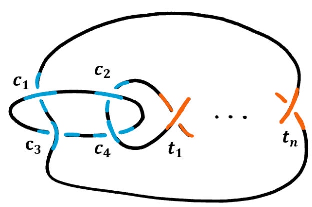

Figure 11: The Whitehead link family we will investigate. The "oval" crossings, c_1 through c_4, are shown in blue. The remaining n twist crossings, labeled $t_1, \ldots, t_n$, are indicated in orange.

Let's now consider what happens when our two players play on this family of Whitehead links. We prove four theorems in order to address all possible combinations of player order and parity of number of twist crossings (in other words, the even/oddness of n in $t_1, t_2, ..., t_n$ in Figure 11). In Theorems 4.1–4.4, we will show that for even n, the second player has a winning strategy, while for odd n, the first player wins.

Before we get started proving these results, we would like to observe that the winning strategies we outline are not the *only* possibilities for success, however, they are straightforward to describe and, thus, they are the ones we have chosen to present. We also want to introduce some language we use when analyzing game strategies. Often during game

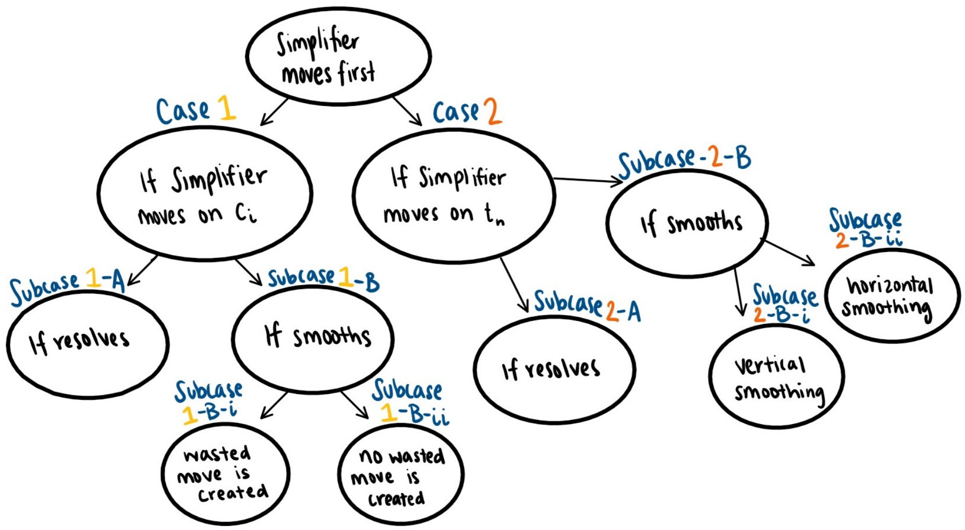

Figure 12: Complicator's winning strategy when playing second on an even twist diagram in Theorem 4.1

play, a smoothing turns one or more crossings in the diagram into reducible crossings—in other words, these crossings do not and cannot affect the knottedness or linkedness of the diagram. Note that resolves also can create reducible crossings. We refer to moves involving these crossings as **wasted moves**.

Note that the proof of Theorem 4.1 is written in great detail, while the remaining three theorems are proven more succinctly. To aid the reader, we've provided an illustration of the Complicator's response moves for Theorem 4.1 (showing that the Complicator has a winning strategy playing second). In Figure 12, we consider all six of the Simplifier's possible moves and clearly outline how the Complicator should respond in order to ensure the diagram will not be completely simplified. This strategy can be applied at any point in the game, and the Complicator's response moves only rely on the Simplifier's most recent move. Let's get started!

Theorem 4.1 *If the KnotLink Game is played on a Whitehead link shadow from the family shown in Figure 11 where there is an even number of crossings in the twist, then the Complicator has a winning strategy when playing second.*

Proof. Let us consider pairs of moves, a move by the Simplifier followed by a Complicator countermove. For what follows, we assume the moves described are played at an arbitrary point in the game. See Figure 11 for crossing labeling conventions. We describe the winning strategy for the Complicator responding to moves of the Simplifier.

Case 1. [Simplifier moves in the oval.] Here, we consider how the Complicator should respond if the Simplifier makes a move in the oval, on a crossing c_i.
Subcase 1-A. [Simplifier resolves a crossing in the oval.] Suppose the Simplifier chooses to resolve a crossing, c_i, in the oval. The Complicator's response should be to resolve

the corresponding crossing, $c_{i\pm2}$, using the same sign the Simplifier used. By resolving with the same sign, the Complicator will be able to avoid the possibility of an RII move occurring in that pair.

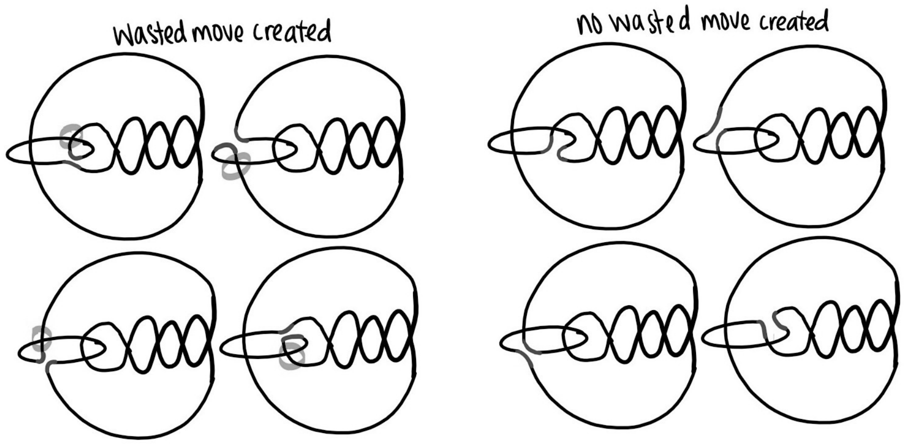

Figure 13: Smoothing options to create, or not create, a wasted move

Subcase 1-B. [Simplifier smooths in the oval.] Suppose the Simplifier uses their smoothing move on a crossing c_i in the oval to turn the 2-component link into a knot. There are two possible outcomes: either a wasted move will be created in the corresponding crossing, $c_{i\pm2}$, or not. (See Figure 13.) In the following two subcases, we will see how the Complicator should respond.

- **Subcase 1-B-i.** [Simplifier smooths in oval to create a wasted move.]

  If a wasted move is created by the Simplifier's smoothing, then the Complicator should resolve this wasted move crossing (in any way). The final diagram will be the figure-eight knot or the trefoil if this strategy and the strategy outlined for the twists is followed. See Figure 14.

- **Subcase 1-B-ii.** [Simplifier smooths in oval without creating a wasted move.] If the Simplifier smooths on c_i without creating a wasted move, the Complicator should respond by smoothing $c_{i\pm2}$ in such a way that the shadow is turned back into a 2-component link. (Following this strategy, it will become the Hopf link by the end of the game. See Figure 15 for a sample game.)

Case 2. [Simplifier moves in the twist.] Here, we consider how the Complicator should respond if the Simplifier makes a move in the twist, on a crossing t_i.

Subcase 2-A. [Simplifier resolves a crossing in twist.] Suppose, for this case, that the Simplifier resolves a crossing in the twist. In order to avoid the twist having a local writhe of zero (which would cause the twist to completely unravel), the Complicator's response should be to resolve another crossing in the twist using the same sign that was chosen for the *first* resolved crossing in the twist. While this strategy may lead to some RII

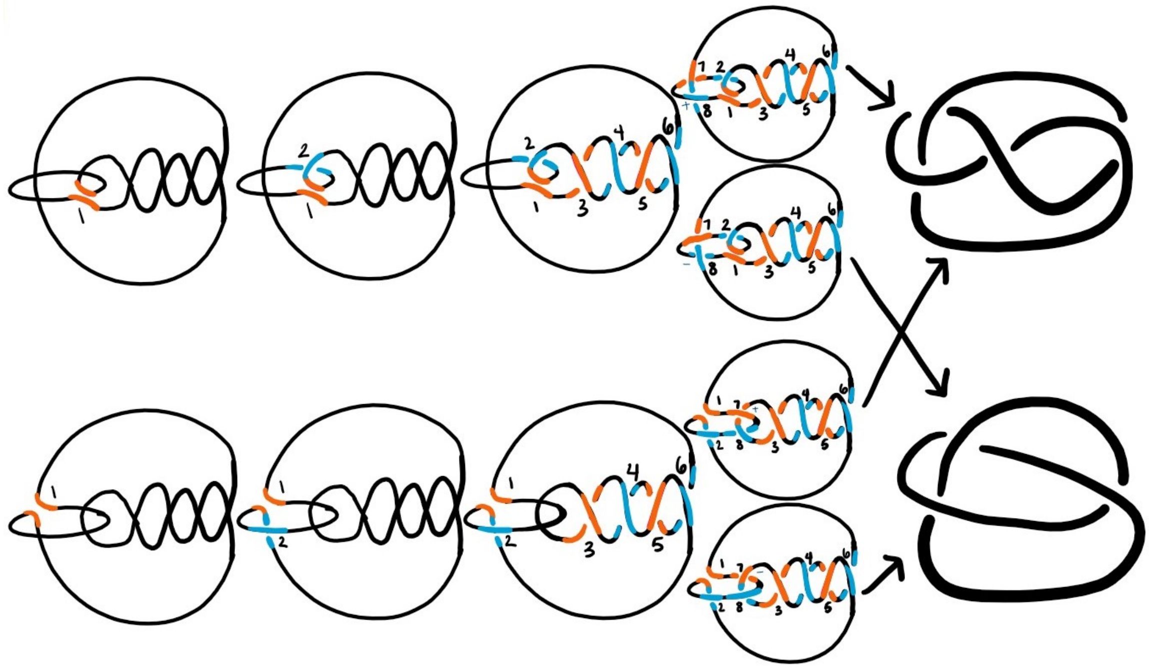

Figure 14: Subcase 1-B-i: In this example, the Simplifier's first move results in a wasted move. The outcome is a trefoil knot or a figure-eight knot.

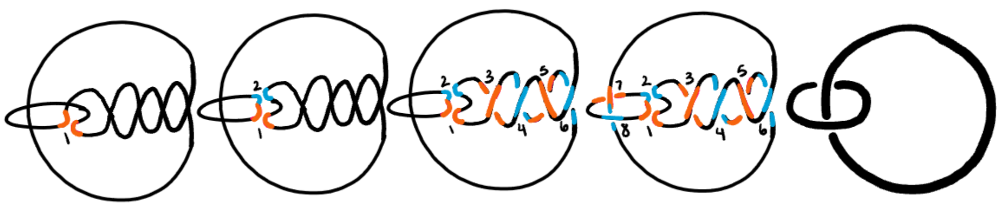

Figure 15: Subcase 1-B-ii: Simplifier's first move does not result in a wasted move

simplifications in the twist, the absolute value of the local writhe of the twist crossings will be nonzero at the end of the game.

Subcase 2-B. [Simplifier smooths in the twist.] Suppose the Simplifier does a horizontal smoothing. The Complicator should respond with a vertical smoothing in the twist. Likewise, if the Simplifier does a vertical smoothing, the Complicator should respond with a horizontal one in the twist. In both cases, the result is a three component link. From here, the Complicator should respond to moves in the oval following the strategy of Subcase 1-A. They should respond to Simplifier resolves of the reducible crossings, t_i, with their own resolves of reducible crossings. Then, the resulting link will be nontrivial. See Figure 16.

□

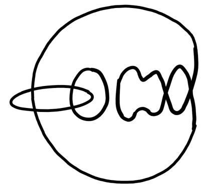

Figure 16: Example of a possible pseudodiagram after a vertical and a horizontal smoothing move have both been played in the twist.

Theorem 4.2 *If the KnotLink Game is played on a Whitehead link shadow where there is an odd number of crossings in the twist, then the Complicator has a winning strategy when playing first.*

Proof. The Complicator should resolve a crossing in the twist with their first move. Since this leaves an even number of crossings to resolve in the twist, the Complicator can from this point forward, adopt the strategy described in proof of Theorem 4.1 with similar outcomes. The Complicator will still be able to have the last move in the twist, forcing the Simplifier to move first in the oval. If the Complicator always matches the sign of their first move for all subsequent moves in the twist, they can force a nonzero local writhe in the twist. Because the local writhe in the twist will necessarily be nonzero and the Simplifier must move first in the oval, the Complicator's winning strategy will succeed in this case as well. We leave it to the reader to reconstruct the details of each case. □

Theorem 4.3 *If the KnotLink Game is played on a Whitehead link shadow where there is an odd number of crossings in the twist, then the Simplifier has a winning strategy when playing first.*

Proof. Let the Simplifier perform a vertical smoothing on t_1 for their opening move. This creates a 3-component link with an even number of wasted moves among the remaining $t_2, t_3, \cdots, t_n$ crossings (since n is odd). This crossing parity will force the Complicator to be the first to move in the oval. When the Complicator moves in the oval crossings, they have three options: they can use a smoothing that does not create a wasted move, they can use a smoothing that does create a wasted move, or they can simply resolve crossings. The Simplifier should always respond to a move in c_i with a resolve in $c_{i\pm2}$. This resolve should be of opposite sign to the Complicator's move when the move is a resolve, but the sign does not matter if the Complicator uses a smoothing move. This will force the Complicator to move first in the other pair of crossings in the oval, where again the Simplifier should respond to a move in c_i with a resolve in $c_{i\pm2}$, of opposite sign if the Complicator's move is not a smoothing move. The result of this strategy, once all crossings in the oval are resolved, will make one pair of crossings in the oval have an RII move, and the other either have an RII move or a smoothing and an RI move.

Representative gameplay flowing from the Simplifier's opening move is shown in Figure 17. □

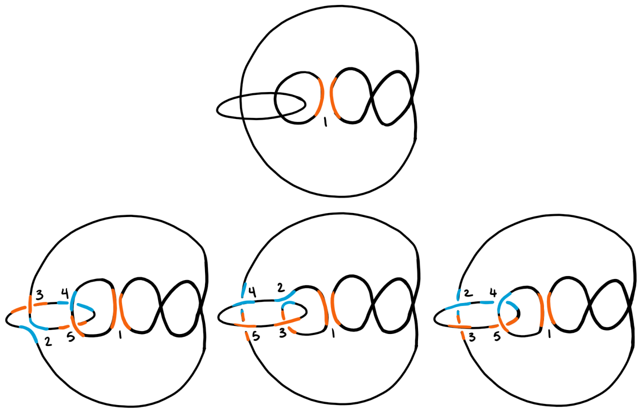

Figure 17: Illustration of Theorem 4.3: Simplifier's opening move is a vertical smoothing. Shown below are three possible game outcomes. Crossings in the twist have been left unresolved since they are irrelevant to the game outcome and their even parity means that the Complicator will be forced to move first in the oval.

Theorem 4.4 *If the KnotLink Game is played on a Whitehead link shadow where there is an even number of crossings in the twist, then the Simplifier has a winning strategy when playing second.*

Proof. Since the diagram has an even number of crossings, we can pair up crossings such that the Complicator moves first in each pair, and the Simplifier has a response move on the second crossing in the pair. We use the following pairs: $\{c_1, c_3\}, \{c_2, c_4\}, \{t_1, t_2\}, \cdots,$ $\{t_{n-1}, t_n\}$. The Simplifier's winning strategy is similar to the Complicator's winning strategy described in the proof of Theorem 4.1, except the Simplifier will opt to respond to any resolve by resolving the paired crossing with the *opposite sign*, so that the number of crossings in the diagram can be reduced by two (using RII moves) after each pair of resolves. On the other hand, if the Complicator uses a smoothing move on one of the c_i crossings and no wasted move is created (as in the first pseudodiagram in Figure 15), then the Simplifier can respond with a smoothing move of their own in the paired crossing in order to transform the diagram into a 2-component link, further simplifying it. The opposite-sign resolve strategy above will ensure this link becomes unlinked. □

Now that we have shown the winning strategies for each case, we should point out a simple way to remember who wins in the KnotLink Game played on the Whitehead link family: whoever has the last move in the game has a winning strategy. If there are an odd number of twists, then the first player has a winning strategy, otherwise the second player has a winning strategy.

5 The KnotLink Game on Pretzel Links

We now turn our attention to another family of links to play the KnotLink Game on: *pretzel* knots and links. While every member of the Whitehead link family has exactly two components, our pretzel link family members have one, two, or three components. Let's take a look at how we can construct a link in this family. We start with some twisting.

Two strings twisted around each other with k crossings is called a $\pm k$-**twist**. If we orient the twist vertically and the crossings in this twist have overstrands with a positive slope, we call it a k-twist. If the crossings have negatively-sloped overstrands, we call the twist a $-k$-twist. We may then represent any pretzel link of the sort we want to study by first setting three such twists vertically parallel. Next, we connect the closest loose ends between the first and second twists and between the second and third twists. Finally, we connect the remaining loose ends at the top and the bottom.

We illustrate two possible outcomes of this construction in Figure 18. If we form a pretzel link by joining a p-twist, a q-twist, and an r-twist, we call the resulting object a (p, q, r) **pretzel link**. So, in our example, the pretzel link on the left is the $(3, 3, 3)$ pretzel link, and the one on the right is the $(4, -2, -3)$ pretzel link.

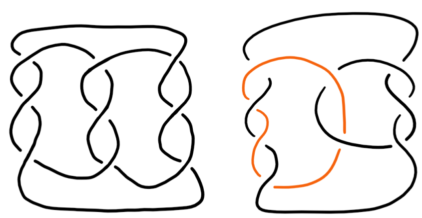

Figure 18: A $(3, 3, 3)$ pretzel knot (left) and $(4, -2, -3)$ pretzel link (right)

The ordering (p, q, r) of twists in a pretzel link is naturally cyclic, given the topology of the link. In other words, the (p, q, r) pretzel link is equivalent to the ones denoted by (q, r, p) and (r, p, q). (Test your visualization skills! Can you imagine grabbing the p twist and pulling it over the other two twists, setting it down to the right of the r twist? What would happen to the arcs connecting these twists? What would happen to the crossings in the p twist? Would any new crossings be created? If so, could they be removed with any Reidemeister moves?)

The parity of the number of crossings in each twist will be quite important for determining winning strategies. The number of components in a pretzel link depends only on the parity of the twists. If all twists have an odd number of crossings or if exactly one twist has an even number of crossings, then our pretzel link has a single component—that is, it is a knot. Otherwise, it is a 2- or 3-component link, depending on the number of even twists.[6] Using natural notation, we can understand the number of components in a pretzel link as follows. If OOE denotes that exactly two of $\{p, q, r\}$ are odd (in any order), for instance, then OOO and OOE pretzel links are knots. EEO pretzel links have two components, and EEE pretzel links have three components. We encourage you to draw some examples of each link type to convince yourselves of this fact!

So, how will this affect the strategies of the players? For the Complicator, more components creates more opportunities to end with a Hopf link or other, more complex, nontrivial link. To see how link components can be created or absorbed, we note that game moves can change the parity of our twists. A vertical smoothing in a twist always changes the parity of that twist. So, it changes OOO to OOE (keeping the knot a knot) and changes EEE to EEO (keeping the link a link, but reducing the number of components). Likewise, a vertical smoothing in a twist can either change pretzel knot OOE into an OOO knot or an EEO link. A vertical smoothing performed on a twist in an EEO 2-component pretzel link either reduces the number of components, transforming it into a OOE knot (i.e., 1-component link), or increases the number of components, producing a EEE 3-component link. This means that players with a smoothing move in reserve can shift the category of knot or link of the game board if that might help them.

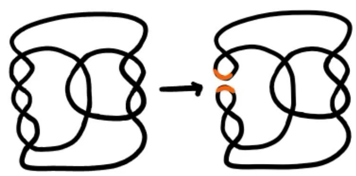

Figure 19: A $(4, 2, 3)$ pretzel link shadow, after a horizontal smoothing in the 4-twist, becomes the shadow of a $(2, 5)$-torus knot (with some additional, extraneous crossings in the former twist).

While vertical smoothing moves can have a big effect, the effect of a horizontal smoothing in a twist can be even more profound. A horizontal smoothing in one twist creates a knot or link pseudodiagram of a $(2, m)$-torus knot (if m is odd) or $(2, m)$-torus link (if m is even) with two components. Here, m equals the sum of the number of crossings in the remaining two twists.[2] (See [1] for more on the KnotLink Game played on $(2, m)$-torus links.) This leads to the following structural facts which are of interest to our players.

1. If m is odd and the local writhe of the crossings in the two remaining twists (i.e. the non-reducible crossings) is ± 1, then the knot is the unknot. This is because such a local writhe means that all but one crossing can be removed by a sequence of RII moves. A knot with only one crossing must be the unknot.

2. If m is odd and the absolute value of the local writhe of the crossings in the two remaining twists is 3 or more, then the knot is not the unknot since it can be represented by a reduced, alternating knot diagram with at least three crossings.[3]

[2]Technically, we might get a pseudodiagram that can only be resolved to a smaller, $(2, m - 2i)$-torus link—but not a $(2, m)$-torus link—if there are crossings that have already been resolved with opposite sign. Pairs of opposite-sign crossings can be reduced using RII moves at the end of the game.

[3]This fact is a direct consequence of the Tait conjecture that states that any reduced, alternating knot or link diagram has the fewest crossings of any diagram of that knot or link. Despite the misleading name, the Tait conjectures are actually theorems. At one time, they were conjectured by Peter Guthrie Tait, but they have since been proven. The Tait conjecture we reference here was proven independently by Kauffman, Murasugi and Thistlethwaite [5, 7, 9].

3. If m is even and the absolute value of the local writhe of the crossings is ≥ 2, then the link is not the unlink, since this condition guarantees a nonzero linking number.

Without further ado, let's see what happens when we play the KnotLink Game on this pretzel link family. Before we begin, we present a summary of our results in Table 1. Note that for Pretzel links (EEE and EEO) we have complete results for who wins, no matter who plays first. However, in the cases of Pretzel knots (OOE and OOO), we have only partial results, not a complete characterization of the game outcomes.

	Simplifier plays first	Complicator plays first
EEE	Complicator wins (Lemma 5.2)	Simplifier wins (Lemma 5.3)
EEO	Simplifier wins (Lemma 5.5)	Complicator wins (Lemma 5.6)
OOE	It's complicated (see Cor. 5.8, Prop. 5.9)	Simplifier wins (Theorem 5.7)
OOO	Simplifier wins (Theorem 5.10)	It's complicated (see Cor. 5.11)

Table 1: Game outcomes for pretzel link.

Let's jump in to proving these results!

Theorem 5.1 *For the KnotLink Game played on a (p, q, r) pretzel link shadow where p, q, and r are even, Player 2 has a winning strategy.*

We will prove Theorem 5.1 by considering the two cases of who plays first in the following two lemmas.

Lemma 5.2 *(EEE, Simplifier moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where p, q, and r are even. If the Simplifier moves first, then the Complicator has a winning strategy.*

Proof. Note that because the Simplifier has only a single smoothing move to use, the Complicator can ensure that the game will end in either EEE (Simplifier only used resolves), EEO (Simplifier used a vertical smoothing), or EE (Simplifier used a horizontal smoothing and removed a twist) configurations. The Complicator can also choose to move last in every twist.

The Complicator's strategy is, thus, relatively simple: ensure that the local writhe in all but one of the even twists is zero, with a local writhe in the other even twist of ± 2. In all such cases, the resulting link will be nontrivial, either a Hopf Link or a disjoint union of the Hopf Link and an unknot. Note that the Complicator will not use any smoothing moves in this process. These cases describe the Complicator's response to any Simplifier move during the course of the game.

Case 1. [Simplifier resolves a crossing in an even twist.] Then the Complicator will respond by resolving a crossing in the same twist with the opposite sign to keep the local writhe in the twist zero, unless this is the last move in the first twist to be completed. If it is the last move in the first twist to be completed, the Complicator will resolve the crossing with the opposite sign to create a twist with a local writhe of ± 2. By following

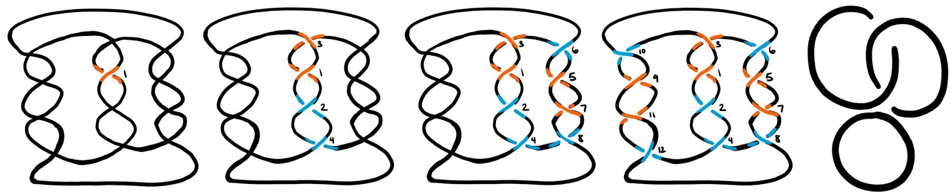

Figure 20: Illustration of Lemma 5.2, Case 1. Note that the local writhes in each twist in the final diagram, left to right, are 0, ± 2, and 0.

this strategy, there will be one even twist with local writhe ± 2 and one or two even twists with local writhe zero at the end of the game.

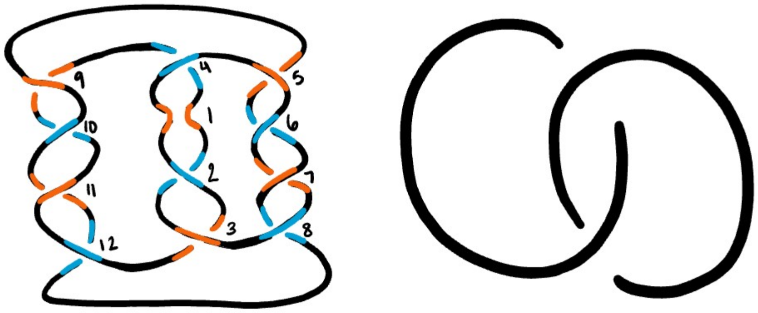

Figure 21: Illustration of Lemma 5.2, Case 2. Note that the local writhes in each twist, left to right, are 0, ± 1, and ± 2.

Case 2. [Simplifier uses a vertical smoothing in any twist.] Note that the Complicator's strategy means that the current local writhe of the resolved crossings in this twist is zero. The Complicator should reply with a resolve in this twist. For the rest of the game, any time the Simplifier resolves a crossing in this twist the Complicator should reply with a resolve of the opposite sign. This will ensure that when all moves in this now-odd twist have been played the local writhe will be ± 1.
Case 3. [Simplifier uses a horizontal smoothing in any twist.] This makes all remaining moves in this twist wasted moves. The Complicator should respond with a resolve in this same twist. For the rest of the game, any time the Simplifier resolves a crossing in this twist, the Complicator should reply with any resolve in the same twist. (The sign does not matter since these are wasted moves.)

By following this strategy, the only possible final configurations are:

- EEE with local writhes of 0, 0, and ± 2, which will yield a disjoint union of the Hopf Link and an unknot.
- EEO with local writhes of 0, ± 2, and ± 1, which will yield the Hopf Link.

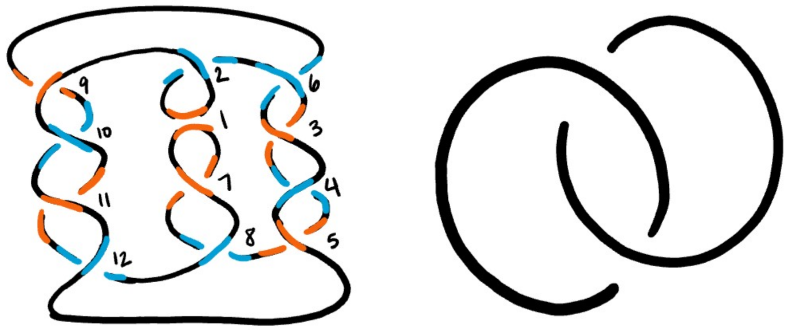

Figure 22: Illustration of Lemma 5.2, Case 3. Note that the local writhes in the twists, left and right, are 0 and ± 2. (We do not sum the middle crossings because the horizontal smoothing makes all crossings in this twist reducible, so they do not affect the game outcome.)

- EE with local writhes of 0 and ± 2, which will yield the Hopf Link.

Thus, the Complicator has won without using a smoothing move.

□

Lemma 5.3 *(EEE, Complicator moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where $p, q,$ and r are even. If the Complicator moves first, then the Simplifier has a winning strategy.*

Proof. The Simplifier will be able to keep a local writhe of zero in every twist with this strategy:
Case 1. [Complicator resolves a crossing.] Anytime the Complicator resolves a crossing in a twist, the Simplifier will resolve a crossing with the opposite sign in the same twist. Since the Simplifier will always be able to move last in each twist, this will ensure a local writhe of zero in each twist if no other types of moves are used. This will result in the unlink.
Case 2. [Complicator uses a vertical smoothing.] If the Complicator chooses to use their smoothing move as a vertical smoothing, the Simplifier can reply with a vertical smoothing in the same twist. Since each twist is even and the Simplifier always responds in the same twist where the Complicator moved, there will always be a crossing left on which the Simplifier can play their vertical smoothing. The effect of these two moves produces another EEE pretzel link, and the game proceeds using only Case 1 above, since no player has a smoothing move left. All twists will end with local writhe zero. This will result in the unlink.
Case 3. [Complicator uses a horizontal smoothing.] If the Complicator chooses to use their smoothing move as a horizontal smoothing, the Simplifier can reply with a resolve in that same twist, as now all remaining moves in that twist are wasted moves. This

preserves the Simplifier's ability to always respond with a move in the same twist in which the Complicator has just played, and thus always to have the last move in a twist. The rest of the game will fall under Case 1, above, with the Simplifier able to ensure a local writhe of zero in the two other twists. This will result in the unlink. □

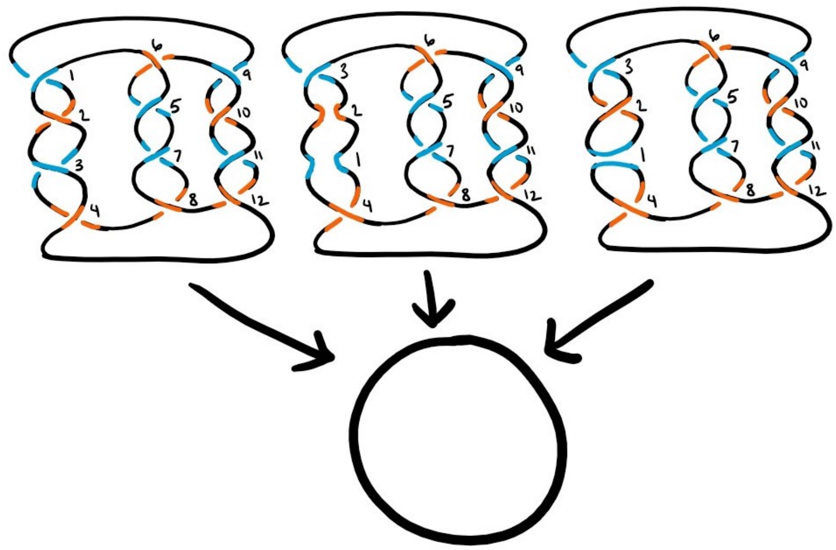

Figure 23: Illustration of Lemma 5.3: Case 1 (left), Case 2 (center), Case 3 (right) all result in the same final diagram of the unknot.

Theorem 5.4 *For the KnotLink Game played on a (p, q, r) pretzel link shadow where p and q are even, and r is odd, Player 1 has a winning strategy.*

As with Theorem 5.1, the proof of this theorem is accomplished by proving two lemmas.

Lemma 5.5 *(EEO, Simplifier moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where p and q are even, and r is odd. If the Simplifier moves first, then the Simplifier has a winning strategy.*

Proof.

The Simplifier begins by resolving a crossing with sign -1 in the odd twist. This leaves an even number of remaining moves in the odd twist, as well as in each of the two even twists. The current local writhe of each of the even twists is zero, and the local writhe in the odd twist is -1.

Case 1. [The Complicator resolves a crossing.] The Simplifier resolves a crossing in the same twist with opposite sign. If the Complicator never uses a smoothing move, this will result in an *EEO* diagram with local writhes of 0, 0, and -1, which will yield the unlink. The Simplifier wins.

Case 2. [The Complicator uses a vertical smoothing in the odd twist.] The Simplifier should resolve a crossing with sign $+1$ in this same twist to make the current local writhe in that twist zero. Note that the diagram is now of the form EEE, which will yield a 3-component link. For the rest of the game, the Simplifier should respond to each of the Complicator's resolves with a resolve in the same twist of the opposite sign. This will result in a diagram with a local writhe in each twist of zero, which results in the unlink. The Simplifier wins.

Case 3. [The Complicator uses a vertical smoothing in one of the even twists.] Note that this shifts the diagram to OOE, which will yield a knot, with the Simplifier making the next move. Further, note that based on the Simplifier's opening move, this OOE diagram must have a current local writhe of zero in both the even twist and in the newly odd twist, along with a local writhe of -1 in the original odd twist. The Simplifier should respond in the newly odd twist with a positive resolve. For the rest of the game, the Simplifier continues to respond to the Complicator's moves with a resolve of opposite sign in the same twist as each Complicator move. By following this strategy, the even twist will finish with local writhe zero, and the odd twists will finish with local writhes of ± 1, one of each sign. This results in the unknot. The Simplifier wins.

Case 4. [The Complicator uses a horizontal smoothing in an even twist.] This leaves an odd number of wasted moves in that twist. Note also that the odd twist has current local writhe -1 and the even twist must have current local writhe zero based on the Simplifier's strategy for replying to the Complicator's resolve moves outlined in Case 1. The Simplifier wastes a move by resolving in the same twist in which the Complicator just used a smoothing. This leaves an even number of wasted moves and an even number of remaining moves in the other two twists as well. For the rest of the game, the Simplifier need only reply to any Complicator resolve with a resolve of opposite sign in the same twist. This will yield a local writhe of zero in the remaining even twist and a local writhe of -1 in the odd twist. Combined with the horizontal smoothing in the other even twist, this will yield the unknot. The Simplifier wins.

Case 5. [The Complicator uses a horizontal smoothing in the odd twist and at least one unresolved crossing remains in the even twists.] Then an even number of unresolved crossings remain in each of the even twists and an odd number of unresolved crossings remain in the (original) odd twist. Also, the local writhe for each even twist is currently zero. The Simplifier responds with a horizontal smoothing in either one of the even twists. This leaves an odd number of unresolved crossings available for wasted moves in that twist. The odd number of wasted moves available in each of the twists, which now include a horizontal smoothing, means there is an even number of available wasted moves overall. For the rest of the game, the Simplifier will respond to a wasted move by the Complicator with another wasted move, and will respond to a resolve in the remaining even twist with a resolve of opposite sign. Since the even number of wasted moves available force the Complicator to resolve first in the even number of unresolved crossings in the remaining even twist, the final diagram will be the unknot. See Figure 24 for an example.

Case 6. [The Complicator uses a horizontal smoothing in the odd twist and no unresolved crossings remain in either of the even twists.] There are no remaining moves in either of the even twists, and the strategy followed by the Simplifier (Case 1) means that all crossings in each even twist were resolved with a final local writhe of zero. The only remaining

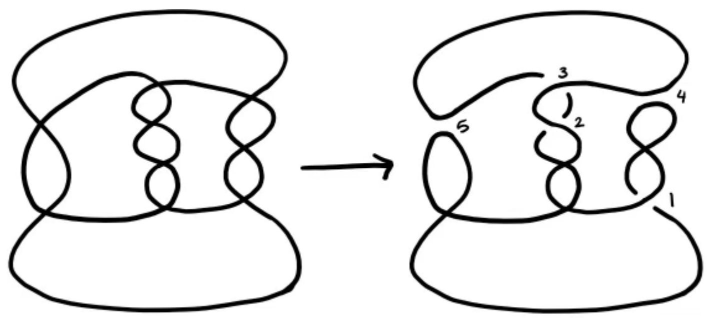

Figure 24: An example of Lemma 5.5, Case 5.

unresolved crossings will now be wasted moves because of the horizontal smoothing move the Complicator just made. The result will thus be the unlink regardless of the Simplifier's next move or any subsequent moves from either player, since only wasted moves remain. See Figure 25 for an example.

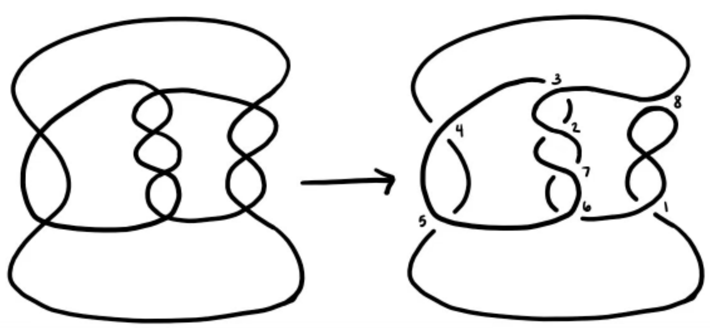

Figure 25: An example of Lemma 5.5, Case 6.

□

Lemma 5.6 *(EEO, Complicator moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where p and q are even, and r is odd. If the Complicator moves first, then the Complicator has a winning strategy.*

Proof. **Case 1.** $[r \geq 3.]$ The Complicator moves first by performing a vertical smoothing move in the odd twist. The configuration is now EEE with the Simplifier moving next.

The Complicator may now use the winning strategy in Lemma 5.2, EEE moving second to win, since that strategy does not require the use of any smoothing moves.

Case 2. $[r = 1.]$ The Complicator moves first by performing a vertical smoothing move in the odd twist. This is now equivalent to a local writhe of zero in that twist. The Complicator may now use the winning strategy in Lemma 5.2, EEE moving second to win. Following that strategy will now ensure that the next twist completed without smoothings has a local writhe of ± 2. The final configuration will be the Hopf Link.

Thus, the Complicator wins.

□

For the KnotLink Game played on a (p, q, r) pretzel link shadow where p and q are odd, and r is even, we only know the outcome for some special cases. We prove that if the Simplifier plays second, then the Simplifier wins. However, if the Complicator plays second, several different outcomes are possible. We'll look at some special cases to see why this case is so vexing.

Theorem 5.7 *(OOE, Complicator moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where p and q are odd, and r is even. If the Complicator moves first, then the Simplifier has a winning strategy.*

Proof.

First, we observe that the Simplifier will get the last move. Also, importantly, because the Simplifier can force the Complicator to move first in the even twist, the Simplifier can make sure to move last in the even twist. With that in mind, let's consider the following strategy.

Strategy for the Even Twist: Any time the Complicator moves in the even twist, the Simplifier will respond as follows in the same even twist:

- If the Complicator resolves a crossing, then the Simplifier resolves with opposite sign.
- If the Complicator uses a vertical smoothing, the Simplifier uses a vertical smoothing also.
- If the Complicator uses a horizontal smoothing, then the Simplifier resolves with any sign in the (formerly) even twist (which now consists only of wasted moves).

Note that by following this part of the strategy, we ensure that the even twist has local writhe zero or has been removed by a horizontal smoothing in the final diagram.

Knowing that the even twist can only turn out to be local writhe zero or removed by horizontal smoothing, the Simplifier need now only ensure that the combined local writhe of the odd twists is zero to know that the final configuration is the unknot or the unlink.

Strategy for the Odd Twists: Any time the Complicator moves in one of the odd twists, the Simplifier will respond as follows:

- If the Complicator resolves a crossing in an odd twist with unresolved crossings remaining, the Simplifier will respond by resolving a crossing with opposite sign in the same twist. This will guarantee that the local writhe of the signs of the crossings in each odd twist is ± 1 by the end of the game if no smoothings are used in the odd twists.

- If the Complicator resolves the last unresolved crossing in one of the odd twists, the Simplifier will resolve a crossing in the other odd twist with opposite sign. Note that there must be an unresolved crossing remaining in the other odd twist because the Simplifier's strategy for the even twist means they must also have the last move in the (combined) odd twists. By resolving with the opposite sign here, the Simplifier ensures that by continuing with their strategy, the local writhe for the last completed odd twist will be opposite of the local writhe for the first completed odd twist. In fact, the two local writhes will be +1 and -1 if no smoothings are used in the odd twists.

- If the Complicator uses a vertical smoothing in an odd twist with remaining unresolved crossings, then the Simplifier will respond with a vertical smoothing in that same odd twist. Note that this simply means the twist is still an odd twist with two fewer unresolved crossings than before.

- If the Complicator uses a vertical smoothing as the last move in an odd twist, then the Simplifier will respond with a vertical smoothing in the other odd twist. Note that based on the Simplifier's strategy, the Complicator must have completed the (first odd, now even) twist with local writhe zero. There also must now be an even number of resolves left in the remaining odd twist, and the Simplifier's strategy will ensure a local writhe of zero in that twist as well.

- If the Complicator uses a horizontal smoothing in either odd twist, the Simplifier can force a win with a horizontal smoothing in any other twist (including the even twist). If no unresolved crossings remain in the even twist or the other odd twist, then the local writhe in the even twist must be zero and the local writhe in the completed odd twist must be ± 1 which will result in the unknot regardless of any remaining moves.

The only possible final configurations following this strategy are local writhes in each twist of $(1, -1, 0)$, $(-1, 1, 0)$, or $(0, 0, 0)$, or configurations with only one or two remaining twists after horizontal smoothing moves. In all cases, the resulting diagram is the unknot or the unlink. Thus, the Simplifier wins.

□

Now, what happens for the KnotLink Game played on a (p, q, r) pretzel link shadow where p and q are odd, and r is even and the Simplifier moves first? It depends! The following partial result in this case can be obtained as a corollary of Theorem 3.2 in [1].

Corollary 5.8 *Suppose the KnotLink Game is played on a $(1, 1, r)$ pretzel link shadow with r even. If the Simplifier moves first, then the Complicator has a winning strategy.*

Proof. In this special case, the pretzel link shadow is equivalent to a rational link shadow of the form $(2, r)$ (with r even) analyzed in [1]. According to Theorem 3.2 in [1], the second player wins on this type of shadow, regardless of their goal. So, for pretzel links of the form $(1, 1, r)$ where the Simplifier moves first, the Complicator has a winning strategy. □

Another special case where we have some knowledge of game outcomes for (p, q, r) where p and q are odd, and r is even is the $(1, q, 2)$ case. Surprisingly, the game outcome for this special case is the opposite of the game outcome for the OOE special case $(1, 1, r)$!

Proposition 5.9 *Suppose the KnotLink Game is played on a $(1, q, 2)$ pretzel link shadow with q odd, $q \geq 3$. If the Simplifier moves first, then the Simplifier has a winning strategy.*

Proof. On the Simplifier's first move they should resolve the single crossing in the first twist with sign $+1$.

Case 1. [Complicator moves in the q-twist.] The first time the Complicator moves in the q-twist, the Simplifier should respond with a horizontal smoothing in the q-twist. This is guaranteed to be possible since $q \geq 3$. For all subsequent moves the Complicator plays in the q-twist, the Simplifier will respond by resolving a crossing in the q-twist with any sign *if a move in this twist is available.*

Case 2. [Complicator moves first in the 2-twist.]

Subcase 2-A. If the Complicator resolves a crossing in the 2-twist, the Simplifier should resolve the other crossing in the 2-twist with opposite sign, creating an RII move.

Subcase 2-B. If the Complicator performs a horizontal or vertical smoothing in the 2-twist, the Simplifier will resolve the other crossing in the 2-twist, say with sign -1.

Case 3. [Simplifier moves first in the 2-twist.] The Simplifier should resolve a crossing with sign -1.

Let's now consider the effect this combination of strategies has on the game outcome. Since, in every case, the q-twist is smoothed horizontally, all q-twist crossings can be undone with RI moves, leaving at most three crossings in the diagram (coming from the 1-twist and the 2-twist). The only precrossings that determine the final game board are the precrossing in the first twist, which was resolved with a $+1$, and the two precrossings in the 2-twist. Based on our strategy, if the 2-twist precrossings are both resolved, then the local writhe of these three crossings is guaranteed to be ± 1. Any 3-crossing knot diagram with writhe ± 1 is the unknot. If, on the other hand, the Complicator chose to smooth a crossing in the 2-twist, a horizontal smoothing would instantly transform the diagram into an unknot (or unlink) diagram. If the Complicator chose to smooth vertically in the 2-twist, the Simplifier would have chosen the other 2-twist crossing to be a -1 crossing, which would produce an unlink. □

The only remaining type of pretzel link to consider is that with all odd twists. Below we show the result for when the Simplifier moves first, as well as a partial result in the case where the Complicator moves first.

Theorem 5.10 *(OOO, Simplifier moves first.) Suppose the KnotLink Game is played on a (p, q, r) pretzel link shadow where $p, q,$ and r are odd. If the Simplifier moves first, then the Simplifier has a winning strategy.*

Proof. The Simplifier can begin with a horizontal smoothing move in any twist. This leaves an even number of wasted moves in that twist and two odd twists in place. This is a 2-component $(2, m)$-torus link shadow, with m even. Since there are an even number of wasted moves, the Simplifier can force the Complicator to move first in the two odd twists.

The Complicator is already quite constrained at this point. If the Complicator only uses resolves in the twist crossings, the Simplifier will always reply with a resolve of opposite sign in one of the twists that creates an RII move. This will leave the unlink at the end of the game, and the Simplifier will win. Thus, we know the Complicator must try a smoothing move at some point to have a chance at winning. But smoothing in either direction on a wasted crossing would not affect the game outcome, and a horizontal smoothing move in either of the other two twists would also grant the Simplifier a win since this would transform the diagram immediately into a twisted-up unknot.

So, we know that the Complicator must use a vertical smoothing move at some point in the game to have any hope of winning against the Simplifier's strategy. But if the Complicator uses a vertical smoothing, this will shift the underlying diagram from a $(2, m)$-torus link pseudodiagram where m is even to a $(2, n)$-torus knot pseudodiagram where n is odd. Furthermore, we know that when the Complicator makes this smoothing move, the Simplifier's strategy has thus far ensured a global writhe of the resolved, non-wasted crossings equal to zero. The Simplifier must move next with a resolve somewhere, which will change the writhe of resolved, non-wasted crossings to ± 1. The Complicator can resolve in a twist using the same sign to push the writhe to ± 2, but the Simplifier will return the writhe back to ± 1 with each reply. This pattern will repeat until the Simplifier's last move, where the Simplifier can ensure the final diagram has a writhe of ± 1. A $(2, n)$-torus knot with writhe ± 1 is the unknot, and so the Simplifier wins. □

While we have a nice result if the Simplifier moves first on a pretzel link with all odd twists, the situation when the Complicator moves first is murky. The following partial result in this case can be obtained as a corollary of Theorem 3.5 in [1].

Corollary 5.11 *Suppose the KnotLink Game is played on a $(1, 1, r)$ pretzel link shadow, with r odd. Then the Simplifier has a winning strategy regardless of who plays first.*

Proof. A pretzel link of the form $(1, 1, r)$ is equivalent to a rational link of the form $(2, r)$ (following the conventions of [1]). Since this case (where r is odd) is a KnotLink Game played on a rational link shadow with one odd and one even twist, Theorem 3.5 in [1] guarantees the Simplifier has a winning strategy regardless of who plays first. □

6 Conclusion

We conclude our investigation by collecting the game outcomes from our work and the work of our predecessors in [1].

Do you notice something interesting? There are knot or link diagrams for all outcome classes *except for* the outcome class where the Complicator wins playing either first or second. This leads to a question we're interested in knowing the answer to:

	C moves first, C wins	C moves first, S wins
S moves first, C wins		E Whitehead EEE pretzel EE rational [1]
S moves first, S wins	O Whitehead EEO pretzel	$(2, p)$-torus link [1] OE rational [1] EO rational [1] OO rational [1]

Table 2: Game outcomes summary from this paper and previous research.

Q: Is there a knot or link shadow for which the Complicator has a winning strategy regardless of whether they move first or second?

As you, dear reader, explore the KnotLink Game on diagrams that have not yet been studied you might consider this question. Good luck, and have fun playing!

Acknowledgments

We would like to thank the Seattle University Mathematics Department for their encouragement of our work.

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Received: June 6, 2023 **Accepted**: January 2, 2025
Communicated by Daphne Der-Fen Liu