Antimagic Labelings of Forests

J. Sierra, D. D.-F. Liu, and J. Toy*

Abstract - An antimagic labeling of a graph $G(V, E)$ is a bijection $f : E \to \{1, 2, \ldots, |E|\}$ so that $\sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e)$ holds for all $u, v \in V(G)$ with $u \neq v$, where $E(v)$ is the set of edges incident to v. We call G antimagic if it admits an antimagic labeling. A forest is a graph without cycles; equivalently, every component of a forest is a tree. It was proved by Kaplan, Lev, and Roditty in 2009, and by Liang, Wong, and Zhu in 2014, that every tree with at most one vertex of degree two is antimagic. A major tool used in the proof is the zero-sum partition introduced by Kaplan, Lev, and Roditty in 2009. In this article, we provide an algorithmic representation for the zero-sum partition method and apply this method to show that every forest with at most one vertex of degree two is also antimagic.

Keywords : antimagic labeling; antimagic graphs; trees; rooted trees; forests

Mathematics Subject Classification (2020) **:** 05C78; 05C05

1 Introduction

The notion of magic graphs was motivated by magic squares. A magic square is an $n \times n$ array of integers $\{1, 2, \ldots, n^2\}$ so that each row, column, and the main and back-main diagonals of the square sum to the same value (see Figure 1 for an example). In 1963, Sedláček [12] extended this concept to graphs by labeling the edges of a graph with numbers and defining the *vertex-sum* of each vertex to be the total of the labels assigned to the edges incident to that vertex. Figure 1 shows how a magic square is used to create a *magic labeling* on a complete bipartite graph, where all the vertex-sums are identical.

An *edge labeling* of a graph G is a function f that assigns to each edge of G a positive integer. For a vertex $u \in V(G)$, denote $E(u)$ the set of edges incident to u. The *vertex-sum* of u is defined as

$$\varphi_f(u) = \sum_{e \in E(u)} f(e). \tag{1}$$

An edge labeling f is called *magic* if all vertices have the same vertex-sum (see Figure 1). Many variations of magic labelings have been studied (cf. [2]). Among them, the *antimagic labeling* has been studied widely in recent decades. The definition is given below. For positive integers $a \leq b$, denote $[a, b] = \{a, a + 1, \ldots, b\}$.

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2	7	6
9	5	1
4	3	8

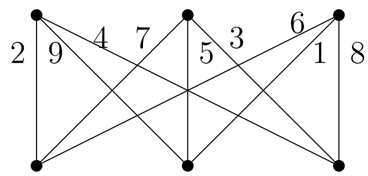

Figure 1: A magic square along with a magic $K_{3,3}$ graph. Each row and column of the magic square sum to 15. Each vertex in $K_{3,3}$ has vertex-sum 15.

Definition 1.1 *Let $G = (V, E)$ be a graph with m edges. A bijective function $f : E \to [1, m]$ is an* **antimagic labeling** *for G if $\varphi_f(u) \neq \varphi_f(v)$ for any two vertices $u \neq v$, where the vertex-sum $\varphi_f(u)$ is defined in Equation* (1)*. If G admits such an antimagic labeling, then G is said to be antimagic.*

By Definition 1.1, it is clear that K_2, the simple graph with only one edge and two vertices, is not antimagic. The notion of antimagic labelings was introduced by Hartsfield and Ringel in [9] in 1990, who conjectured that every connected graph other than K_2 is antimagic. Since then, this conjecture has been studied extensively. Many families of graphs are known to be antimagic, yet the conjecture remains open (cf. [1–4, 6–8]).

A *tree* is a connected graph without cycles. For a graph G and a vertex $v \in V(G)$, the *degree* of v, denoted by $\deg(v)$, is the number of edges incident to v. The following result was due to Kaplan, Lev, and Roditty [10], and Liang, Wong, and Zhu [11]:

Theorem 1.2 ([10, 11]) *A tree $T \neq K_2$ with at most one degree two vertex is antimagic.*

A major tool used in proving Theorem 1.2 is called the *zero-sum partition*. It was first devised by Kaplan, Lev, and Roditty in [10]. In Section 2, we provide a step-by-step process of this method.

A *forest* is a graph without cycles. Thus, every component of a forest is a tree, called a *component tree*. We aim to investigate antimagic labelings of forests by utilizing the zero-sum partition method of antimagic labelings for trees. The main result of this article is:

Theorem 1.3 *A forest F with at most one degree two vertex and without K_2 component trees is antimagic.*

The proof of Theorem 1.3 is presented in Section 3. In Section 2, we introduce the zero-sum partition method that will be used in the proof of Theorem 1.3. In Section 4, we discuss possible directions and open problems for future study.

2 The Zero-Sum Partition Method

The zero-sum partition method is based on the following two results. Here we present proofs that provide a step-by-step partition algorithm.

Lemma 2.1 ([10]) *Let s, l be non-negative integers and let $k = 2s + 6l$. Then there is a partition of $[1, k]$ into subsets $Q_1, Q_2, \dots, Q_{s+2l}$ such that the following hold:*

$$\begin{array}{lll} \textit{For } i \in [1, l]\text{:} & |Q_i| = 3 \textit{ and } \sum\limits_{a \in Q_i} a = k + 1. \\ \textit{For } i \in [1 + l, s + l]\text{:} & |Q_i| = 2 \textit{ and } \sum\limits_{a \in Q_i} a = k + 1. \\ \textit{For } i \in [s + l + 1, s + 2l]\text{:} & |Q_i| = 3 \textit{ and } \sum\limits_{a \in Q_i} a = 2(k + 1). \end{array}$$

Proof. The proof in [10] provides formulas of sets Q_i directly. To assist in understanding, we provide a step-by-step method to obtain these formulas. In general, Lemma 2.1 partitions the set of numbers in $[1, k]$ ($k = 2s + 6l$) into three types of sets, called A-, B-, and C-sets, where each A-set has three elements with sum $k + 1$, each B-set has two elements with sum $k + 1$, and each C-set has three elements with sum $2(k + 1)$. Precisely, there are l A-sets (denoted as $A_1, A_2, \dots, A_l$), s B-sets (denoted as $B_1, B_2, \dots, B_s$), and l C-sets (denoted as $C_1, C_2, \dots, C_l$).

The strategy of getting these A-, B-, and C-sets is expressed in the following four steps. Along the process, we use the following Example 2.2 to illustrate each step.

Example 2.2 Suppose $s = 5$ and $l = 2$. Then $s + 3l = 11$ and $k = 2(s + 3l) = 22$. By Lemma 2.1, we can partition the numbers in $[1, 22]$ into subsets $A_1, A_2, B_1, \dots, B_5, C_1, C_2$.

Step 1: Arranging all the labels into a two-row matrix. List the numbers in $[1, k]$ as a $2 \times (\frac{k}{2})$ matrix M where the first row has numbers $1, 2, \dots, \frac{k}{2}$ in increasing order, and the second row has numbers $k, k - 1, \dots, \frac{k}{2} + 1$ in decreasing order. Precisely, $a_{1,i} = i$ and $a_{2,i} = k - i + 1$, for $i \in [1, \frac{k}{2}]$. See Figure 2 as an illustration for Example 2.2. Consequently, the two numbers in each column sum up to $k + 1$. Note that there are $\frac{k}{2} = s + 3l$ columns.

$$\left[\begin{array}{ccccccccccc} \multicolumn{2}{c}{\overbrace{}^{l}} & \multicolumn{5}{c}{\overbrace{}^{s}} & \multicolumn{2}{c}{\overbrace{}^{l}} & \multicolumn{2}{c}{\overbrace{}^{l}} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \end{array}\right]$$

Figure 2: Arrange $[1, 22]$ into a two-row matrix M. Group the columns as shown. In total, there are $s + 3l$ columns.

Step 2: Determining the B-sets. Fix the columns of $[l + 1, l + s]$ in M as the B-sets. Precisely,

$$\textbf{B-sets}: B_i = \{a_{1,l+i},\ a_{2,l+i}\} = \{l + i,\ k - l - i + 1\},\ i \in [1, s].$$

Note that the sum of each B-set is $k + 1$. See Figure 3 for an illustration of Step 2 for Example 2.2.

$$\left[\begin{array}{ccccccccccc} 1 & 2 & \overbrace{3 & 4 & 5 & 6 & 7}^{s} & 8 & 9 & 10 & 11 \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \end{array}\right]$$

Figure 3: Use these s columns to create the B-sets. The B-sets are $B_1 = \{3, 20\}, B_2 = \{4, 19\}, B_3 = \{5, 18\}, B_4 = \{6, 17\}$, and $B_5 = \{7, 16\}$.

Step 3: Determining the A-sets. We use the first l elements on the first row of M to create A-sets. Recall that each column $i \in [1, l]$ in M, the sum of the two elements is $k+1$, since $a_{1,i} + a_{2,i} = (i) + (k - i + 1) = k + 1$. We replace $a_{2,i}$ by two elements, $a_{1,l+s+i}$ and $a_{2,l+s+2i}$, since:

$$a_{1,l+s+i} + a_{2,l+s+2i} = l + s + i + [k - (l + s + 2i) + 1] = k - i + 1 = a_{2,i}.$$

Hence, we obtain the l A-sets as follows:

$$\textbf{A-sets} : A_i = \{a_{1,i},\ a_{1,s+l+i},\ a_{2,s+l+2i}\} = \{i,\ l + s + i,\ k - (l + s + 2i) + 1\},\ i \in [1, l].$$

See Figure 4 for an illustration of this step for Example 2.2.

$$\left[\begin{array}{ccccccccccc} \overbrace{1 & 2}^{l} & 3 & 4 & 5 & 6 & 7 & \overbrace{8 & 9}^{l} & \overbrace{10 & 11}^{l} \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \end{array}\right]$$

Figure 4: The orange boxes are used to create A-sets. The process is to exchange the bottom numbers from the first l columns with numbers in the last $2l$ columns. Notice that, 22=8+14 and 21 = 9+12. The A-sets are $A_1 = \{1, 8, 14\}$ and $A_2 = \{2, 9, 12\}$.

After Steps 1, 2, and 3, the remaining unused labels are

$$\begin{array}{ll} a_{1,i}, & i \in [s + 2l + 1, s + 3l] \\ a_{2,i}, & i \in [1, l] \cup \{s + l + 1, s + l + 3, \dots, s + 3l - 1\}. \end{array}$$

Step 4: Determining the C-sets. For $i \in [1, l]$, first fix $a_{2,i} = k + 1 - i \in C_i$. Next, we combine the unused labels $a_{2,s+3l-2i}$ and $a_{1,s+3l-i}$ to form C_i:

$$\textbf{C-sets} : C_i = \{a_{2,i}, a_{1,s+3l+1-i},\ a_{2,s+3l+1-2i}\} = \{k-i+1,\ s+3l+1-i,\ k-(s+3l+1-2i)+1\}.$$

One can easily check that C_i is a C-set since the sum of the three elements is $2(k + 1)$. See Figure 5 for an illustration of Step 4 and the final partition for Example 2.2.

After the four steps, each number in $[1, k]$ belongs to exactly one of the A-, B-, C-sets. It is clear that the numbers in the first row and the numbers in the second row up to $a_{2,l+s}$ are used exactly once. For the remaining numbers, $a_{2,i}$, $i \in [s+l+1, s+3l]$, exactly half are in the A-sets while the other half are in the C-sets, according to the parity of i. See Figure 6 for an illustration. □

The next corollary is an immediate consequence of Lemma 2.1, which allows us to partition the set of integers $[1, k]$ into subsets of any sizes greater than one.

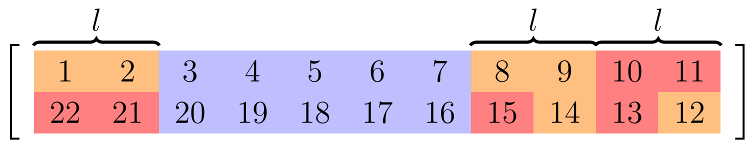

Figure 5: The remaining numbers are used to create C-sets. The C-sets are $C_1 = \{22, 11, 13\}$ and $C_2 = \{21, 10, 15\}$.

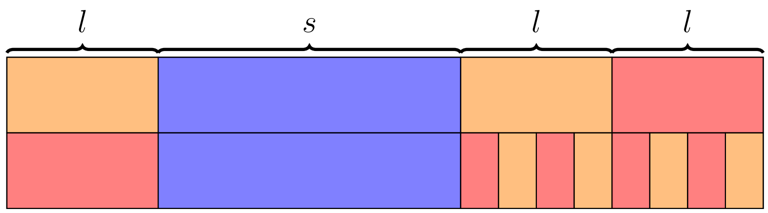

Figure 6: This figure helps visualize how $[1, k]$ is partitioned. A-sets cover the blocks in orange, B-sets cover the blocks in blue, and C-sets cover the blocks in red.

Corollary 2.3 ([10, 11]) *Let $k = r_1 + r_2 + \cdots + r_t$ be a partition of the positive integer k, where $r_i \geq 2$ for $i \in [1, t]$. Then the set $[1, k]$ can be partitioned into pairwise disjoint subsets, $D_1, D_2, \ldots, D_t$, such that for every $1 \leq i \leq t$, $|D_i| = r_i$ and $\sum_{a \in D_i} a \equiv 0 \pmod{k'}$, where $k' = k + 1$ if k is even, and $k' = k$ if k is odd.*

Proof. Let $k = r_1 + r_2 + \cdots + r_t$, where each $r_i \geq 2$ for $i = 1, 2, \ldots, t$. First, assume k is even. Write each r_i as the sum of multiples of 2 and multiples of 3, $r_i = 2m_i + 3n_i$, $m_i, n_i \geq 0$. Then $k = \sum_{i=1}^{t}(2m_i + 3n_i)$. Because k is even, $n_1 + n_2 + \cdots + n_t$ must be even. Denote

$$s = m_1 + m_2 + \cdots + m_t \quad \text{and} \quad l = \frac{n_1 + n_2 + \cdots + n_t}{2}.$$

Then, $k = 2s + 6l$. By Lemma 2.1, $[1, k]$ can be partitioned into sets $Q_1, Q_2, \ldots, Q_{s+2l}$.

We denote this partition of $[1, k]$ as $\mathcal{Q}$. To create the set D_i with $|D_i| = r_i = 2m_i + 3n_i$, take m_i sets in $\mathcal{Q}$ that are 2-element sets, and take n_i sets in $\mathcal{Q}$ that are 3-element sets. With the selected sets in $\mathcal{Q}$, let D_i be the union of those sets. Since each 2- or 3-element set in $\mathcal{Q}$ has the elements sum to $0 \pmod{k+1}$, it implies that the elements in D_i also sum to $0 \pmod{k+1}$.

Now assume $k = r_1 + r_2 + \cdots + r_t$ is odd. Then there exists some odd r_j, $r_j \geq 3$. Then $k - 1 = r_1 + r_2 + \cdots (r_j - 1) + \cdots + r_t$. Note that $r_j - 1 \geq 2$. As $k - 1$ is even, following the procedure described above, we can partition the set $[1, k-1]$ into the union of sets D_i, $i \in [1, t]$, where $|D_i| = r_i$ for $i \neq j$ and $|D_j| = r_j - 1$, and the sum of each set D_i is a multiple of k, $i \in [1, t]$. Once the sets are created, replace D_j by $\{k\} \cup D_j$ and keep the other D_i, $i \neq j$, to create a partition for $[1, k]$ which satisfies the statement. □

The following definitions show how we apply Corollary 2.3 to obtain an antimagic labeling for some trees. A *leaf* of a graph is a degree one vertex; a non-leaf vertex is called an *internal vertex*.

Definition 2.4 *An* **(oriented) rooted tree** *is a tree where one vertex is designated to be the root. We draw the root at the top of the tree and orient edges from top to down. Define the root to be at level 0. A vertex v is at level-i if the distance from v to the root is i. All the edges of the tree are oriented from vertices in level-i to vertices in level-$(i+1)$. For a vertex v, the* **children of** *v are the vertices that are adjacent to v and are one level higher than v. We call v the parent of its children. We say the tree is oriented from parent to children.*

Definition 2.5 *Let T be a rooted tree. For every non-root vertex v, we denote e^v as the* **incoming edge** *(the edge from the parent) of v. This is well-defined since each vertex (other than the root) has exactly one incoming edge.*

Definition 2.6 *Let T be a rooted tree with m edges. For $v \in V(T)$, denote $E^+(v)$ the set of* **outgoing edges** *from v and denote $|E^+(v)| = n_v$.*

Let $v_1, v_2, \ldots, v_t$ be the vertices of T with $n_{v_i} \geq 1$. It readily follows that

$$m = |E(T)| = n_{v_1} + n_{v_2} + \cdots + n_{v_t}. \tag{2}$$

Definition 2.7 *For a rooted tree T with m edges, we call a bijective mapping $f : E(T) \to [1, m]$ a* **zero-sum labeling** *of T if for each $i \in [1, t]$,*

$$\sum_{e \in E^+(v_i)} f(e) \equiv 0 \ (mod\ m'),$$

where $m' = m$ if m is odd; and $m' = m + 1$ if m is even.

Proposition 2.8 [10, 11] *Let T be a tree with an even number of edges and at most one vertex of degree two. Then there exists a zero-sum labeling for T that is antimagic.*

Proof. Let T be a tree with m edges, where m is even. Root T at the degree two vertex if T has a degree two vertex; otherwise, root T at any internal vertex. Denote this root as w. Let $v_1, v_2, \ldots, v_t$ be the internal vertices of T and n_i the number of outgoing edges of v_i. By our assumption, for $i \in [1, t]$, $n_i \geq 2$. By Equation (2) and Corollary 2.3, there exists a zero-sum labeling f for T by assigning numbers in D_i to the outgoing edges of v_i, $i \in [1, t]$. Further, for every non-root vertex $v \in V(T)$, $\varphi_f(v) = f(e^v) + \sum_{e \in E^+(v)} f(e) \equiv f(e^v) \pmod{m+1}$ (if v is a leaf then $E^+(v) = \emptyset$). For the root vertex w, $\varphi_f(w) \equiv 0 \not\equiv f(e^v) \pmod{m+1}$ for any v. Since each v has a distinct incoming edge, no two internal vertices will have congruent sums (mod $m + 1$). Thus, all vertices have distinct vertex sums, implying f is an antimagic labeling for T. □

See Figure 7 for an illustration of a zero-sum labeling of a tree by using Corollary 2.3. In the proof of Theorem 1.2 in [10, 11] the tree is carefully rooted so that a zero-sum labeling is also an antimagic labeling. Following this idea, in the proof of Theorem 1.3 (next section), we carefully root each component tree of a given forest F at a vertex and then identify all these roots as a single vertex to form a single tree T. We find a zero-sum labeling for T using the zero-sum partition method. Then we use this labeling to produce an antimagic labeling for F.

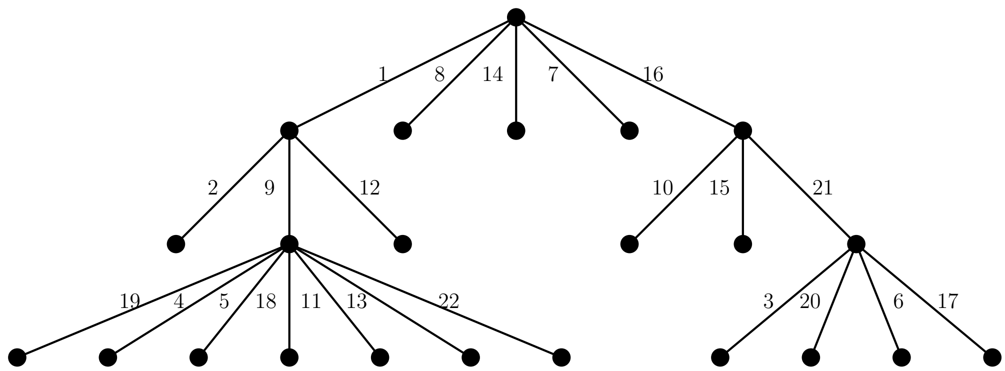

Figure 7: A zero-sum antimagic labeling of a tree using the partition of Example 2.2 in Figure 5.

3 Proof of Theorem 1.3

Let F be a forest with s component trees and $m = |E(F)|$. Denote the component trees of F by T_i with root at w_i for $i \in [1, s]$. If $s = 1$, then F is a tree, and the result follows by Theorem 1.2. Henceforth we assume $s \geq 2$. We proceed with the proof by considering cases.

Case 1: m **is odd.** Consider two sub-cases.

Sub-case 1.1: F **has no degree two vertex.** Let $w_i \in V(T_i)$ be a leaf of T_i for $i \in [1, s]$. Root each T_i at w_i and orient from parents to children as defined in Definition 2.4. Denote w_i' to be the child of w_i and denote the edge between them as $e_i = w_i w_i'$.

Let T be the tree obtained by identifying the vertices $w_1, w_2, \ldots, w_s$ into a single vertex w, where w is the root of T. Let $v_1, v_2, \ldots, v_t$ be the non-leaf vertices of T. Assume each vertex v_i has n_i outgoing edges, $n_i \geq 2$. Recall from Equation (2), the number of edges of T is $m = n_1 + n_2 + \cdots + n_t$. Since m is odd, there exists some odd n_j so that $n_j \geq 3$. By Corollary 2.3 one can partition $[1, m]$ into sets $D_1, D_2, \ldots, D_t$, where $|D_i| = n_i \geq 2$, and the elements in each D_i sum up to a multiple of m. Further, the label m is assigned to an outgoing edge $v_j v_{j'}$ of v_j with $n_j \geq 3$. This gives a zero-sum labeling f for T, where $\varphi_f(v) \equiv f(e^v) \pmod m$ holds for all vertex, except the root w. Thus all values of $\varphi_f(v_i)$, $i \in [1, t]$, are distinct, except that $\varphi_f(w) \equiv \varphi_f(v_{j'}) \equiv 0 \pmod m$, as $f(v_j v_{j'}) = m$.

Let g be the labeling for F defined by

$$g(e) = \begin{cases} f(ww_i') & \text{if } e = w_i w_i', \\ f(e) & \text{otherwise.} \end{cases}$$

Note that $v_{j'}$ is the only vertex with $\varphi_g(v_{j'}) = \varphi_f(v_{j'}) \equiv 0 \pmod m$. For every non-root vertex u of F (i.e., $u \neq w_i$ for all $i \in [1, s]$), we have $\varphi_g(u) = \varphi_f(u) \equiv f(e^u) \pmod m$. Thus $\varphi_g(u) \pmod m$ are pair-wisely distinct. If w_i and w_j are two root vertices of the component trees, $i \neq j$, then $\varphi_g(w_i) = f(e_i) \neq f(e_j) = \varphi_g(w_j)$. Thus the roots have different vertex-sums. Further, $\varphi_g(w_i) \equiv \varphi_g(w_i') \equiv f(e_i) \pmod m$. Since w_i' has at least two children, we have $\varphi_g(w_i') \geq f(e_i) + m > f(e_i) = \varphi_g(w_i)$, implying $\varphi_g(w_i) \neq \varphi_g(w_i')$. Hence, g is an antimagic labeling for F. See Figure 8 as an example.

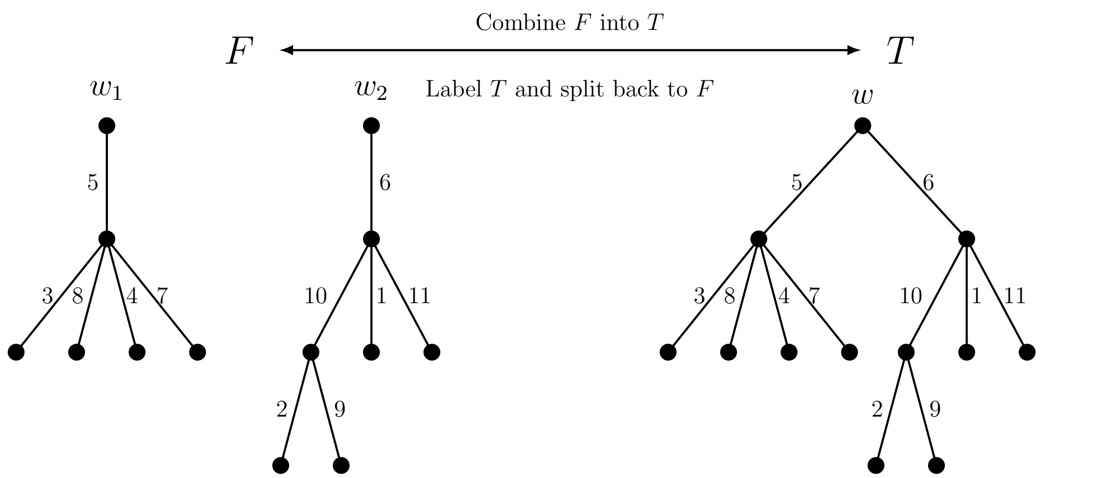

Figure 8: An example for Sub-case 1.1. The forest F to the left has two component trees, T_1 and T_2, and $m = 11$ edges. Root T_1 and T_2 at a leaf, w_1 and w_2, and identify w_1 and w_2 into a single root w to form a tree T shown on the right. A zero-sum labeling on T gives an antimagic labeling on F when T is split back into the components of F.

Sub-case 1.2: F **contains a degree two vertex.** Suppose F has a degree two vertex, v'. Without loss of generality, assume $v' \in V(T_1)$. Root each T_i at a leaf $w_i \in V(T_i)$ for $i \in [1, s]$. Let w_i' be the child of w_i and let v'' be the (only) child of v'. Let T be the tree obtained by identifying $w_1, w_2, \ldots, w_s$ into a vertex w, which is the root of T. Note that it is possible that v' is the child of w_1 in T_1. Let $w = v_1, v_2, \ldots, v_t$ be the internal vertices of $T \setminus \{v'\}$. By Corollary 2.3, we can partition the numbers in the set $[1, m-1]$ into sets that sum up to multiples of m and assign these sets to the outgoing edges of v_i, $i \in [1, t]$. Finally, we assign m to the edge $v'v''$. This labeling, denoted by f, is a zero-sum labeling for T, where $\varphi_f(u) \equiv f(e^u) \pmod{m}$ holds for every non-root vertex u and $f(v'v'') = m$. Note that $\varphi_f(v') \equiv f(e^{v'}) \pmod{m}$. Let g be the labeling for F defined as follows:

$$g(e) = \begin{cases} f(ww_i') & \text{if } e = w_iw_i', \\ f(e) & \text{otherwise.} \end{cases}$$

Then $\varphi_g(u) \equiv g(e^u) \pmod{m}$ if $u \neq w_i$. Note that v'' is the only vertex with $\varphi_g(v'') \equiv 0 \pmod{m}$. If $u \neq v$, then $\varphi_g(u) \not\equiv \varphi_g(v) \pmod{m}$ since u and v have different incoming edge labels. However, we have $\varphi_g(w_i) \equiv \varphi_g(w_i') \pmod{m}$. Each w_i' has at least two children or one child if $w_i' = v'$ (the degree two vertex), so $\varphi_g(w_i') \geq g(w_iw_i') + m > g(w_iw_i') = \varphi_g(w_i)$. Thus g is an antimagic labeling for F. See Figure 9 as an example.

Case 2: m **is even.**

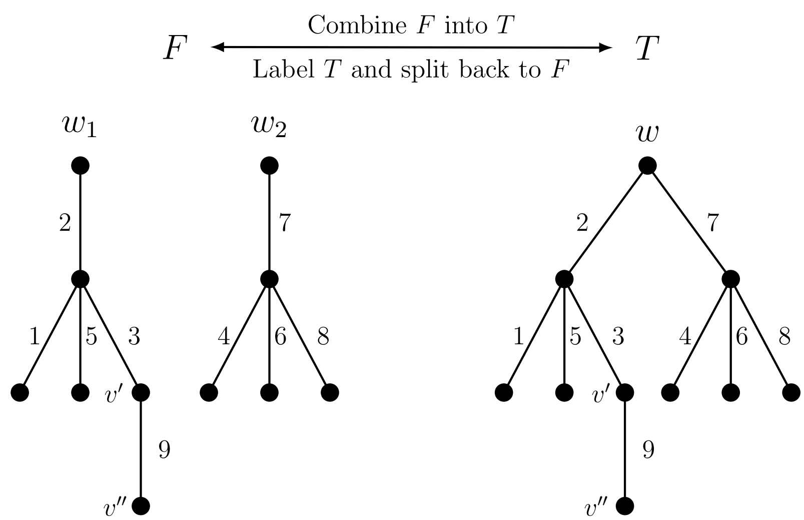

Figure 9: An example for Sub-case 1.2. The forest F on the left consists of two trees T_1 and T_2, and $m = 9$ edges, where T_1 has a degree two vertex v' and roots of T_1 and T_2 are leaves. We identify these two roots to create a tree T (on the right) and find a zero-sum labeling on F using Corollary 2.3 and labeling $v'v''$ with m. Afterwards, split T back into T_1 and T_2 to get an antimagic labeling for F as shown.

Sub-case 2.1: F has no degree two vertex. For $i \in [1, s]$, root T_i at a leaf w_i. Let w_i' be the child of w_i. Let T be the tree obtained by identifying the vertices $w_1, w_2, \ldots, w_s$ into a single vertex w. Denote $v_1, v_2, \ldots, v_t$ the non-leaf vertices of T. Assume each vertex v_i has r_i children. Then $|E(T)| = m = r_1 + r_2 + \cdots + r_t$, where $r_i \geq 2$ for $i \in [1, t]$. By Proposition 2.8 there is a zero-sum labeling f for T, which is antimagic.

Let g be the labeling for F defined by

$$g(e) = \begin{cases} f(ww_i') & \text{for } e = w_iw_i',\ i \in [1, s]; \\ f(e) & \text{otherwise.} \end{cases}$$

Similar to Case 1, one can easily show that g is an antimagic labeling of F.

Sub-case 2.2. F contains a degree two vertex. Assume F has a degree two vertex u. Without loss of generality, assume u is located in T_1. Consider two possibilities.

$\diamond$ **Sub-case 2.2.1.** $s = 2$. Root T_1 at $w_1 = u$. Root T_2 at w_2 where $\deg_{T_2}(w_2) \geq 3$. By Corollary 2.3 there is a zero-sum labeling f for F. In this labeling, we assign a B-set to edges of $E^+(w_1)$. The vertex-sums are distinct in modulo $m + 1$, excluding the two roots w_1 and w_2. Since w_1 is degree two, $\varphi_f(w_1) = m + 1$. Further, as $\deg(w_2) \geq 3$, we can choose the labels for edges incident to w_2 to sum up to at least $2(m + 1)$ (if $\deg(w_1) = 3$, we use a C-set from Lemma 2.1). Thus the vertex-sums are all distinct. Therefore f is an antimagic labeling for F. See Figure 10.

⋄ **Sub-case 2.2.2.** $s \geq 3$. Let $w_1 = u$ be the root of T_1, and $E^+(u) = \{e', e''\}$. For the remaining component trees T_i, $i \in [2, s]$, root T_i at a leaf $w_i \in V(T_i)$. Let T be the tree obtained by identifying $w_1, w_2, \ldots, w_s$ into a single vertex w, and root T at w. By Corollary 2.3, there exists a zero-sum labeling for T such that $f(e')$ and $f(e'')$ belong to the same B-set in the partition, $f(e') + f(e'') = m + 1$. This is possible as $s \geq 3$, and so $\deg(w) \geq 4$.

Let g be the labeling for F defined by

$$g(e) = \begin{cases} f(wv) & e = w_i v,\ i \in [1, s]; \\ f(e) & \text{otherwise.} \end{cases}$$

Then $\varphi_g(w_1) = m + 1 \equiv 0 \pmod{m+1}$. Similarly to the above cases, one can show that g is an antimagic labeling for F. See Figure 11 as an example. This completes the proof of Theorem 1.3. □

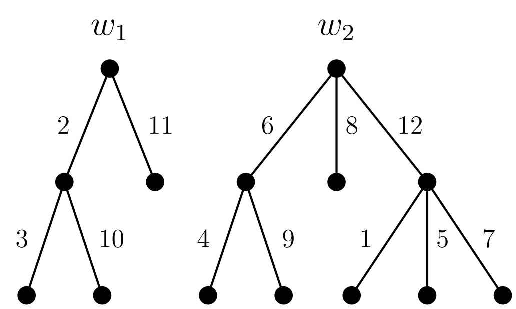

Figure 10: An example of Sub-case 2.2.1. This forest has two components, T_1 and T_2, $m = 12$ edges, and a degree two vertex w_1. Root T_1 at w_1, and root T_2 at a vertex w_2 of degree-3 or higher. By Corollary 2.3 there exists a zero-sum antimagic labeling so that $\varphi(w_1) = m + 1$ and $\varphi(w_2) \geq 2(m + 1)$.

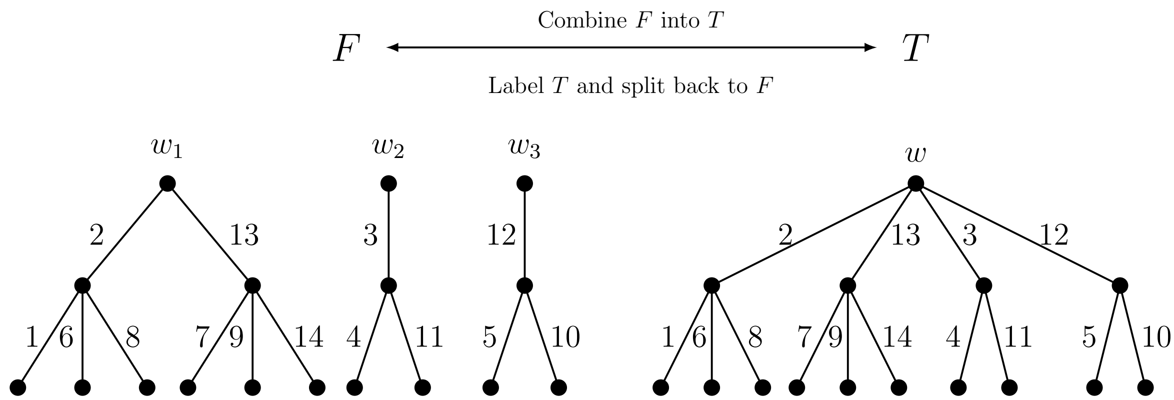

Figure 11: An example of Sub-case 2.2.2. The forest F has 3 components, T_1, T_2, and T_3. Root T_1 at the degree two vertex w_1, and root the other trees at a leaf. When labeling the combined tree T, the edges incident to w_1 get a B-set, so that it is the only vertex with sum 0 (mod 15).

4 Future Study

In this article, we proved that every forest without K_2 as a component tree and having at most one vertex of degree two is antimagic. It would be interesting to investigate the forests where each component tree contains at most one degree two vertex.

Suppose G is a graph and $e = uv$ is an edge of G. A subdivision of e is the operation of replacing $e = uv$ with a path (u, w_e, v), where w_e is a new vertex. For a tree T, denote by T^* the tree obtained by subdividing each edge in T. The following result was proved in [11].

Theorem 4.1 ([11]) *If T is a tree without degree two vertices, then T^* is antimagic.*

Define F^* in a similar way to T^*. Suppose F is a forest. Denote by F^* the forest obtained by subdividing each edge in F.

Question. For any forest F without a degree two vertex and without K_2 as a component, is it true that F^* is antimagic?

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