Using A Subset of Orthonormalized Bernstein Polynomials to Solve Boundary Value Problems

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Abstract - An explicit formula for an orthornomalized subset Bernstein polynomial basis is derived in order to solve linear boundary value problems with Dirichlet conditions via the Galerkin method.

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1 Introduction

We are interested in studying numerical solutions to second-order boundary value problems consisting of the equation

$$u'' = f(t, u, u'), \quad t \in (0, 1) \tag{1.1}$$

and boundary conditions (BCs)

$$u(0) = 0, \quad u(1) = 0, \tag{1.2}$$

by utilizing the Galerkin method involving Bernstein polynomials.

The Bernstein polynomials of degree n, defined by

$$B_{i,n}(t) := \begin{cases} \binom{n}{i} t^i (1-t)^{n-i}, & n \in \mathbb{N}, \quad 0 \le i \le n, \\ 0, & i < 0 \text{ or } i > n, \end{cases}$$

were introduced in 1912 by Sergei Natanovich Bernstein in a constructive proof of the Weierstrass approximation theorem [2]. Since then, Bernstein polynomials have been used in many areas of pure and applied mathematics as well as computer science. For example, Bernstein polynomials provide an explicit polynomial representation of Bézier curves and surfaces, which have been used extensively in modern robotics and computer aided geometric design. Specifically, Bézier curves have found many applications in models of smooth curves in computer graphics and animation, the design and development of new fonts, and the production of mechanical components for large-scale industrial use [15, 17].

A more comprehensive survey of Bézier curves and the role of Bernstein polynomials can be found in [4]. Bernstein polynomials have been also used in the design of filter sharpening functions in signal processing [12], the modeling of intermolecular potential energy surfaces [7], as well as stability analysis for polynomial dynamical systems [13, 14] [15]. Of particular interest in this paper is the use of Bernstein polynomials in numerical methods for solving both ordinary and partial differential equations [8, 11, 18].

The authors in [8] considered BVP (1.1), (1.2) by applying the Galerkin method using the full Bernstein polynomial basis, $\{B_{i,n}(t)\}_{i=0}^{n}$ to create polynomial approximate solutions to BVP (1.1), (1.2). In particular, they studied the nonhomgeneous, second-order, linear differential equation of the form

$$u'' + u = f(t), \quad t \in (0,1),$$

with BC (1.2). By assuming an approximation to the solution of the form

$$u(t) = \sum_{i=0}^{n} c_i B_{i,n}(t), \quad n \geq 1,$$

they showed that such an approximate solution must satisfy

$$\sum_{i=0}^{n} \left\{ \int_0^1 [-B'_{i,n}(t)B'_{j,n}(t) + B_{i,n}(t)B_{j,n}(t)]\, dt \right\} c_i - \int_0^1 f(t)B_{j,n}(t)\, dt = 0.$$

This is equivalent to solving an $n+1$ by $n+1$ system $\mathbf{BC} = \mathbf{b}$ for $\mathbf{C}$, where $\mathbf{B}$ has entries

$$B_{i,j} = \int_0^1 [-B'_{i,n}(t)B'_{j,n}(t) + B_{i,n}(t)B_{j,n}(t)]\, dt$$

and $\mathbf{b}$ has entries

$$b_j = \int_0^1 f(t)B_{j,n}(t)\, dt.$$

The boundary conditions were then applied by deleting the first row, first column, last row, last column on the matrix $\mathbf{B}$.

The authors in [5] applied the Galerkin method using orthonormalized Bernstein polynomials. In their method, they used the orthonormalized version of $\{B_{i,n}(t)\}_{i=0}^{n}$, i.e.,

$$\phi_{j,n}(t) = \sqrt{2(n-j)+1} \sum_{k=0}^{j} (-1)^k \frac{\binom{2n+1-k}{j-k}\binom{j}{k}}{\binom{n-k}{j-k}} B_{j-k,n-k}(t), \quad j = 0, \ldots, n,$$

that is stated in [1].

It is noted that the $B_{0,n}(t)$ and $B_{n,n}(t)$ do not satisfy BC (1.2). However, the subset $\{B_{i,n}(t)\}_{i=1}^{n-1}$ do satisfy BC (1.2). Motivated by this fact and the results in [8] as well as [5], we are interested in the effects of using an orthonormalized version of $\{B_{i,n}(t)\}_{i=1}^{n-1}$ as the basis functions for our approximating solution for BVP (1.1), (1.2) using the Galerkin method.

While the Gram-Schmidt process is an effective tool for orthonormalization, having an explicit formula for the orthonormalized subset of Bernstein polynomials, i.e., $\{B_{i,n}(t)\}_{i=1}^{n-1}$, would avoid numerical integration, reducing roundoff error, and can be implemented quicker in many programming languages. As a result, we proceed with first introducing the explicit formula of the orthonormalized set $\{B_{i,n}(t)\}_{i=1}^{n-1}$ (cf. Theorem 2.4) and showing two different interesting proofs of that result, one involving Jacobi polynomials, and the other involving a coefficient extraction method. We next apply our result to BVP (1.1), (1.2) to study the effects of using orthonormal basis functions.

2 Orthonormalized Bernstein Basis

Let Π^n be the inner product space of polynomials defined on $[0,1]$ with real coefficients and are of degree at most n or less. The inner product is defined by

$$\langle p, q\rangle_{\Pi^n} = \int_0^1 p(t)q(t)\,dt, \qquad p, q \in \Pi^n.$$

We equip Π^n with the induced norm $\|p\|_{\Pi^n} = \sqrt{\langle p, p\rangle_{\Pi^n}}$.

Lemma 2.1 $\{B_{i,n}(t)\}_{i=0}^{n}$ *forms a basis for* Π^n.

Proof. Since Π^n is of dimension $n+1$, it is enough to show that $\{B_{i,n}(t)\}_{i=0}^{n}$ are linearly independent. By induction, it easy to see that $\{B_{0,0}(t)\} = \{1\}$ is linearly independent for Π^0. It is, therefore, a basis for Π^0.
Assuming that $\{B_{i,n-1}(t)\}_{i=0}^{n-1}$ is a basis for Π^{n-1}, consider

$$0 = \sum_{i=0}^{n} c_i B_{i,n}(t)$$

where $c_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n+1$. Taking the derivative of both sides and simplifying, we have

$$\begin{aligned}
0 &= \sum_{i=0}^{n} c_i B'_{i,n}(t) \\
&= \sum_{i=0}^{n} c_i n(B_{i-1,n-1}(t) - B_{i,n-1}(t)) \\
&= n\sum_{i=1}^{n-1} c_i B_{i-1,n-1}(t) - \sum_{i=0}^{n-1} c_i B_{i,n-1}(t) \\
&= n\sum_{i=0}^{n-2}(c_{i+1} - c_i)B_{i,n-1}(t) - c_{n-1}B_{n-1,n-1}(t).
\end{aligned}$$

Since $\{B_{i,n-1}(t)\}_{i=0}^{n-1}$ is a basis for Π^{n-1}, then it follows that $c_{n-1} = 0$ and $c_{i+1} - c_i = 0$ for $i = 0, 1, \ldots, n-2$. This implies that $c_i = 0$ for $i = 1, 2, \ldots, n-1$. This leaves $c_n B_{n,n}(t) = 0$

implying that $c_n = 0$. Hence, $\{B_{i,n}(t)\}_{i=0}^{n}$ is linearly independent, and thus, a basis for Π^n. This completes the proof. □

Lemma 2.2 *For* $0 \le i \le n$,

$$B'_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

Proof. For any $0 \le i \le n$, we have

$$\begin{aligned} B'_{i,n}(t) &= \binom{n}{i}[it^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}] \\ &= n\left(\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-1-i)!}t^i(1-t)^{n-i-1}\right) \\ &= n\left(\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i} - \binom{n-1}{i}t^i(1-t)^{n-i-1}\right) \\ &= n(B_{i-1,n-1}(t) - B_{i,n-1}(t)). \end{aligned}$$

□

Define $^\circ\Pi^n := \{p \in \Pi^n : p(0) = p(1)\}$. Clearly, this is a subspace of Π^n and $\langle p, q\rangle_{^\circ\Pi^n} = \langle p, q\rangle_{\Pi^n}$.

Lemma 2.3 $\{B_{i,n}(t)\}_{i=1}^{n-1}$ *forms a basis for* $^o\Pi^n$.

Proof. The dimension of $^o\Pi^n$ is $n-1$. Since $\{B_{i,n}(t)\}_{i=1}^{n-1}$ are linearly independent and each satisfy $B_{i,n}(0) = B_{i,n}(1) = 0$, then $\{B_{i,n}(t)\}_{i=1}^{n-1}$ is a basis for $^o\Pi^n$. □

Let $\{H_{i,n}(t)\}_{i=1}^{n-1}$ be the set of orthonormal basis functions produced by applying the Gram-Schmidt process to $\{B_{i,n}(t)\}_{i=1}^{n-1}$. The following theorem gives the explicit form of $H_{i,n}(t)$, for $1 \le i \le n-1$.

Theorem 2.4 *For* $n \ge 2$, *the set of orthonormal basis functions* $\{H_{i,n}(t)\}_{i=1}^{n-1}$ *can be written as*

$$H_{i,n}(t) = \sqrt{\frac{(2n+1-i)(2n+2-i)(2n+1-2i)}{i(i+1)}} \sum_{k=0}^{i-1} (-1)^k \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{n-k}{i-k}} B_{i-k,n-k}(t).$$

While certainly one proof is enough, there are two interesting proofs we provide that demonstrate Theorem 2.4. We provide both for enlightenment. The first proof utilizes Jacobi polynomials and will require integration. The second proof we provide of Theorem 2.4 utilizes a coefficient extraction approach and will require combinatorics.

2.1 Jacobi Polynomial Approach

We next state the definition of the Jacobi polynomials along with two properties used in the proof of Theorem 2.4.

Definition 2.5 *Given $\alpha, \beta > -1$ and $n \in \mathbb{N}_0$, the Jacobi polynomial $P_n^{\alpha,\beta}$ is defined by*

$$P_n^{\alpha,\beta}(t) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{t-1}{2}\right)^m.$$

Lemma 2.6

$$\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_m^{\alpha,\beta}(t) P_n^{\alpha,\beta}(t)\, dt = \delta_{mn},$$

where

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases},$$

and $\alpha, \beta > -1$.

Proof. See [16, Corollary 3.6]. □

Lemma 2.7

$$P_n^{\alpha,\beta+1}(t) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{l=0}^{n} \frac{(2l+\alpha+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)} P_l^{\alpha,\beta}(t)$$

Proof. See [16, Remark 3.3 and Theorem 3.18]. □

We state Rodrigues' formula for clarity, see [16, Theorem 3.17].

Lemma 2.8

$$(1-t)^\alpha (1+t)^\beta P_n^{\alpha,\beta}(t) = \frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dt^n} \left[(1-t)^{n+\alpha}(1+t)^{n+\beta}\right]$$

The following lemma gives a relationship between Jacobi polynomials and Bernstein polynomials.

Lemma 2.9 *For $1 \leq i \leq n-1$, we have*

$$(-1)^{i-1} t(1-t)^{n-i} P_{i-1}^{2,2n-2i+1}(1-2t) = \sum_{l=0}^{i-1} \frac{\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{\binom{n-l}{i-l}} B_{i-l,n-l}(t).$$

Proof.

$$\sum_{l=0}^{i-1}(-1)^l \frac{\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{\binom{n-l}{i-l}} B_{i-l,n-l}(t) = \sum_{l=0}^{i-1}(-1)^l \binom{2n+1-l}{i-1-l}\binom{i+1}{l} t^{i-l}(1-t)^{n-i}$$
$$= t^i(1-t)^{n-i}\sum_{l=0}^{i-1}(-1)^l \binom{2n+1-l}{i-1-l}\binom{i+1}{l} t^{-l}$$
$$= (-1)^{i-1}t^i(1-t)^{n-i}\sum_{k=0}^{i-1}(-1)^k \binom{2n+2-i+k}{k}\binom{i+1}{i-1-k} t^{k-i+1}$$
$$= (-1)^{i-1}t(1-t)^{n-i}\sum_{k=0}^{i-1} \binom{2n+2-i+k}{k}\binom{i+1}{i-1-k} (-t)^{k}$$
$$= (-1)^{i-1}t(1-t)^{n-i}\sum_{k=0}^{i-1} \frac{(2n+2-i+k)!(i+1)!}{k!(2n+2-i)!(i-1-k)!(k+2)!}(-t)^{k}$$
$$= (-1)^{i-1}t(1-t)^{n-i}\cdot\frac{(i+1)!}{(i-1)!(2n+2-i)!}\times$$
$$\sum_{k=0}^{i-1}\frac{(i-1)!(2n+2-i+k)!}{k!(k+2)!(i-1-k)!}\left(\frac{(1-2t)-1}{2}\right)^k$$
$$= (-1)^{i-1}t(1-t)^{n-i}P_{i-1}^{2,2n-2i+1}(1-2t)$$

□

We now provide the proof of Theorem 2.4 using Jacobi polynomials.

Proof of Theorem 2.4. Consider $\langle H_{i,n}, H_{j,n}\rangle_{\circ\Pi^n}$, for $1 \leq i, j \leq n-1$. We first show that for $i = j$, $\|H_{i,n}\|^2_{\circ\Pi^n} = 1$. We have by Lemma 2.9.

$$I_1 = \frac{i(i+1)}{(2n-i+1)(2n-i+2)(2n+1-2i)}\|H_{i,n}\|^2_{\circ\Pi^n}$$
$$= \int_0^1\left(\sum_{k=0}^{i-1}(-1)^k\frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{n-k}{i-k}}B_{i-k,n-k}(t)\right)^2 dt$$
$$= \int_0^1 (1-t)^{2n-2i}t^2[P_{i-1}^{2,2n-2i+1}(1-2t)]^2.$$

Let $u = 1 - 2t$. Then from Lemmas 2.7 and 2.6, we have

$$\begin{aligned}
I_1 &= 2^{-2n+2i-1} \int_{-1}^{1} (1-u)^2 (1+u)^{2n-2i} [P_{i-1}^{2,2n-2i+1}(u)]^2 \, du \\
&= 2^{-2n+2i-1} \int_{-1}^{1} (1-u)^2 (1+u)^{2n-2i} \times \\
&\quad \left(\frac{(i+1)!}{(2n-i+2)!} \sum_{k=0}^{i-1} (-1)^{i-1-k} \frac{(2n+2k-2i+3)(2n+k-2i+2)!}{(k+2)!} P_k^{2,2n-2i}(u) \right)^2 du. \\
&= \frac{2^{-2n+2i-1}[(i+1)!]^2}{[(2n-i+2)!]^2} \times \\
&\quad \sum_{k=0}^{i-1} \left[\frac{(2n+2k-2i+3)(2n+k-2i+2)!}{(k+2)!} \right]^2 \int_{-1}^{1} (1-u)^2 (1+u)^{2n-2i} [P_k^{2,2n-2i}]^2 \, du \\
&= \frac{2^{-2n+2i-1}[(i+1)!]^2}{[(2n-i+2)!]^2} \times \\
&\quad \sum_{k=0}^{i-1} \left[\frac{(2n+2k-2i+3)(2n+k-2i+2)!}{(k+2)!} \right]^2 \frac{2^{2n-2i+1}(k+2)!(2n-2i+k)!}{(2n+2k-2i+3)k!(2n-2i+2+k)!} \\
&= \left[\frac{(i+1)!}{(2n-i+2)!} \right]^2 \sum_{k=0}^{i-1} \frac{(2n-2i+2k+3)(2n-2i+k+2)!(2n-2i+k)!}{k!(k+2)!} \\
&= \left[\frac{(i+1)!(2n-2i)!}{(2n-i+2)!} \right]^2 \cdot \sum_{k=0}^{i-1} (2n-2i+2k+3) \binom{2n-2i+k+2}{k+2} \binom{2n-2i+k}{k} \\
&= \left[\frac{(i+1)!(2n-2i)!}{(2n-i+2)!} \right]^2 \frac{i(i+2)}{2n-2i+1} \cdot \binom{2n-i}{i} \binom{2n-i+2}{i+2} \\
&= \frac{i(i+1)}{(2n-i+1)(2n-i+2)(2n+1-2i)},
\end{aligned}$$

which completes the proof for this case.

Now assume $i \neq j$. Without loss of generality, further assume that $i > j$. For simplicity, let

$$ {}_n q_i = \sqrt{\frac{i(i+1)}{(2n-i+1)(2n-i+2)(2n+1-2i)}}. $$

We define ${}_nq_j$ similarly. By Lemma 2.9 and letting $u = 1 - 2t$, we have

$$
\begin{aligned}
J_1 &= {}_nq_i \cdot {}_nq_j \int_0^1 \left(\sum_{k=0}^{i-1} (-1)^k \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{n-k}{i-k}} B_{i-k,n-k}(t) \right) \times \\
&\qquad\qquad \left(\sum_{l=0}^{j-1} (-1)^l \frac{\binom{2n+1-l}{j-1-l}\binom{j+1}{l}}{\binom{n-l}{j-l}} B_{j-l,n-l}(t) \right) dt \\
&= {}_nq_i \cdot {}_nq_j \int_0^1 (1-t)^{2n-(i+j)} t^2 P_{i-1}^{2,2n-2i+1}(1-2t)\, P_{j-1}^{2,2n-2j+1}(1-2t)\, dt \\
&= {}_nq_i \cdot {}_nq_j 2^{-2n+(i+j)-1} \int_{-1}^1 (1-u)^2 (1+u)^{2n-(i+j)} P_{i-1}^{2,2n-2i+1}(u)\, P_{j-1}^{2,2n-2j+1}(u)\, dt.
\end{aligned}
$$

Using integration by parts along with Lemma 2.8 we have

$$
\begin{aligned}
J_1 &= {}_nq_i \cdot {}_nq_j 2^{-2n+(i+j)-1} \int_{-1}^1 \left[(1-u)^2(1+u)^{2n-i} P_{i-1}^{2,2n-2i+1}(u)\right] (1+u)^{-j} P_{j-1}^{2,2n-2j+1}(u)\, du \\
&= {}_nq_i \cdot {}_nq_j 2^{-2n+(i+j)-1} \int_{-1}^1 \left[(1-u)^2(1+u)^{2n-2i+1} P_{i-1}^{2,2n-2i+1}(u)\right] \times \\
&\qquad\qquad (1+u)^{i-j-1} P_{j-1}^{2,2n-2j+1}(u)\, du \\
&= {}_nq_i \cdot {}_nq_j \frac{(-1)^{i-1} 2^{-2n+j}}{(i-1)!} \int_{-1}^1 \partial_u^{i-1} \left\{(1-u)^{i+1}(1+u)^{2n-i}\right\} (1+u)^{i-j-1} P_{j-1}^{2,2n-2j+1}(u)\, du \\
&= {}_nq_i \cdot {}_nq_j \frac{(-1)^{i-1} 2^{-2n+j}}{(i-1)!} \int_{-1}^1 (1-u)^{i+1}(1+u)^{2n-i} \partial_u^{i-1} \left\{(1+u)^{i-j-1} P_{j-1}^{2,2n-2j+1}(u)\right\} du \\
&= 0.
\end{aligned}
$$

The last line is the result of taking $i-1$ derivatives of a degree $i-j-1+j-1 = i-2$ polynomial. ∎

2.2 The Coefficient Extraction Approach

We define for any power series $A(z) = \sum_{k=0}^\infty a_k z^k$

$$[z^k]A(z) = a_k, k \in \mathbb{N}_0,$$

where $[\cdot]$ denotes the "coefficient of" operator as introduced in [10] and [6, Section 5.4]. That is, $[z^n]A(z)$ denotes the coefficient of z^n in $A(z)$. In particular, we note that

$$[z^k](1+z)^n = \binom{n}{k}, \qquad k, n \in \mathbb{N}_0,\ k \le n. \tag{2.3}$$

Lemma 2.10 *Let $n, m,$ and k be nonnegative integers satisfying $n \ge 2$ and $0 \le k \le m \le n-2$. Then*

$$\sum_{l=0}^m (-1)^l \binom{m}{l} \binom{2n+1-l}{k} \binom{2m+2-k-l}{m-k} = \binom{m}{k}.$$

Proof. As defined in (2.3), let $z, w > 0$ be such that

$$[w^k z^{m-k}]\left((1+w)^{2n+1-l}(1+z)^{2m+2-k-l}\right) = \binom{2n+1-l}{k}\binom{2m+2-k-l}{m-k}. \tag{2.4}$$

It follows that

$$\begin{aligned}
&\sum_{l=0}^{m}(-1)^l\binom{m}{l}\binom{2n+1-l}{k}\binom{2m+2-k-l}{m-k} \\
&\quad= \sum_{l=0}^{m}(-1)^l\binom{m}{l}[w^k z^{m-k}]\left((1+w)^{2n+1-l}(1+z)^{2m+2-k-l}\right) \\
&\quad= [w^k z^{m-k}]\left((1+w)^{2n+1}(1+z)^{2m+2-k}\sum_{l=0}^{m}\binom{m}{l}\left(\frac{-1}{(1+w)(1+z)}\right)^l\right) \\
&\quad= [w^k z^{m-k}]\left((1+w)^{2n+1}(1+z)^{2m+2-k}\left(1-\frac{1}{(1+w)(1+z)}\right)^m\right) \\
&\quad= [w^k z^{m-k}]\left((1+w)^{2n-m+1}(1+z)^{m+2-k}(w(1+z)+z)^m\right) \\
&\quad= [w^k z^{m-k}]\left((1+w)^{2n-m+1}(1+z)^{m+2-k}\sum_{r=0}^{m}\binom{m}{r}w^r(1+z)^r z^{m-r}\right) \\
&\quad= [w^k z^{m-k}]\sum_{r=0}^{m}\binom{m}{r}w^r z^{m-r}(1+w)^{2n-m+1}(1+z)^{m+2-k+r} \\
&\quad= \sum_{r=0}^{m}\binom{m}{r}[w^{k-r}z^{r-k}](1+w)^{2n-m+1}(1+z)^{m+2-k+r} \\
&\quad= \sum_{r=0}^{m}\binom{m}{r}\binom{2n-m+1}{k-r}\binom{m+2-k+r}{r-k} \\
&\quad= \binom{m}{k}
\end{aligned}$$

where the last summation is only valid when $r = k$. □

Lemma 2.11 *Let n and m be positive integers satisfying $n \geq 2$ and $0 \leq m \leq n-2$. Then*

$$\sum_{k=0}^{m}(-1)^k\binom{m+2}{k}\binom{2n+1-k}{m-k} = \binom{2n-m-1}{m}.$$

Proof. Let $s > 0$ satisfy

$$[s^{2n-m+1}](1+s)^{2n+1-k} = \binom{2n+1-k}{m-k}.$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k\binom{m+2}{k}\binom{2n+1-k}{m-k} \\
&\quad= \sum_{k=0}^{m}(-1)^k\binom{m+2}{k}[s^{2n-m+1}](1+s)^{2n+1-k} \\
&\quad= [s^{2n-m+1}](1+s)^{2n+1}\sum_{k=0}^{m}\binom{m+2}{k}(1+s)^{-k} \\
&\quad= [s^{2n-m+1}](1+s)^{2n+1}\ \times \\
&\qquad\left[\left(1-\frac{1}{1+s}\right)^{m+2}-\left((m+2)\left(\frac{-1}{1+s}\right)^{m+1}+\left(\frac{-1}{1+s}\right)^{m+2}\right)\right] \\
&\quad= [s^{2n-m+1}](1+s)^{2n+1}\left[\left(\frac{s}{1+s}\right)^{m+2}-\left(\frac{-1}{1+s}\right)^{m+1}\left((m+2)-\frac{1}{1+s}\right)\right] \\
&\quad= [s^{2n-m+1}](1+s)^{2n-m-1}\left[s^{m+2}+(-1)^{m+2}((m+2)(1+s)-1)\right] \\
&\quad= [s^{2n-m+1}](1+s)^{2n-m-1}s^{m+2}+(-1)^{m+2}[s^{2n-m+1}](1+s)^{2n-m-1}((m+1)(1+s)+s) \\
&\quad= [s^{2n-2m-1}](1+s)^{2n-m-1}\ + \\
&\qquad(-1)^{m+2}\left([s^{2n-m+1}](m+1)(1+s)^{2n-m}+[s^{2n-m}](1+s)^{2n-m-1}\right) \\
&\quad= \binom{2n-m-1}{m},
\end{aligned}$$

where the second term vanishes as the result of taking the coefficients of s^{2n-m+1} and s^{2n-m} in polynomials of degree $2n-m$ and $2n-m-1$ in s, respectively. ☐

Lemma 2.12 *Assume* $0 \le k < i < j \le n-1$, *then*

$$\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{2n+1-l}{k}\binom{i+j-k-l}{i-1-k} = 0.$$

Proof. Define $u > 0$ and $v > 0$ such that

$$[u^{i-1-k}](1+u)^{i+j-k-l} = \binom{i+j-k-l}{i-1-k}$$

and

$$[v^k](1+v)^{2n+1-l} = \binom{2n+1-l}{k}.$$

$$
\begin{aligned}
\sum_{l=0}^{j-1}&(-1)^l\binom{j-1}{l}\binom{2n+1-l}{k}\binom{i+j-k-l}{i-1-k}\\
&=\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{2n+1-l}{k}[u^{i-1-k}](1+u)^{i+j-k-l}\\
&=[u^{i-1-k}](1+u)^{i+j-k}\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}[v^k](1+v)^{2n+1-l}(1+u)^{-l}\\
&=[u^{i-1-k}](1+u)^{i+j-k}[v^k](1+v)^{2n+1}\sum_{l=0}^{j-1}\binom{j-1}{l}\left(\frac{-1}{(1+u)(1+v)}\right)^l\\
&=[u^{i-1-k}](1+u)^{i+j-k}[v^k](1+v)^{2n+1}\left(1-\frac{1}{(1+u)(1+v)}\right)^{j-1}\\
&=[u^{i-1-k}](1+u)^{i+1-k}[v^k](1+v)^{2n-j+2}\left(u(1+v)+v\right)^{j-1}\\
&=[u^{i-1-k}](1+u)^{i+1-k}[v^k](1+v)^{2n-j+2}\sum_{r=0}^{j-1}\binom{j-1}{r}u^r(1+v)^rv^{j-1-r}\\
&=\sum_{r=0}^{j-1}\binom{j-1}{r}[u^{i-1-k-r}](1+u)^{i+1-k}[v^{k-j+1+r}](1+v)^{2n-j+2+r} \qquad (2.5)\\
&=0,
\end{aligned}
$$

since the assumption that $0 \le k < i < j$ in (2.5) implies that $k-j+1+r<0$ whenever $r<j-k-1$, and $i-1-k-r<0$ whenever $r \ge j-k-1$.

□

Proof of Theorem 2.4. Consider $\langle H_{i,n}(t), H_{j,n}(t)\rangle_{\circ\Pi^n}$ for $1 \le i, j \le n-1$. We first show that if $i=j$, then $\|H_{i,n}(t)\|^2_{\circ\Pi^n}=1$. By the inner product condition in [9, Lemma 1],

$$
\begin{aligned}
I_2 =&\frac{i(i+1)}{(2n-i+1)(2n-i+2)}\|H_{i,n}(t)\|^2_{\circ\Pi^n}\\
&=(2n-2i+1)\sum_{k=0}^{i-1}\sum_{l=0}^{i-1}(-1)^{k+l}\frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{(2n+1-k-l)\binom{2n-k-l}{2i-k-l}}\\
&=\sum_{k=0}^{i-1}\sum_{l=0}^{i-1}(-1)^{k+l}\frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{\binom{2n+1-k-l}{2i-k-l}}\\
&=\frac{1}{\binom{2n-1+2}{i+1}}\sum_{k=0}^{i-1}(-1)^k\frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{i-1}{k}}\sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\binom{2n+1-l}{k}\binom{2i-k-l}{i-1-k}.
\end{aligned}
$$

By Lemma 2.10, with $m = i - 1$, we have

$$I_2 = \frac{1}{\binom{2n-1+2}{i+1}} \sum_{k=0}^{i-1} (-1)^k \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{i-1}{k}} \binom{i-1}{k}$$
$$= \frac{1}{\binom{2n-1+2}{i+1}} \sum_{k=0}^{i-1} (-1)^k \binom{2n+1-k}{i-1-k}\binom{i+1}{k}.$$

By Lemma 2.11, with $m = i - 1$, we have

$$I_2 = \frac{\binom{2n-i}{i-1}}{\binom{2n-1+2}{i+1}} = \frac{i(i+1)}{(2n-i+1)(2n-i+2)},$$

which completes the proof for this case.

Now, suppose that $i \neq j$. Without loss of generality, also assume that $j > i$. Following similarly to the proof for the case when $i = j$, we have

$$J_2 = (2n-i-j+1) \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} (-1)^{k+l} \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{(2n+1-k-l)\binom{2n-k-l}{i+j-k-l}}$$
$$= \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} (-1)^{k+l} \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}\binom{2n+1-l}{i-1-l}\binom{i+1}{l}}{\binom{2n+1-k-l}{i+j-k-l}}$$
$$= \frac{j(j+1)}{\binom{2n-j+2}{i-1}} \sum_{k=0}^{i-1} (-1)^k \frac{\binom{2n+1-k}{i-1-k}\binom{i+1}{k}}{\binom{i-1}{k}} \times$$
$$\sum_{l=0}^{j-1} (-1)^l \binom{j-1}{l}\binom{2n+1-l}{k}\binom{i+j-k-l}{i-1-k}$$

By the assumption that $0 \leq k < i < j \leq n-1$, we have that the inner sum vanishes by Lemma 2.12, which completes the proof for this case. □

3 Application to Boundary Value Problems

The orthonormal subset of Bernstein polynomials can be useful in solving BVP (1.1), (1.2) when a scaled linear term is present. This is due to the orthogonality property of the basis polynomials. The following examples highlight this.

Example 3.1 Consider the nonhomogeneous, second-order, linear BVP consisting of the equation

$$u'' + qu = f(t), \quad t \in (0,1), \tag{3.6}$$

with BC (1.2), where $q \in \mathbb{R}$. We seek approximate solutions of the form

$$u_n(t) = \sum_{i=1}^{n-1} c_i H_{i,n}(t), \quad n \geq 2.$$

Substituting this into the equation given in BVP (3.6), multiplying through by $H_{j,n}(t)$, for some $j = 1, \dots, n-1$, and applying integration by parts, we have

$$\int_0^1 \sum_{i=1}^{n-1} c_i [H''_{i,n}(t) H_{j,n}(t) + q H_{i,n}(t) H_{j,n}(t)]\, dt = \int_0^1 H_{j,n}(t) f(t)\, dt$$
$$\sum_{i=1}^{n-1} c_i \int_0^1 [-H'_{i,n}(t) H'_{j,n}(t) + q H_{i,n}(t) H_{j,n}(t)]\, dt = \int_0^1 H_{j,n}(t) f(t)\, dt.$$

This is equivalent to solving the $(n-1) \times (n-1)$ system $\mathbf{Bc} = \mathbf{b}$, where

$$\mathbf{B} = [b_{ij}] = \begin{cases} \displaystyle\int_0^1 q - [H'_{i,n}(t)]^2\, dt, & i = j \\ \displaystyle\int_0^1 -H'_{i,n}(t) H'_{j,n}(t)\, dt, & i \neq j \end{cases}$$

with

$$\int_0^1 q H_{i,n}(t) H_{j,n}(t)\, dt = \begin{cases} q, & i = j \\ 0, & i \neq j \end{cases}$$

due to the orthonormal property of $H_{i,j}(t)$;

$$\mathbf{c} = (c_1, \dots, c_{n-1})^T; \tag{3.7}$$

and

$$\mathbf{b} = [b_j] = \int_0^1 H_{j,n}(t) f(t)\, dt, \quad j = 1, 2, \dots, n-1. \tag{3.8}$$

As in [8], if we let $q(t) \equiv 1$ and $f(t) = t^2 e^{-t}$ it is easy to show that the solution to BVP (3.6) is

$$u(t) = -\frac{1}{2}\cos(t) + \frac{\cos(1) - 4e^{-1}}{2\sin(1)}\sin(t) + \frac{1}{2}e^{-t}(1+t)^2.$$

Using only 10 orthonormal basis functions, we can use $u_{11}(t)$ to approximate $u(t)$ with very little difference between them as Figure 1a demonstrates. In fact, Figure 1b shows the absolute error is less than 1.14×10^{-14}. This is an improvement to what was done in [8] as the authors used 45 Bernstein polynomials to achieve a similar result.

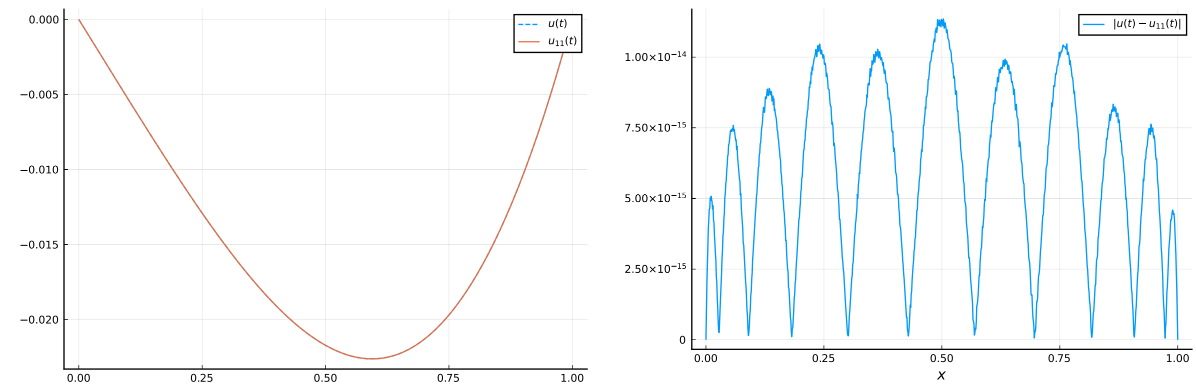
Example 3.2 Consider the BVP consisting of the equation

$$u'' + p(t)u' + qu = f(t), \quad t \in (0, 1), \tag{3.9}$$

with BC (1.2), where $p(t) \in C[0, 1]$ and $q \in \mathbb{R}$.

Again, we seek approximate solutions of the form

$$u_n(t) = \sum_{i=1}^{n-1} c_i H_{i,n}(t), \quad n \geq 2.$$



(a) Plot of $u(t)$ and $u_{11}(t)$ (b) Absolute Difference of $|u(t) - u_{11}(t)|$

Figure 1: Analysis of $u_{11}(t)$ for BVP (3.6), (1.2)

Substituting this into the equation given in BVP (3.9), multiplying through by $H_{j,n}(t)$, for some $j = 1, \dots, n-1$, and applying integration by parts, we have

$$\sum_{i=1}^{n-1} c_i \int_0^1 [-H'_{i,n}(t)H'_{j,n}(t) + p(t)H'_{i,n}H_{j,n}(t) + qH_{i,n}(t)H_{j,n}(t)]\,dt = \int_0^1 H_{j,n}(t)f(t)\,dt.$$

Using the orthonormal property of $H_{i,n}(t)$, this is equivalent to solving the $(n-1) \times (n-1)$ system $\mathbf{Bc} = \mathbf{b}$, where

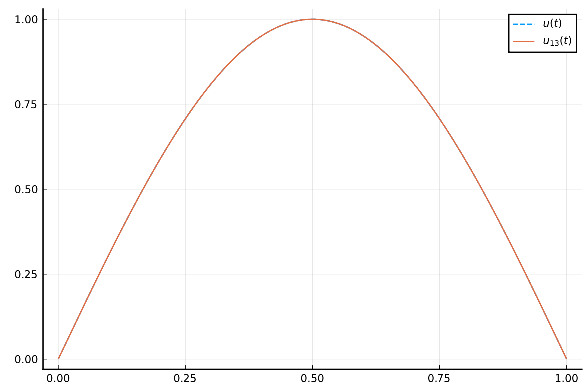
$$\mathbf{B} = [b_{ij}] = \begin{cases} \displaystyle\int_0^1 -[H'_{i,n}(t)]^2 + p(t)H'_{i,n}(t)H_{i,n}(t) + q, & i = j, \\ \displaystyle\int_0^1 -[H'_{i,n}(t)H_{j,n}(t)]^2 + p(t)H'_{j,n}(t)H_{i,n}(t), & i \neq j; \end{cases}$$

and $\mathbf{c}$ and $\mathbf{b}$ are defined the same as in (3.7) and (3.8), respectively.

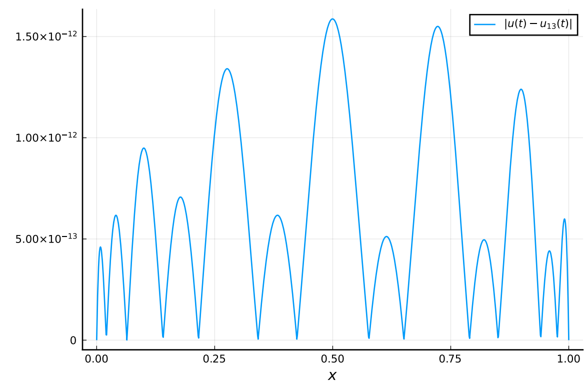
It can be shown that $u(t) = \sin(\pi x)$ is a solution to BVP (3.9), where $p(t) = \sin(x)$, $q \equiv 4$, and $f(t) = -\pi^2 \sin(\pi x) + \sin(x)\cos(\pi x)\pi + 4\sin(\pi x)$, with BC (1.2). For such a problem, by using 12 orthonormal Bernstein basis functions, $u_{13}(t)$, we obtained an absolute error of less than 1.6×10^{-12}. Figures 2a and 2b demonstrate this.

While the selected problems contain a linear term, the application of the orthonormal subset of Bernstein polynomials can be applied to for general equations such as (1.1), as long as BC (1.2) is satisfied. All calculations in these applications were done using Julia programming language, see [3].

The use of the subset of orthonormal Bernstein polynomials is mainly beneficial whenever there is scaled linear term in the ODE. When there is no scaled linear term, utilizing the subset orthonormal Bernstein basis functions only adds to the complexity of basis functions implemented. The full basis, i.e., $\{B_{i,n}(t)\}_{i=0}^{n}$ appears to do just as well as our subset orthonormal basis as seen in [8].



(a) Plot of $u(t)$ and $u_{13}(t)$



(b) Absolute Difference of $|u(t) - u_{13}(t)|$

Figure 2: Analysis of $u_{13}(t)$ for BVP (3.9), (1.2)

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