Antimagic Labelings of Forests

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Abstract - An antimagic labeling of a graph \( G(V, E) \) is a bijection \( f : E \rightarrow \{1, 2, \ldots, |E|\} \) so that \( \sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e) \) holds for all \( u, v \in V(G) \) with \( u \neq v \), where \( E(v) \) is the set of edges incident to \( v \). We call \( G \) antimagic if it admits an antimagic labeling. A forest is a graph without cycles; equivalently, every component of a forest is a tree. It was proved by Kaplan, Lev, and Roditty in 2009, and by Liang, Wong, and Zhu in 2014, that every tree with at most one vertex of degree two is antimagic. A major tool used in the proof is the zero-sum partition introduced by Kaplan, Lev, and Roditty in 2009. In this article, we provide an algorithmic representation for the zero-sum partition method and apply this method to show that every forest with at most one vertex of degree two is also antimagic.

Keywords : antimagic labeling; antimagic graphs; trees; rooted trees; forests

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1 Introduction

The notion of magic graphs was motivated by magic squares. A magic square is an \( n \times n \) array of integers \( \{1, 2, \ldots, n^2\} \) so that each row, column, and the main and back-main diagonals of the square sum to the same value (see Figure 1 for an example). In 1963, Sedláček [12] extended this concept to graphs by labeling the edges of a graph with numbers and defining the vertex-sum of each vertex to be the total of the labels assigned to the edges incident to that vertex. Figure 1 shows how a magic square is used to create a magic labeling on a complete bipartite graph, where all the vertex-sums are identical.

An edge labeling of a graph \( G \) is a function \( f \) that assigns to each edge of \( G \) a positive integer. For a vertex \( u \in V(G) \), denote \( E(u) \) the set of edges incident to \( u \). The vertex-sum of \( u \) is defined as

\[
\varphi_f(u) = \sum_{e \in E(u)} f(e). \tag{1}
\]

An edge labeling \( f \) is called magic if all vertices have the same vertex-sum (see Figure 1). Many variations of magic labelings have been studied (cf. [2]). Among them, the antimagic labeling has been studied widely in recent decades. The definition is given below. For positive integers \( a \leq b \), denote \([a, b] = \{a, a+1, \ldots, b\}\).

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Definition 1.1 Let $G = (V, E)$ be a graph with $m$ edges. A bijective function $f : E \to [1, m]$ is an antimagic labeling for $G$ if $\varphi_f(u) \neq \varphi_f(v)$ for any two vertices $u \neq v$, where the vertex-sum $\varphi_f(u)$ is defined in Equation (1). If $G$ admits such an antimagic labeling, then $G$ is said to be antimagic.

By Definition 1.1, it is clear that $K_2$, the simple graph with only one edge and two vertices, is not antimagic. The notion of antimagic labelings was introduced by Hartsfield and Ringel in [9] in 1990, who conjectured that every connected graph other than $K_2$ is antimagic. Since then, this conjecture has been studied extensively. Many families of graphs are known to be antimagic, yet the conjecture remains open (cf. [1–4, 6–8]).

A tree is a connected graph without cycles. For a graph $G$ and a vertex $v \in V(G)$, the degree of $v$, denoted by $\deg(v)$, is the number of edges incident to $v$. The following result was due to Kaplan, Lev, and Roditty [10], and Liang, Wong, and Zhu [11]:

**Theorem 1.2 ([10, 11])** A tree $T \neq K_2$ with at most one degree two vertex is antimagic.

A major tool used in proving Theorem 1.2 is called the zero-sum partition. It was first devised by Kaplan, Lev, and Roditty in [10]. In Section 2, we provide a step-by-step process of this method.

A forest is a graph without cycles. Thus, every component of a forest is a tree, called a component tree. We aim to investigate antimagic labelings of forests by utilizing the zero-sum partition method of antimagic labelings for trees. The main result of this article is:

**Theorem 1.3** A forest $F$ with at most one degree two vertex and without $K_2$ component trees is antimagic.

The proof of Theorem 1.3 is presented in Section 3. In Section 2, we introduce the zero-sum partition method that will be used in the proof of Theorem 1.3. In Section 4, we discuss possible directions and open problems for future study.

2 The Zero-Sum Partition Method

The zero-sum partition method is based on the following two results. Here we present proofs that provide a step-by-step partition algorithm.

Figure 1: A magic square along with a magic $K_{3,3}$ graph. Each row and column of the magic square sum to 15. Each vertex in $K_{3,3}$ has vertex-sum 15.
Lemma 2.1 ([10]) Let \( s, l \) be non-negative integers and let \( k = 2s + 6l \). Then there is a partition of \([1, k]\) into subsets \( Q_1, Q_2, \ldots, Q_{s+2l} \) such that the following hold:

For \( i \in [1, l] \):
\[
|Q_i| = 3 \quad \text{and} \quad \sum_{a \in Q_i} a = k + 1.
\]

For \( i \in [1 + l, s + l] \):
\[
|Q_i| = 2 \quad \text{and} \quad \sum_{a \in Q_i} a = k + 1.
\]

For \( i \in [s + l + 1, s + 2l] \):
\[
|Q_i| = 3 \quad \text{and} \quad \sum_{a \in Q_i} a = 2(k + 1).
\]

Proof. The proof in [10] provides formulas of sets \( Q_i \) directly. To assist in understanding, we provide a step-by-step method to obtain these formulas. In general, Lemma 2.1 partitions the set of numbers in \([1, k]\) \((k = 2s + 6l)\) into three types of sets, called \( A \)-, \( B \)-, and \( C \)-sets, where each \( A \)-set has three elements with sum \( k + 1 \), each \( B \)-set has two elements with sum \( k + 1 \), and each \( C \)-set has three elements with sum \( 2(k + 1) \). Precisely, there are \( l \) \( A \)-sets (denoted as \( A_1, A_2, \ldots, A_l \)), \( s \) \( B \)-sets (denoted as \( B_1, B_2, \ldots, B_s \)), and \( l \) \( C \)-sets (denoted as \( C_1, C_2, \ldots, C_l \)).

The strategy of getting these \( A \)-, \( B \)-, and \( C \)-sets is expressed in the following four steps. Along the process, we use the following Example 2.2 to illustrate each step.

Example 2.2 Suppose \( s = 5 \) and \( l = 2 \). Then \( s + 3l = 11 \) and \( k = 2(s + 3l) = 22 \). By Lemma 2.1 we can partition the numbers in \([1, 22]\) into subsets \( A_1, A_2, B_1, \ldots, B_5, C_1, C_2 \).

Step 1: Arranging all the labels into a two-row matrix. List the numbers in \([1, k]\) as a \( 2 \times (\frac{k}{2}) \) matrix \( M \) where the first row has numbers \( 1, 2, \ldots, \frac{k}{2} \) in increasing order, and the second row has numbers \( k, k-1, \ldots, \frac{k}{2} + 1 \) in decreasing order. Precisely, \( a_{1,i} = i \) and \( a_{2,i} = k - i + 1 \), for \( i \in [1, \frac{k}{2}] \). See Figure 2 as an illustration for Example 2.2. Consequently, the two numbers in each column sum up to \( k + 1 \). Note that there are \( \frac{k}{2} = s + 3l \) columns.

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\
\end{array}
\]

Figure 2: Arrange \([1, 22]\) into a two-row matrix \( M \). Group the columns as shown. In total, there are \( s + 3l \) columns.

Step 2: Determining the \( B \)-sets. Fix the columns of \([l + 1, l + s]\) in \( M \) as the \( B \)-sets. Precisely,

\[
\text{\( B \)-sets : } B_i = \{a_{1,l+i}, a_{2,l+i}\} = \{l + i, k - l - i + 1\}, \; i \in [1, s].
\]

Note that the sum of each \( B \)-set is \( k + 1 \). See Figure 3 for an illustration of Step 2 for Example 2.2.
$s$ = \[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\
\end{array}
\]

Figure 3: Use these $s$ columns to create the $B$-sets. The $B$-sets are $B_1 = \{3, 20\}$, $B_2 = \{4, 19\}$, $B_3 = \{5, 18\}$, $B_4 = \{6, 17\}$, and $B_5 = \{7, 16\}$.

**Step 3: Determining the $A$-sets.** We use the first $l$ elements on the first row of $M$ to create $A$-sets. Recall that each column $i \in [1, l]$ in $M$, the sum of the two elements is $k+1$, since $a_{1,i} + a_{2,i} = (i) + (k-i+1) = k+1$. We replace $a_{2,i}$ by two elements, $a_{1,i+s+i}$ and $a_{2,i+s+2i}$, since:

\[
a_{1,i+s+i} + a_{2,i+s+2i} = l + s + i + [k - (l + s + 2i) + 1] = k - i + 1 = a_{2,i}.
\]

Hence, we obtain the $l$ $A$-sets as follows:

$A$-sets: $A_i = \{a_{1,i}, a_{1,i+l+i}, a_{2,i+l+2}\} = \{i, l + s + i, k - (l + s + 2i) + 1\}, \ i \in [1, l]$.

See Figure 4 for an illustration of this step for Example 2.2.

\[
\begin{array}{cccccccccc}
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
l & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12
\end{array}
\]

Figure 4: The orange boxes are used to create $A$-sets. The process is to exchange the bottom numbers from the first $l$ columns with numbers in the last $2l$ columns. Notice that, $22 = 8+14$ and $21 = 9+12$. The $A$-sets are $A_1 = \{1, 8, 14\}$ and $A_2 = \{2, 9, 12\}$.

After Steps 1, 2, and 3, the remaining unused labels are

\[
a_{1,i}, \ i \in [s + 2l + 1, s + 3l] \\
a_{2,i}, \ i \in [1, l] \cup \{s + l + 1, s + l + 3, \ldots, s + 3l - 1\}.
\]

**Step 4: Determining the $C$-sets.** For $i \in [1, l]$, first fix $a_{2,i} = k + 1 - i \in C_i$. Next, we combine the unused labels $a_{2,i+s+3l-2i}$ and $a_{1,s+3l-i}$ to form $C_i$:

$C$-sets: $C_i = \{a_{2,i}, a_{1,i+s+3l-1-i}, a_{2,i+s+3l+1-2i}\} = \{k-i+1, s+3l+1-i, k-(s+3l+1-2i)+1\}$.

One can easily check that $C_i$ is a $C$-set since the sum of the three elements is $2(k+1)$. See Figure 5 for an illustration of Step 4 and the final partition for Example 2.2.

After the four steps, each number in $[1, k]$ belongs to exactly one of the $A$-, $B$-, $C$-sets. It is clear that the numbers in the first row and the numbers in the second row up to $a_{1,i+s}$ are used exactly once. For the remaining numbers, $a_{2,i}, i \in [s + l + 1, s + 3l]$, exactly half are in the $A$-sets while the other half are in the $C$-sets, according to the parity of $i$.

See Figure 6 for an illustration.

The next corollary is an immediate consequence of Lemma 2.1 which allows us to partition the set of integers $[1, k]$ into subsets of any sizes greater than one.
Let $k = r_1 + r_2 + \cdots + r_t$ be a partition of the positive integer $k$, where $r_i \geq 2$ for $i \in [1, t]$. Then the set $[1, k]$ can be partitioned into pairwise disjoint subsets, $D_1, D_2, \ldots, D_t$, such that for every $1 \leq i \leq t$, $|D_i| = r_i$ and $\sum_{a \in D_i} a \equiv 0 \pmod{k'}$, where $k' = k + 1$ if $k$ is even, and $k' = k$ if $k$ is odd.

**Proof.** Let $k = r_1 + r_2 + \cdots + r_t$, where each $r_i \geq 2$ for $i = 1, 2, \ldots, t$. First, assume $k$ is even. Write each $r_i$ as the sum of multiples of 2 and multiples of 3, $r_i = 2m_i + 3n_i$, $m_i, n_i \geq 0$. Then $k = \sum_{i=1}^{t}(2m_i + 3n_i)$. Because $k$ is even, $n_1 + n_2 + \cdots + n_t$ must be even. Denote

$$s = m_1 + m_2 + \cdots + m_t 	ext{ and } l = \frac{n_1 + n_2 + \cdots + n_t}{2}.$$ 

Then, $k = 2s + 6l$. By [Lemma 2.1], $[1, k]$ can be partitioned into sets $Q_1, Q_2, \ldots, Q_{s+2l}$.

We denote this partition of $[1, k]$ as $Q$. To create the set $D_i$ with $|D_i| = r_i = 2m_i + 3n_i$, take $m_i$ sets in $Q$ that are 2-element sets, and take $n_i$ sets in $Q$ that are 3-element sets. With the selected sets in $Q$, let $D_i$ be the union of those sets. Since each 2- or 3-element set in $Q$ has the elements sum to 0 (mod $k+1$), it implies that the elements in $D_i$ also sum to 0 (mod $k+1$).

Now assume $k = r_1 + r_2 + \cdots + r_t$ is odd. Then there exists some odd $r_j$, $r_j \geq 3$. Then $k - 1 = r_1 + r_2 + \cdots (r_j - 1) + \cdots + r_t$. Note that $r_j - 1 \geq 2$. As $k - 1$ is even, following the procedure described above, we can partition the set $[1, k - 1]$ into the union of sets $D_i$, $i \in [1, t]$, where $|D_i| = r_i$ for $i \neq j$ and $|D_j| = r_j - 1$, and the sum of each set $D_i$ is a multiple of $k$, $i \in [1, t]$. Once the sets are created, replace $D_j$ by $\{k\} \cup D_j$ and keep the other $D_i$, $i \neq j$, to create a partition for $[1, k]$ which satisfies the statement. \qed

The following definitions show how we apply Corollary 2.3 to obtain an antimagic labeling for some trees. A leaf of a graph is a degree one vertex; a non-leaf vertex is called an internal vertex.
Definition 2.4 An (oriented) rooted tree is a tree where one vertex is designated to be the root. We draw the root at the top of the tree and orient edges from top to down. Define the root to be at level 0. A vertex \( v \) is at level-i if the distance from \( v \) to the root is \( i \). All the edges of the tree are oriented from vertices in level-i to vertices in level-(i+1).

For a vertex \( v \), the children of \( v \) are the vertices that are adjacent to \( v \) and are one level higher than \( v \). We call \( v \) the parent of its children. We say the tree is oriented from parent to children.

Definition 2.5 Let \( T \) be a rooted tree. For every non-root vertex \( v \), we denote \( e^v \) as the incoming edge (the edge from the parent) of \( v \). This is well-defined since each vertex (other than the root) has exactly one incoming edge.

Definition 2.6 Let \( T \) be a rooted tree with \( m \) edges. For \( v \in V(T) \), denote \( E^+(v) \) the set of outgoing edges from \( v \) and denote \( |E^+(v)| = n_v \).

Let \( v_1, v_2, \ldots, v_t \) be the vertices of \( T \) with \( n_v \geq 1 \). It readily follows that

\[
m = |E(T)| = n_{v_1} + n_{v_2} + \cdots + n_{v_t}.
\]

(2)

Definition 2.7 For a rooted tree \( T \) with \( m \) edges, we call a bijective mapping \( f : E(T) \to [1, m] \) a zero-sum labeling of \( T \) if for each \( i \in [1,t] \),

\[
\sum_{e \in E^+(v_i)} f(e) \equiv 0 \pmod{m'},
\]

where \( m' = m \) if \( m \) is odd; and \( m' = m+1 \) if \( m \) is even.

Proposition 2.8 [10, 11] Let \( T \) be a tree with an even number of edges and at most one vertex of degree two. Then there exists a zero-sum labeling for \( T \) that is antimagic.

Proof. Let \( T \) be a tree with \( m \) edges, where \( m \) is even. Root \( T \) at the degree two vertex if \( T \) has a degree two vertex; otherwise, root \( T \) at any internal vertex. Denote this root as \( w \). Let \( v_1, v_2, \ldots, v_t \) be the internal vertices of \( T \) and \( n_i \) the number of outgoing edges of \( v_i \). By our assumption, for \( i \in [1,t] \), \( n_i \geq 2 \). By Equation (2) and Corollary 2.3 there exists a zero-sum labeling \( f \) for \( T \) by assigning numbers in \( D_i \) to the outgoing edges of \( v_i \), \( i \in [1,t] \). Further, for every non-root vertex \( v \in V(T) \), \( \varphi_f(v) = f(e^v) + \sum_{e \in E^+(v)} f(e) \equiv f(e^v) \pmod{m+1} \) (if \( v \) is a leaf then \( E^+(v) = \emptyset \)). For the root vertex \( w \), \( \varphi_f(w) \equiv 0 \neq f(e^w) \pmod{m+1} \) for any \( v \). Since each \( v \) has a distinct incoming edge, no two internal vertices will have congruent sums (mod \( m+1 \)). Thus, all vertices have distinct vertex sums, implying \( f \) is an antimagic labeling for \( T \). \( \square \)

Figure 6 for an illustration of a zero-sum labeling of a tree by using Corollary 2.3. In the proof of Theorem 1.2 in [10, 11] the tree is carefully rooted so that a zero-sum labeling is also an antimagic labeling. Following this idea, in the proof of Theorem 1.3 (next section), we carefully root each component tree of a given forest \( F \) at a vertex and then identify all these roots as a single vertex to form a single tree \( T \). We find a zero-sum labeling for \( T \) using the zero-sum partition method. Then we use this labeling to produce an antimagic labeling for \( F \).
Figure 7: A zero-sum antimagic labeling of a tree using the partition of Example 2.2 in Figure 3.

3 Proof of Theorem 1.3

Let $F$ be a forest with $s$ component trees and $m = |E(F)|$. Denote the component trees of $F$ by $T_i$ with root at $w_i$ for $i \in [1, s]$. If $s = 1$, then $F$ is a tree, and the result follows by Theorem 1.2. Henceforth we assume $s \geq 2$. We proceed with the proof by considering cases.

Case 1: $m$ is odd. Consider two sub-cases.

Sub-case 1.1: $F$ has no degree two vertex. Let $w_i \in V(T_i)$ be a leaf of $T_i$ for $i \in [1, s]$. Root each $T_i$ at $w_i$ and orient from parents to children as defined in Definition 2.4. Denote $w_i'$ to be the child of $w_i$ and denote the edge between them as $e_i = w_i w_i'$.

Let $T$ be the tree obtained by identifying the vertices $w_1, w_2, \ldots, w_s$ into a single vertex $w$, where $w$ is the root of $T$. Let $v_1, v_2, \ldots, v_t$ be the non-leaf vertices of $T$. Assume each vertex $v_i$ has $n_i$ outgoing edges, $n_i \geq 2$. Recall from Equation (2) the number of edges of $T$ is $m = n_1 + n_2 + \cdots + n_t$. Since $m$ is odd, there exists some odd $n_j$ so that $n_j \geq 3$. By Corollary 2.3 one can partition $[1, m]$ into sets $D_1, D_2, \ldots, D_t$, where $|D_i| = n_i \geq 2$, and the elements in each $D_i$ sum up to a multiple of $m$. Further, the label $m$ is assigned to an outgoing edge $v_i v_j$ for $v_j$ with $n_j \geq 3$. This gives a zero-sum labeling $f$ for $T$, where $\varphi_f(v) \equiv f(e^v) \pmod{m}$ holds for all vertex, except the root $w$. Thus all values of $\varphi_f(v_i), i \in [1, t]$, are distinct, except that $\varphi_f(w) \equiv \varphi_f(v_j') \equiv 0 \pmod{m}$, as $f(v_j v_j') = m$.

Let $g$ be the labeling for $F$ defined by

$$g(e) = \begin{cases} f(w w_i)' & \text{if } e = w_i w_i', \\ f(e) & \text{otherwise.} \end{cases}$$

Note that $v_j'$ is the only vertex with $\varphi_g(v_j') = \varphi_f(v_j') \equiv 0 \pmod{m}$. For every non-root vertex $u$ of $F$ (i.e., $u \neq w_i$ for all $i \in [1, s]$), we have $\varphi_g(u) = \varphi_f(u) \equiv f(e^u) \pmod{m}$. Thus $\varphi_g(u) \pmod{m}$ are pair-wisely distinct. If $w_i$ and $w_j$ are two root vertices of the component trees, $i \neq j$, then $\varphi_g(w_i) = f(e_i) \neq f(e_j) = \varphi_g(w_j)$. Thus the roots have different vertex-sums. Further, $\varphi_g(w_i) \equiv \varphi_g(w_i') \equiv f(e_i) \pmod{m}$. Since $w_i'$ has at least
two children, we have \( \varphi_g(w_i') \geq f(e_i) + m > f(e_i) = \varphi_g(w_i) \), implying \( \varphi_g(w_i) \neq \varphi_g(w_i') \). Hence, \( g \) is an antimagic labeling for \( F \). See Figure 8 as an example.

Figure 8: An example for Sub-case 1.1. The forest \( F \) to the left has two component trees, \( T_1 \) and \( T_2 \), and \( m = 11 \) edges. Root \( T_1 \) and \( T_2 \) at a leaf, \( w_1 \) and \( w_2 \), and identify \( w_1 \) and \( w_2 \) into a single root \( w \) to form a tree \( T \) shown on the right. A zero-sum labeling on \( T \) gives an antimagic labeling on \( F \) when \( T \) is split back into the components of \( F \).

**Sub-case 1.2:** \( F \) contains a degree two vertex. Suppose \( F \) has a degree two vertex, \( v' \). Without loss of generality, assume \( v' \in V(T_1) \). Root each \( T_i \) at a leaf \( w_i \in V(T_i) \) for \( i \in [1,s] \). Let \( w_i' \) be the child of \( w_i \) and let \( v'' \) be the (only) child of \( v' \). Let \( T \) be the tree obtained by identifying \( w_1, w_2, \ldots, w_s \) into a vertex \( w \), which is the root of \( T \). Note that it is possible that \( v' \) is the child of \( w_1 \) in \( T_1 \). Let \( w = v_1, v_2, \ldots, v_t \) be the internal vertices of \( T \setminus \{v'\} \). By Corollary 2.3, we can partition the numbers in the set \([1, m-1]\) into sets that sum up to multiples of \( m \) and assign these sets to the outgoing edges of \( v_i, i \in [1,t] \). Finally, we assign \( m \) to the edge \( v'v'' \). This labeling, denoted by \( f \), is a zero-sum labeling for \( T \), where \( \varphi_f(u) \equiv f(e^u) \pmod{m} \) holds for every non-root vertex \( u \) and \( f(v'v'') = m \). Note that \( \varphi_f(v') \equiv f(e^{v'}) \pmod{m} \). Let \( g \) be the labeling for \( F \) defined as follows:

\[
g(e) = \begin{cases} 
    f(ww_i') & \text{if } e = w_iw_i', \\
    f(e) & \text{otherwise.}
\end{cases}
\]

Then \( \varphi_g(u) \equiv g(e^u) \pmod{m} \) if \( u \neq w_i \). Note that \( v'' \) is the only vertex with \( \varphi_g(v'') \equiv 0 \pmod{m} \). If \( u \neq v \), then \( \varphi_g(u) \neq \varphi_g(v) \pmod{m} \) since \( u \) and \( v \) have different incoming edge labels. However, we have \( \varphi_g(w_i) \equiv \varphi_g(w_i') \pmod{m} \). Each \( w_i' \) has at least two children or one child if \( w_i' = v' \) (the degree two vertex), so \( \varphi_g(w_i') \geq g(w_iw_i') + m > g(w_iw_i') = \varphi_g(w_i) \). Thus \( g \) is an antimagic labeling for \( F \). See Figure 9 as an example.

**Case 2:** \( m \) is even.
Sub-case 2.1: $F$ has no degree two vertex. For $i \in [1, s]$, root $T_i$ at a leaf $w_i$. Let $w'_i$ be the child of $w_i$. Let $T$ be the tree obtained by identifying the vertices $w_1, w_2, \ldots, w_s$ into a single vertex $w$. Denote $v_1, v_2, \ldots, v_t$ the non-leaf vertices of $T$. Assume each vertex $v_i$ has $r_i$ children. Then $|E(T)| = m = r_1 + r_2 + \cdots + r_t$, where $r_i \geq 2$ for $i \in [1, t]$. By Proposition 2.8 there is a zero-sum labeling $f$ for $T$, which is antimagic.

Let $g$ be the labeling for $F$ defined by

$$g(e) = \begin{cases} f(ww'_i) & \text{for } e = w_iw'_i, \ i \in [1, s]; \\ f(e) & \text{otherwise.} \end{cases}$$

Similar to Case 1, one can easily show that $g$ is an antimagic labeling of $F$.

Sub-case 2.2. $F$ contains a degree two vertex. Assume $F$ has a degree two vertex $u$. Without loss of generality, assume $u$ is located in $T_1$. Consider two possibilities.

$\diamond$ Sub-case 2.2.1. $s = 2$. Root $T_1$ at $w_1 = u$. Root $T_2$ at $w_2$ where $\deg_{T_2}(w_2) \geq 3$. By Corollary 2.3 there is a zero-sum labeling $f$ for $F$. In this labeling, we assign a $B$-set to edges of $E^+(w_1)$. The vertex-sums are distinct in modulo $m + 1$, excluding the two roots $w_1$ and $w_2$. Since $w_1$ is degree two, $\varphi_f(w_1) = m + 1$. Further, as $\deg(w_2) \geq 3$, we can choose the labels for edges incident to $w_2$ to sum up to at least $2(m + 1)$ (if $\deg(w_1) = 3$, we use a $C$-set from Lemma 2.1). Thus the vertex-sums are all distinct. Therefore $f$ is an antimagic labeling for $F$. See Figure 10.
\textbf{Sub-case 2.2.2.} \( s \geq 3 \). Let \( w_1 = u \) be the root of \( T_1 \), and \( E^+(u) = \{e', e''\} \). For the remaining component trees \( T_i, i \in [2, s] \), root \( T_i \) at a leaf \( w_i \in V(T_i) \). Let \( T \) be the tree obtained by identifying \( w_1, w_2, \ldots, w_s \) into a single vertex \( w \), and root \( T \) at \( w \). By Corollary 2.3, there exists a zero-sum labeling for \( T \) such that \( f(e') \) and \( f(e'') \) belong to the same \( B \)-set in the partition, \( f(e') + f(e'') = m + 1 \). This is possible as \( s \geq 3 \), and so \( \deg(w) \geq 4 \).

Let \( g \) be the labeling for \( F \) defined by
\[
g(e) = \begin{cases} 
  f(wv) & e = w_i v, \ i \in [1, s]; \\
  f(e) & \text{otherwise}.
\end{cases}
\]

Then \( \varphi_g(w_1) = m + 1 \equiv 0 \pmod{m + 1} \). Similarly to the above cases, one can show that \( g \) is an antimagic labeling for \( F \). See Figure 11 as an example. This completes the proof of Theorem 1.3. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10}
\caption{An example of Sub-case 2.2.1. This forest has two components, \( T_1 \) and \( T_2 \), \( m = 12 \) edges, and a degree two vertex \( w_1 \). Root \( T_1 \) at \( w_1 \), and root \( T_2 \) at a vertex \( w_2 \) of degree-3 or higher. By Corollary 2.3 there exists a zero-sum antimagic labeling so that \( \varphi(w_1) = m + 1 \) and \( \varphi(w_2) \geq 2(m + 1) \).}
\end{figure}
Figure 11: An example of Sub-case 2.2.2. The forest $F$ has 3 components, $T_1$, $T_2$, and $T_3$. Root $T_1$ at the degree two vertex $w_1$, and root the other trees at a leaf. When labeling the combined tree $T$, the edges incident to $w_1$ get a $B$-set, so that it is the only vertex with sum $0 \pmod{15}$.

4 Future Study

In this article, we proved that every forest without $K_2$ as a component tree and having at most one vertex of degree two is antimagic. It would be interesting to investigate the forests where each component tree contains at most one degree two vertex.

Suppose $G$ is a graph and $e = uv$ is an edge of $G$. A subdivision of $e$ is the operation of replacing $e = uv$ with a path $(u, w, v)$, where $w$ is a new vertex. For a tree $T$, denote by $T^*$ the tree obtained by subdividing each edge in $T$. The following result was proved in [11].

**Theorem 4.1** ([11]) If $T$ is a tree without degree two vertices, then $T^*$ is antimagic.

Define $F^*$ in a similar way to $T^*$. Suppose $F$ is a forest. Denote by $F^*$ the forest obtained by subdividing each edge in $F$.

**Question.** For any forest $F$ without a degree two vertex and without $K_2$ as a component, is it true that $F^*$ is antimagic?

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