# Clumsy Packing of Polyominoes in Finite Space

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**Abstract -** Clumsy packing is considered an inefficient packing, meaning we find the minimum number of objects we can pack into a space so that we can not pack any more object. Thus, we are effectively spacing the objects as far apart as possible so that we cannot fit another object. In this paper, we consider the clumsy packing of polyominoes in a finite space, which must consider boundary conditions. We examine rectangle, $L$, $T$, and plus polyominoes of various sizes.

**Keywords :** clumsy packing; polyomino

**Mathematics Subject Classification** (2020) **:** 05C70; 05B50

## 1 Introduction

In 1971, Sands [4] proposed the following problem: *Given an $n \times m$ checkerboard, what is the minimum number of dominoes we can place on the board, assuming each domino covers exactly two adjacent squares, so that no additional domino can fit?* In other words, what is the maximum number of $1 \times 1$ 'holes' that can be created? Sands showed that if $nm$ is a multiple of 3, then the maximum number of holes is $\frac{nm}{3}$. In 1988, Gyárfás, Lehel, and Tuza [3] called this type of packing a *clumsy packing* which requires the density of packed polyominoes to be as small as possible relative to the size of the board. They were able to generate results for general board sizes with dominoes, which are $1 \times 2$ rectangular polyominoes. Goddard [1] considered a clumsy packing density for *hooks* (3 squares in the shape of an $L$), *$m \times m$ squares*, and *$1 \times m$ longs* on infinite board. More recently, Walzer ([5], [6]) considered other clumsy packings of the infinite plane for more complicated polyominoes.

In this paper, we consider the clumsy packing of polyominoes in a finite board. The introduction of boundary conditions leads to additional restrictions for clumsily packing polyominoes. However, as the size of the board increases, the impact of the boundary decreases. As an initial first step to understanding boundary conditions, we will consider $n \times n$ boards for polyominoes which are made of $n$ squares. Thus the board size is relative to the size of the polyomino. We hope this will evolve into understanding boundary conditions on any finite $n \times m$ board, but generalized boards are beyond the scope of this paper.

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## 2 Background and Definitions

We will assume throughout that $a, b$, and $n$ are positive integers. We will pack our polyominoes on a square grid of size $n \times n$ where $n$ is the size of the polyomino. We will adapt more of our notation and definitions from Chapter 14 of [2], however, to help with notation and terminology, we will make the following definitions.

### 2.1 The Board

The board will be a square $n \times n$ grid where we place our polyominoes. Therefore, an $8 \times 8$ board can be visualized as a checkerboard. A $1 \times 1$ closed square on our grid whose sides are parallel to the coordinate axes and corners are at integer coordinates will be called a *cell* and we will denote the cell in column $i$ and row $j$ by $C_{i,j}$. For $1 \leq i, j \leq n$ define $X_i = \{C_{i,j} : 1 \leq j \leq n\}$ and $Y_j = \{C_{i,j} : 1 \leq i \leq n\}$. We will refer to $X_i$ as column $i$ and $Y_j$ as row $j$. We will use the convention that $X_1$ is the left column, $X_n$ is the right column, $Y_1$ is the top row, and $Y_n$ is the bottom row. Thus, cell $C_{1,1}$ is the cell in the first column and first row which would be the upper left cell of the grid. The square $n \times n$ grid will be called the board, $\mathbb{B}$, so that $\mathbb{B} = \{C_{i,j} : 1 \leq i \leq n \text{ and } 1 \leq j \leq n \text{ where } i, j \in \mathbb{Z}\}$.

### 2.2 General Polyominoes

We will assume throughout that $a, b, c$, and $d$ are all integers. A *polyomino*, $\mathcal{P}$, is a finite set of cells. The number of cells in a polyomino will be the *size* of the polyomino and will be denoted $|\mathcal{P}|$. Note that two polyominoes are considered disjoint if they do not have any cells in common, but visually they may share edges or corners. Polyominoes will be located on the board based on an *anchor* which will be a specific cell that uniquely determines the location of the polyomino on the board. Informally, for example, for an $L$ polyomino (Definition 3.7) we will define the anchor as the cell located on the intersection of the horizontal and vertical legs.

For integers $c$ and $d$ a *shift* of a polyomino $\mathcal{P}$ by $(c, d)$ is a polyomino $\mathcal{P}_1$ so that $\mathcal{P}_1 = \{C_{x+c,y+d} : C_{x,y} \in \mathcal{P}\}$. Thus $\mathcal{P}_1$ is a shift of $\mathcal{P}$ $c$ units horizontally and $d$ units vertically. We will use the notation $\mathcal{P}_1 = \mathcal{P} + (c, d)$ to represent a shift by $(c, d)$.

For certain polyominoes, we will consider a clockwise rotation of our polyomino by an integer multiple of 90°. For any integer $m$, if $\mathcal{P}$ is a polyomino, then $\mathcal{P}R^m$ is a rotation of $\mathcal{P}$ by $(m \cdot 90)°$ clockwise. This idea will become more formal for each individual polyomino. Note that a rotation or shift may produce a polyomino which is no longer on the board.

In this paper, we will not consider reflections of polyominoes that could be of particular importance for $L$ polyominoes. We will be working with two different types of packing, free and fixed. For *fixed packing*, we will not be allowed to rotate the polyomino, we will only be allowed to shift the polyomino. In *free packing*, we will be allowed to rotate and shift the polyomino.

We will call two polyominoes $\mathcal{P}_1$ and $\mathcal{P}_2$ *fixed equivalent* if there is an integer pair $(c, d)$ so that $\mathcal{P}_2 = \mathcal{P}_1 + (c, d)$. We will denote fixed equivalent polyominoes by $\mathcal{P}_1 \approx \mathcal{P}_2$.

For free equivalent polyominoes, we are allowed to rotate and shift the polyominoes. We will call two polyominoes $\mathcal{P}_1$ and $\mathcal{P}_2$ *free equivalent* if there is a nonnegative integer $m$ and a pair of integers $(c, d)$ such that $\mathcal{P}_2 = \mathcal{P}_1 R^m + (c, d)$. We will denote free equivalent polyominoes by $\mathcal{P}_1 \sim \mathcal{P}_2$.

Our objective is to pack as few copies of our polyomino on the board so that we can not include another polyomino. To do so, we need to define how polyominoes fit onto a board and what makes a valid arrangement.

Given a board $\mathbb{B}$ and a set of polyominoes $\mathbb{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$, we say $\mathbb{P}$ is a *valid arrangement* if $\bigcup_{i=1}^{k} \mathcal{P}_i \subseteq \mathbb{B}$ and if $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for all $1 \leq i < j \leq k$. Otherwise, we call $\mathbb{P}$ an *invalid arrangement.* Therefore a valid arrangment simply means the set of polyominoes is pairwise disjoint and fits onto the board.

A set of polyominoes composed of polyominoes which are free equivalent polyominoes to $\mathcal{P}$ will be called a *free polyomino set of* $\mathcal{P}$*.* A free polyomino set of $\mathcal{P}$, denoted $\mathbb{P}$, is a *free packing of* $\mathcal{P}$ on a board $\mathbb{B}$ if $\mathbb{P}$ is a valid arrangement such that for any free equivalent polyomino $\mathcal{P}^* \sim \mathcal{P}$ we have that $\mathcal{P}^* \cup \mathbb{P}$ is an invalid arrangement. Thus, we can think of a free packing as a maximal set of $\mathcal{P}$. So for a set of polyominoes to be a free packing, they must be a valid packing and they must be maximal in the sense that no additional polyomino can be placed onto the board without overlapping an already placed polyomino. The number of polyominoes in the set is the *free packing number* of the set which will be denoted by $pn^{free}(\mathbb{P})$. Finally, the *clumsy free packing number of* $\mathcal{P}$, denoted $cp^{free}(\mathcal{P})$ is the minimum free packing number over all free polyomino sets of $\mathcal{P}$ on a board $\mathbb{B}$. In essence, the clumsy free packing number is the minimum of the maximals. Any set of polyominoes that attain this minimum will be called a *clumsy free packing.*

We can easily extend the previous definitions to fixed packings. This includes *fixed polyomino set, fixed packing, fixed packing number, clumsy fixed packing number,* and *clumsy fixed packing.*

Unless otherwise specified, we will always consider packings (free or fixed) of a polyomino $\mathcal{P}$ on an $n \times n$ board $\mathbb{B}$ where $n = |\mathcal{P}|$. The generalization of these results for any finite board size would be an ideal expansion of these results by first considering square boards and then generalizing to any $n \times m$ board. However, this generalization is beyond the scope of this paper.

# 3 Results

This section will consist of subsections for *rectangular* polyominoes, $L$ polyominoes, $T$ polyominoes, and *plus* polyominoes. In each section, we define the polyomino and consider fixed and free packings.

## 3.1 Rectangular Polyominoes

*Rectangular polyominoes* are a popular polyomino to study because of their simplicity. The classic domino is a $2 \times 1$ rectangular polyomino. Although they may seem simple, there are not many results known for clumsy packing of rectangular polyominoes. In fact, in 1995 Goddard [1] conjectured that the clumsy packing ratio for a $1 \times m$ polyomino on an infinite board is $2m/(m^2+1)$ for any $m \geq 3$ but were only able to show that the ratio is at least $2/m+1$. Even for $m=3$ the result is unknown.

**Definition 3.1** *The rectangular polyomino, $R_{a,b}$, is the polyomino consisting of all cells $C_{i,j}$ where $1 \leq i \leq a$ and $1 \leq j \leq b$. Thus, $R_{a,b}$, is a rectangular polyomino of size $ab$.*

To know the location of a rectangular polyomino, we need to define the anchor.

**Definition 3.2** *Given a rectangular polyomino, $R_{a,b}$, with $a, b > 1$, we define anchor of the polyomino as the cell $C_{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.*

Note that if $a=b=1$ then $R_{1,1}$ is just an individual cell, which is a trivial case since $cp^{fix}(R_{1,1}) = cp^{free}(R_{1,1}) = 1$.

*Straight polyominoes* are a special case of rectangular polyominoes. We will use straight vertical and straight horizontal polyominoes for our clumsy free packing.

**Definition 3.3** *For $n > 1$, the straight vertical polyomino, $SV_n$, is the rectangular polyomino $R_{1,n}$. We will define the anchor as $C_{1,\lfloor \frac{n}{2} \rfloor}$.*

**Definition 3.4** *For $n > 1$, the straight horizontal polyomino, $SH_n$, is the rectangular polyomino $R_{n,1}$. We will define the anchor as $C_{\lfloor \frac{n}{2} \rfloor, 1}$*

The following theorem shows clumsy packing number for fixed and free straight polyominoes. The proof is very straightforward and is therefore omitted.

**Theorem 3.5** *For any $n \geq 1$,*

$$cp^{free}(SV_n) = cp^{free}(SH_n) = cp^{fix}(SV_n) = cp^{fix}(SH_n) = n.$$

The following theorem finds the clumsy fixed packing number for *rectangular* polyominoes of size at least $2 \times 2$.

**Theorem 3.6** *For any rectangular polyomino, $R_{a,b}$ with $a, b \geq 2$, the clumsy fixed packing is $cp^{fix}(R_{a,b}) = \left\lceil \frac{ab-a+1}{2a-1} \right\rceil \left\lceil \frac{ab-b+1}{2b-1} \right\rceil$.*

**Proof.** Note that $n = ab$. In order to find the clumsy fixed packing number for rectangular polyominoes, we will first show that there exists a fixed packing of a set of $R_{a,b}$, $\mathbb{P}$, so that $pn^{fixed}(\mathbb{P}) = \left\lceil \frac{ab-a+1}{2a-1} \right\rceil \left\lceil \frac{ab-b+1}{2b-1} \right\rceil$.

Consider tiling the board by having $b-1$ empty rows, $Y_1$ through $Y_{b-1}$ and $a-1$ empty columns, $X_1$ through $X_{a-1}$. Then place $R_{a,b}$ polyominoes at centers $C_{x,y}$ where $x = \min\{a-1+(k-1)(2a-1)+\left\lfloor \frac{a}{2} \right\rfloor, ab - \left\lceil \frac{a}{2} \right\rceil\}$ for $1 \leq k \leq \left\lceil \frac{ab-(a-1)}{2a-1} \right\rceil$ and $y = \min\{b-1+(l-1)(2b-1)+\left\lfloor \frac{b}{2} \right\rfloor, ab - \left\lceil \frac{b}{2} \right\rceil\}$ for $1 \leq l \leq \left\lceil \frac{ab-(b-1)}{2b-1} \right\rceil$. See Figure 1.

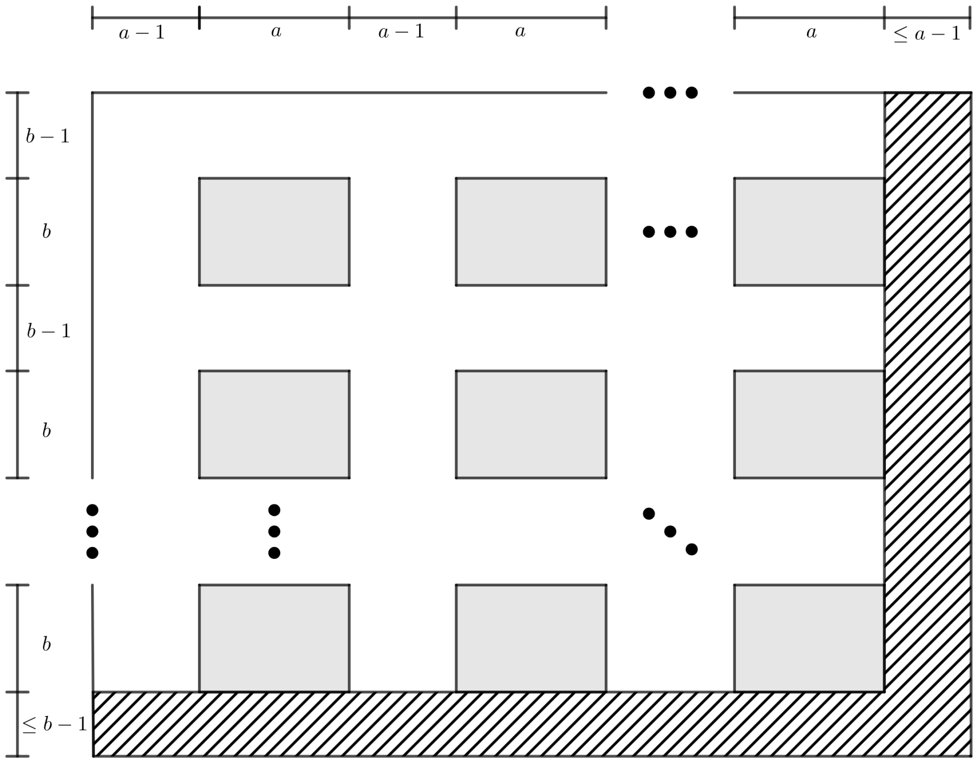


Figure 1: This figure depicts a general example of a clumsy fixed packing of some $R_{a,b}$. The shaded areas are the polyominoes. The hatched region represents the area that may or may not include an additional row or column of polyominoes.

The number of polyominoes placed along a non-empty row will be $\left\lceil \frac{ab-(a-1)}{2a-1} \right\rceil$. The number of polyominoes placed along a non-empty column will be $\left\lceil \frac{ab-(b-1)}{2b-1} \right\rceil$. So the total number of polyominoes in this particular tiling will be $\left\lceil \frac{ab-a+1}{2a-1} \right\rceil \left\lceil \frac{ab-b+1}{2b-1} \right\rceil$. Therefore

$$cp^{fix}\left(R_{a,b}\right) \leq pn^{fixed}(\mathbb{P}) = \left\lceil \frac{ab-a+1}{2a-1} \right\rceil \left\lceil \frac{ab-b+1}{2b-1} \right\rceil .$$

Now we will show that this is also a lower bound for $cp^{fix}\left(R_{a,b}\right)$ and we will do this based on the parity of $a$ and $b$.

**Even/Odd Case**: Assume we have a clumsy fixed packing of $R_{a,b}$, where $a, b \geq 2$ and $a = 2k$ and $b = 2l+1$ for non-negative integers $k$ and $l$.

It is easy to see that there must be an anchor in the top $3l+1$ rows and the bottom $3l+1$ rows and every $4l+1$ rows in between. Similarly, there must also be an anchor in the left $3k-1$ column, the right $3k$ columns, and every $4k-1$ columns in between. Note that every region of $4l+1$ rows and $4k-1$ columns must contain an anchor. Otherwise, in any region of such size without an anchor, we can fit an additional polyomino fixed equivalent to $R_{a,b}$. This is demonstrated in Figure 2 across the top edge of the board. Therefore, there are $2+\left\lfloor \frac{ab-2(3l+1)}{4l+1} \right\rfloor = 2+\left\lfloor \frac{ab-(3b-1)}{2b-1} \right\rfloor$ pairwise disjoint sets of rows, each of

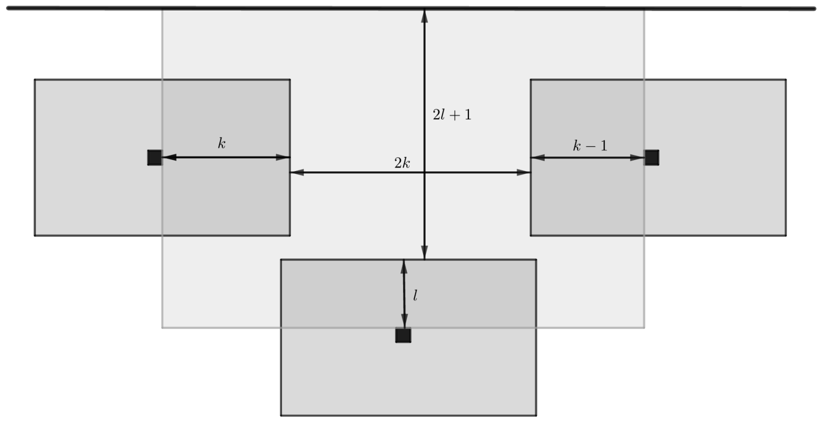


Figure 2: Even/Odd Case: This shows that a $(4k-1)\times(3l+1)$ rectangular region along the top of the board must contain an anchor which are represented by the black cells. If not, there is a $(2k)\times(2l+1)$ region which can contain another polyomino.

which must contain an anchor in each of the $2+\left\lfloor\frac{ab-(3k-1+3k)}{4k-1}\right\rfloor = 2+\left\lfloor\frac{ab-(3a-1)}{2a-1}\right\rfloor$ pairwise disjoint sets of columns. This leaves us with a total of $\left\lfloor 2+\frac{ab-(3b-1)}{2b-1}\right\rfloor\left\lfloor 2+\frac{ab-(3a-1)}{2a-1}\right\rfloor$ disjoint regions on the board, each of which must contain an anchor.

(Note that if $\frac{ab-2(3l+1)}{4l+1}<0$, then the top $3l+1$ rows and the bottom $3l+1$ rows are not disjoint, so we can place an anchor that will be in the top $3l+1$ rows and the bottom $3l+1$ rows. Also, notice that since $a\geq 2$ we know $ab\geq 2l+1$ which implies $\frac{ab-2(3l+1)}{4l+1}\geq -1$. Similarly for columns.)

This shows that for the even/odd case we have

$$cp^{fix}\left(R_{a,b}\right)\geq\left\lfloor 2+\frac{ab-(3b-1)}{2b-1}\right\rfloor\left\lfloor 2+\frac{ab-(3a-1)}{2a-1}\right\rfloor.$$

**Even/Even Case**: Assume $a,b\geq 2$ and $a=2k$ and $b=2l$. Following a similar logic as above, we can see that there must be an anchor in the top $3l-1$ rows, bottom $3l$ rows, and every $4l-1$ rows in between. There must also be an anchor in the left $3k-1$ columns, the right $3k$ columns, and every $4k-1$ columns in between. Note that every region of $4k-1$ columns and $4l-1$ rows must contain an anchor.

Therefore, there are $2+\left\lfloor\frac{ab-(3l-1)-3l}{4l-1}\right\rfloor = 2+\left\lfloor\frac{ab-(3b-1)}{2b-1}\right\rfloor$ pairwise disjoint sets of rows, each of which must contain an anchor in each of the $2+\left\lfloor\frac{ab-3k-(3k-1)}{4k-1}\right\rfloor = 2+\left\lfloor\frac{ab-(3a-1)}{2a-1}\right\rfloor$ pairwise disjoint sets of columns. This leaves a total of $\left\lfloor 2+\frac{ab-(3b-1)}{2b-1}\right\rfloor\left\lfloor 2+\frac{ab-(3a-1)}{2a-1}\right\rfloor$ disjoint regions on the board, each of which must contain an anchor. Notice that this is the same as in the even/odd case.

**Odd/Odd Case**: Assume $a,b\geq 2$ and $a=2k+1$ and $b=2l+1$. Following a similar logic as above, we can see that there must be an anchor in the top $3l+1$ rows, bottom

$3l+1$ rows, and every $4l+1$ rows in between. There must also be an anchor in the left $3k+1$ columns, the right $3k+1$ columns, and every $4k+1$ columns in between. Note that every region of $4k+1$ columns and $4l+1$ rows must contain an anchor.

Therefore, there are $2+\left\lfloor \frac{ab-(3l+1)-(3l+1)}{4l+1} \right\rfloor = 2+\left\lfloor \frac{ab-(3b-1)}{2b-1} \right\rfloor$ pairwise disjoint sets of rows, each of which must contain an anchor in each of the $2+\left\lfloor \frac{ab-(3k+1)-(3k+1)}{4k+1} \right\rfloor = 2+\left\lfloor \frac{ab-(3a-1)}{2a-1} \right\rfloor$ pairwise disjoint sets of columns. This leaves a total of $\left\lfloor 2+\frac{ab-(3b-1)}{2b-1} \right\rfloor \left\lfloor 2+\frac{ab-(3a-1)}{2a-1} \right\rfloor$ disjoint regions on the board, each of which must contain an anchor. Notice this is the same as in the previous two cases.

Since all three cases give the same inequality, we will now proceed independently of the parity of $a$ and $b$.

Now we need to show the equality of our upper and lower bounds on the clumsy fixed packing number.

To accomplish this, we call upon a well-known equality for floor and ceiling functions:

$$\left\lceil \frac{T}{B} \right\rceil = \left\lfloor \frac{T+B-1}{B} \right\rfloor.$$

From this we can easily show that

$$\left\lceil \frac{T}{B} \right\rceil = \left\lfloor \frac{T-1-B}{B} \right\rfloor + 2.$$

With this we are able to see

$$\begin{aligned} \left\lceil \frac{ab-a+1}{2a-1} \right\rceil &= \left\lfloor \frac{(ab-a+1)-1-(2a-1)}{2a-1} \right\rfloor + 2 \\ &= \left\lfloor \frac{ab-(3a-1)}{2a-1} \right\rfloor + 2 \end{aligned}$$

and similarly

$$\left\lceil \frac{ab-b+1}{2b-1} \right\rceil = \left\lfloor \frac{ab-(3b-1)}{2b-1} \right\rfloor + 2.$$

Thus we are able to justify that

$$cp^{fix}\left(R_{a,b}\right) \geq \left\lceil \frac{ab-a+1}{2a-1} \right\rceil \left\lceil \frac{ab-b+1}{2b-1} \right\rceil.$$

□

Clumsy **free** packing for general *rectangular* polyominoes turned out to be more difficult due to the edge effects. For example, a packing of $R_{3,6}$ using no rotations shows $cp^{free}\left(R_{3,6}\right) \leq cp^{fix}\left(R_{3,6}\right) \leq 8$ and we think these are equality. However, a similar packing shows $cp^{free}\left(R_{3,9}\right) \leq cp^{fix}\left(R_{3,9}\right) \leq 10$ but we have constructed a different packing so that $cp^{free}\left(R_{3,9}\right) \leq 9$ which keeps $7 \times 8$ rectangular regions empty in the four corners.

## 3.2 $L$ Polyominoes

As noted in the introduction, previous work has been completed for *hooks* which is a special case of an $L$ polyomino.

**Definition 3.7** *An $L$ polyomino, $L_{a,b}$, is a polyomino where $0 < a \leq b$ such that $L_{a,b} = \{C_{i,1} : 1 \leq i \leq a+1\} \cup \{C_{1,j} : 1 < j \leq b+1\}$. We will define the anchor of the polyomino to be the cell $C_{1,1}$. Notice that $|L_{a,b}| = a+b+1$.*

Thus, $L$ polyominoes have legs of length $a+1$ and $b+1$ and have size $a+b+1$.

We will define $LR_{a,b}$, $LR^2_{a,b}$, and $LR^3_{a,b}$ to be a $90°, 180°$, and $270°$ clockwise rotation of $L_{a,b}$ respectively. So we have:

- $LR_{a,b} = \{C_{i,1} : 1 \leq i \leq b+1\} \cup \{C_{b+1,j} : 1 \leq j \leq a+1\}$ which will have anchor at cell $C_{b+1,1}$,
- $LR^2_{a,b} = \{C_{i,b+1} : 1 \leq i \leq a+1\} \cup \{C_{a+1,j} : 1 \leq j \leq b+1\}$ which will have anchor at cell $C_{a+1,b+1}$, and
- $LR^3_{a,b} = \{C_{i,a+1} : 1 \leq i \leq b+1\} \cup \{C_{1,j} : 1 \leq j \leq a+1\}$ which will have anchor at cell $C_{1,a+1}$.

An example of an $L$ polyomino rotated is provided in Figure 3.

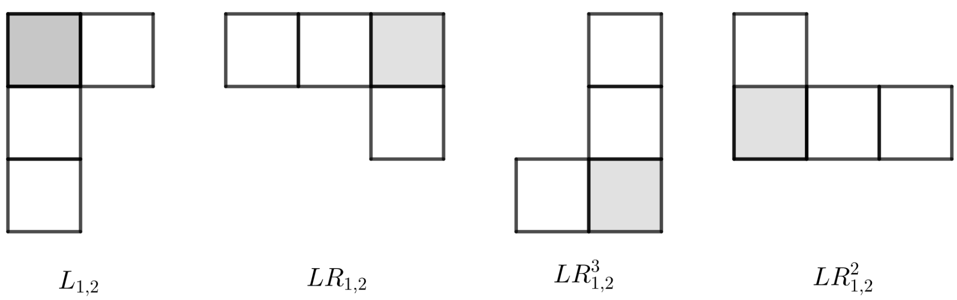


Figure 3: This figure demonstrates $L_{1,2}$, $LR_{1,2}$, $LR^2_{1,2}$, and $LR^3_{1,2}$. The shaded cell is the anchor.

For simplicity, we will use $L_{a,b}, LR_{a,b}$, etc. to describe the size and orientation of the polyomino and use the anchor to describe its location, which is just a shift of $L_{a,b}, LR_{a,b}$, etc.

The packing of $L$ polyominoes may seem straight forward, but there are some intricacies that we must consider. The case where $a = b$ is a special case, as seen in Theorem 3.8 for clumsy fixed packing and Theorem 3.10 for clumsy free packing.

**Theorem 3.8** *For an $L$ polyomino, $L_{a,a}$, $cp^{fix}(L_{a,a}) = 1$.*

**Proof.** Place the $L$ polyomino so that the anchor is in the cell $C_{a+1,1}$ as seen in Figure 4. Notice that if we want to include another $L$ polyomino, we must have the anchor in

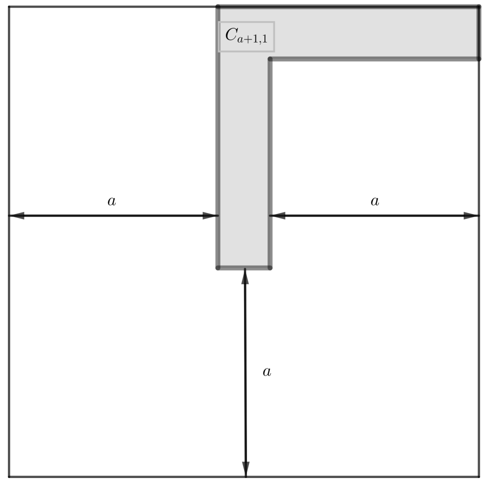


Figure 4: This figures shows the size of the empty regions after placing the anchor of $L_{a,a}$ in $C_{a+1,1}$.

the first $a$ rows and $a$ columns. However, we must have at least $a+1$ consecutive empty horizontal cells which does not happen in the first $a$ rows.

□

Notice that $cp^{fix}(L_{a,b})$ could be small, as in the case where $a = b$. However when $a$ and $b$ are dramatically different, for example $a = 1$ and $b \gg a$, then every set of three adjacent columns must contain an anchor. So our clumsy fixed packing number is at least $\lfloor \frac{b}{3} \rfloor$.

Now, we turn our attention to free packing. We provide bounds for $L_{a,b}$ as well as show that $cp^{free}(L_{a,a}) = 2$.

**Lemma 3.9** *For an L polyomino, $L_{a,b}$, $cp^{free}(L_{a,b}) \geq 2$.*

**Proof.** Notice a single $L$-polyomino cannot intersect the first and last columns. Similarly, an $L$-polyomino cannot intersect the top and bottom rows. Therefore, if we put $L$-polyominoes with anchors in the corners of the board, we cannot intersect more than three of these with any one polyomino and the result follows.

□

**Theorem 3.10** *For an L polyomino, $L_{a,a}$, $cp^{free}(L_{a,a}) = 2$.*

**Proof.** By Lemma 3.9, $cp^{free}(L_{a,a}) \geq 2$. Consider placing an $L$ polyomino $L_{a,a}$, call it $L^1$, with an anchor in the cell $C_{a+1,a+1}$. Place another $L$ polyomino $L_{a,a}$, call it $L^2$, with anchor at cell $C_{a,1}$. Notice that to the left of $L^1$ and $L^2$ there are at most $a$ empty columns, which is not enough for another $L$ polyomino. Similarly, notice to the right of $L^1$ and $L^2$, there are two regions and each has at most $a$ consecutive rows which is not enough for another $L$ polyomino. As seen in Figure 5.

□

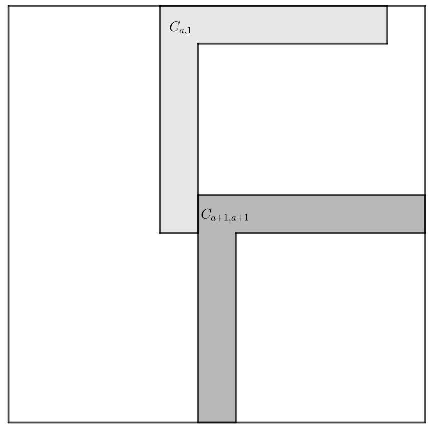


Figure 5: Free packing showing $cp^{free}(L_{a,a}) = 2$.

For a general $L$ polyomino, there exists a free packing of five $L$ polyominoes, however this may not be a clumsy packing.

**Theorem 3.11** *For an $L$ polyomino, $L_{a,b}$, $2 \leq cp^{free}(L_{a,b}) \leq 5$.*

**Proof.** By Lemma 3.9, $2 \leq cp^{free}(L_{a,b})$. A valid arrangement of five $L$ polyominoes can always be constructed in the manner shown in Figure 6, where the four $L$ polyominoes on the left are embedded into a square of side length $b+2$. Thus, there cannot be another $L$ polyomino in the center of these four $L$ polyominoes. The bottom $a-1$ rows and right $a-1$ columns will be empty. However, placing an $LR^2_{a,b}$ with anchor in $C_{a+b+1,b+3}$ will create free packing since $a \leq b$. Therefore, $2 \leq cp^{free}(L_{a,b}) \leq 5$.

□

We know that the clumsy free packing number is often less than the upper bound. For example, consider $a = 3, b = 6$ and $n = 10$. We can place three copies $L_{3,6}$ with anchor at $C_{2,1}, C_{3,4}$, and $C_{7,4}$ (see Figure 7) This creates a free packing of $L_{3,6}$ whose packing number is 3. We can also find a free packing of $L_{2,7}$ whose packing number is 4 (see Figure 8). We conjecture that these are the clumsy free packing numbers for $L_{3,6}$ and $L_{2,7}$ respectively.

**Corollary 3.12** *For an $L$ polyomino, $L_{1,b}$, $2 \leq cp^{free}(L_{1,b}) \leq 4$.*

**Proof.** Note that if $a = b = 1$ then $cp^{free}(L_{1,1}) = 2$ via Theorem 3.10. Given $1 \neq b$, a free packing of four $L$ polyominoes can always be constructed in the manner shown in Figure 9, which has at most $b$ consecutive empty horizontal cells and $b$ consecutive vertical empty cells. Therefore, $2 \leq cp^{free}(L_{1,b}) \leq 4$.

□

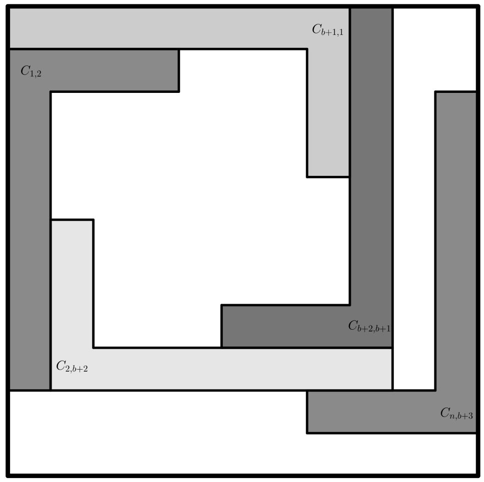


Figure 6: General pattern to show $cp^{free}(L_{a,b}) \leq 5$.

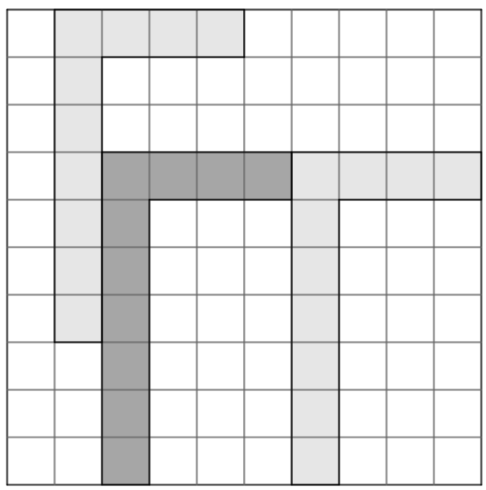

Figure 7: Shows $cp^{free}(L_{3,6}) \leq 3$.

### 3.3 $T$ Polyominoes

A $T$ polyomino is a natural extension of the $L$ polyomino.

**Definition 3.13** *A $T$ polyomino, $T_{a,b}$, is a polyomino such that $T_{a,b} = \{C_{i,1} : 1 \leq i \leq 2a+1\} \cup \{C_{a+1,j} : 2 \leq j \leq b+1\}$. Notice that $|T_{a,b}| = 2a+b+1$. We define the anchor of the polyomino as the cell $C_{a+1,1}$.*

We will define $TR_{a,b}$, $TR^2_{a,b}$, and $TR^3_{a,b}$ as a clockwise rotation of $90°, 180°$, and $270°$ respectively. So we have:

- $TR_{a,b} = \{C_{b+1,i} : 1 \leq i \leq 2a+1$ and $(j, a+1) : 1 \leq j \leq b\}$ which will have anchor at cell $C_{b+1,a+1}$,

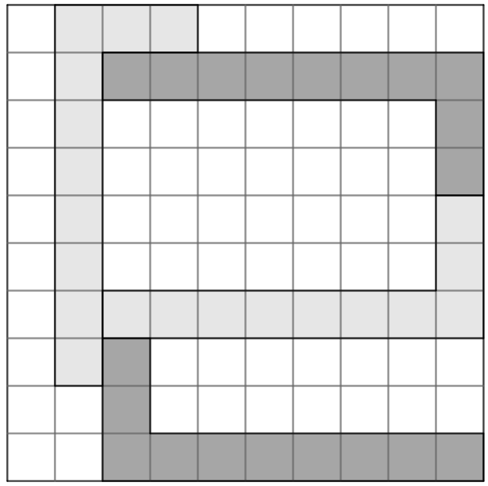

Figure 8: Shows $cp^{free}(L_{2,7}) \leq 4$.

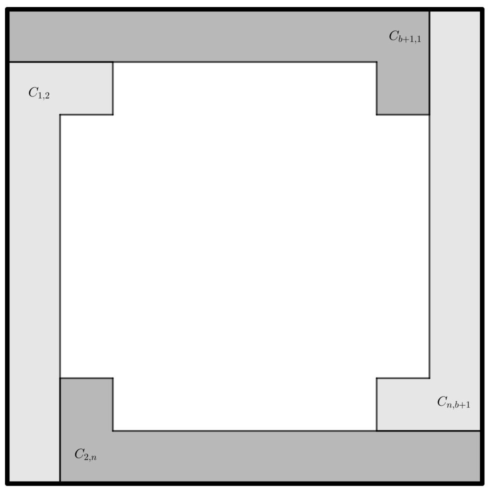


Figure 9: Shows $cp^{free}(L_{1,b}) \leq 4$.

- $TR^2_{a,b} = \{C_{i,b+1} : 1 \leq i \leq 2a+1 \text{ and } (a+1, j) : 1 \leq j \leq b\}$ which will have anchor at cell $C_{a+1,b+1}$, and
- $TR^3_{a,b} = \{C_{1,i} : 1 \leq i \leq 2a+1 \text{ and } (j, a+1) : 2 \leq j \leq b+1\}$ that will have an anchor in cell $C_{1,a+1}$.

As before, we will use $T_{a,b}, TR_{a,b}$, etc. to describe the size and orientation of the polyomino and use the anchor to describe its location, which is just a shift of $T_{a,b}, TR_{a,b}$, etc.

To proceed, we consider two cases for fixed $T$ polyominoes. The first case, $b \leq 2a$, implies that the $T$ polyomino is as wide or wider than it is tall. The second case, $b > 2a$ implies that the $T$ polyomino is taller than it is wide.

**Theorem 3.14** *For a $T$ polyomino, $T_{a,b}$, if $b \leq 2a$ then*

$$cp^{fix}\left(T_{a,b}\right) = \left\lceil \frac{2a+1}{2b+1} \right\rceil .$$

**Proof.** Place $T$ polyominoes, $T_{a,b}$, with anchors $C_{\left\lfloor \frac{b}{2} \right\rfloor + a+1, y}$, where $y = b+1 \pmod{2b+1}$. In addition, if $n \mod (2b+1) \geq b+1$, add another $T$ polyomino with center $C_{\left\lfloor \frac{b}{2} \right\rfloor + a+1, n-(b+1)}$. Note that this will put $T$ polyominoes, centered horizontally, so that the centers have exactly $2b$ rows between them, with the exception being at the bottom of the board where the centers may have less than $2b$ rows between them. If there are at least $2b+1$ rows below the bottommost anchor, then we include another $T$ polyomino with anchor $C_{\left\lfloor \frac{b}{2} \right\rfloor + a+1, n-(b+1)}$. See Figure 10.

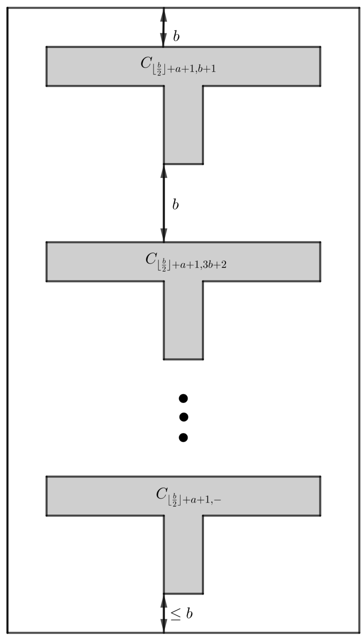


Figure 10: Pattern for $cp^{fix}\left(T_{a,b}\right)$ when $b \leq 2a$.

Therefore, the number of polyominoes that we now have in our arrangement is $\left\lceil \frac{n-b}{2b+1} \right\rceil = \left\lceil \frac{2a+1}{2b+1} \right\rceil$. This shows

$$cp^{fix}\left(T_{a,b}\right) \leq \left\lceil \frac{2a+1}{2b+1} \right\rceil .$$

To show that this is optimal, notice that we must have an anchor in the top $b+1$ rows, Otherwise we can include another $T$ polyomino with an anchor in the top row. Next, notice that, for every $2b+1$ rows thereafter, we must have an anchor. If not, we will have at least $b+1$ consecutive empty rows. Therefore, the number of $T$ polyominoes must be at least

$$cp^{fix}(T_{a,b}) \geq \left\lfloor \frac{n-(b+1)}{2b+1} \right\rfloor + 1$$
$$= \left\lceil \frac{n-b}{2b+1} \right\rceil .$$

The last equality follows since if $x$ and $y$ are positive integers then $\left\lfloor \frac{x}{y} \right\rfloor = \left\lceil \frac{x-y+1}{y} \right\rceil$. □

**Theorem 3.15** *For a fixed $T$ polyomino, $T_{a,b}$, if $b > 2a$, then*

$$cp^{fix}(T_{a,b}) = \left\lceil \frac{b+1}{2a+1} \right\rceil .$$

**Proof.** The proof here is similar to the proof of Theorem 3.14 above, but consider anchors in columns rather than rows. □

For free $T$ polyominoes, we can find an exact value when $a = b$ and the bounds for $T_{a,b}$.

**Theorem 3.16** *For a free $T$ polyomino, $T_{a,b}$, if $a = b$, then*

$$cp^{free}(T_{a,a}) = 2.$$

**Proof.** Assume $a = b$. Consider placing a $T$ polyomino with anchor $C_{a+1,a+1}$ and another $TR$ polyomino with anchor $C_{2a+2,2a+1}$. See Figure 11

Notice that this creates three pairwise disjoint regions which cannot contain another $T$ polyomino. Therefore $cp^{free}(T_{a,a}) \leq 2$.

Assume by way of contradiction that $cp^{free}(T_{a,a}) = 1$. Note that $X_1$ or $X_n$ must be empty if $cp^{free}(T_{a,a}) = 1$. Assume, without loss of generality, a single $T$ polyomino, unrotated, is placed on the board so that $X_n$ is empty. If our single $T$ polyomino does not intersect $Y_n$, then we can place another $TR^2$ polyomino with anchor $C_{n,2a+1}$. If our single $T$ polyomino does intersect $X_n$, then we can place another $T$ polyomino with anchor $C_{1,a+1}$ since there are $2a$ consecutive empty rows above $T$. Therefore $cp^{free}(T_{a,a}) \geq 2$ which completes the proof. □

**Theorem 3.17** *For a free $T$ polyomino, $T_{a,b}$, $2 \leq cp^{free}(T_{a,b}) \leq 4$.*

**Proof.** Assume by way of contradiction that $cp^{free}(T_{a,b}) = 1$. Since $X_n$ or $X_1$ (the left or right column) must be empty, we will assume without loss of generality a single $T$ polyomino, unrotated, is placed on the board with anchor $C_{x,y}$, so that $X_n$ is empty.

If $Y_n$ (the bottom row) is empty, then we can place a $TR^2$ polyomino on the board with anchor $C_{x+1,n}$. Therefore, the $T$ polyomino must intersect $Y_n$ and therefore the anchor must be in row $y = 2a + 1$. Since $X_n$ is empty, we can place a polyomino $TR$ with an anchor $C_{n,a+1}$. This implies $cp^{free}(T_{a,b}) \geq 2$.

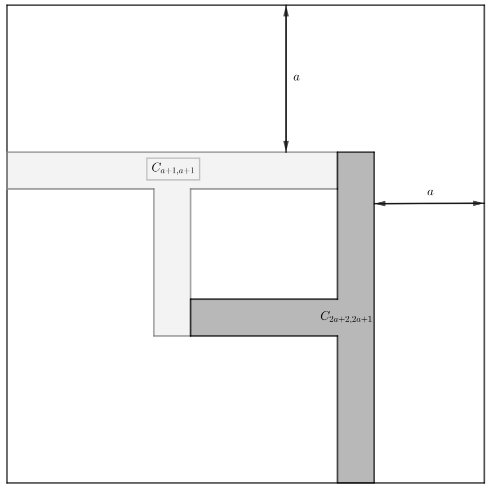


Figure 11: Shows $cp^{free}(T_{a,a}) = 2$.

Let $c = \min\{a, b\}$. Consider placing the following four $T$ polyominoes on the board: $T_{a,b}$ with anchor $C_{a+1,c}$, $TR_{a,b}$ with anchor $C_{n-c+1,a+1}$, $TR^2_{a,b}$ with anchor $C_{n-a,n-c+1}$, and $TR^3_{a,b}$ with anchor $C_{c,n-a}$.

Case 1: Assume $a < b$. Note that $T$ and $TR$ will always be adjacent. This is because $T$ would occupy cells $\{C_{x,y} : 1 \leq x \leq 2a+1, y = a\}$ and $TR$ would occupy cells $\{C_{x,y} : a+2 \leq x \leq n-a, y = a+1\}$. This implies that the four polyominoes will create five disjoint empty regions, four along the outside of the board and one on the interior of the board. The empty interior region will be a square with side length $b-1$, and therefore we cannot fit another $T$ polyomino in the region. Each empty region on the outside of the board will be subsets of a $a \times n$ rectangle and therefore cannot fit another $T$ polyomino. Thus, we have a clumsy packing with four $T$ polyominoes. See Figure 12.

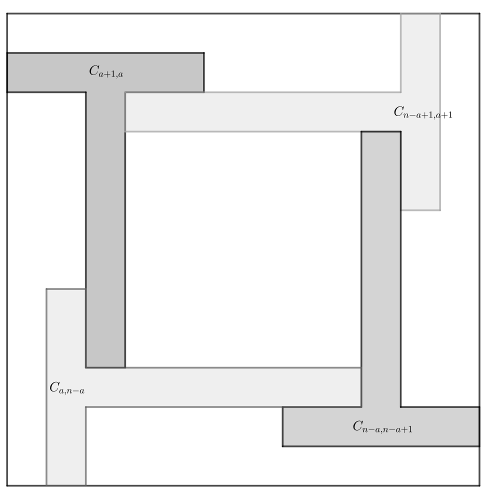


Figure 12: Pattern for $cp^{free}(T_{a,b})$ when $a < b$.

Case 2: Assume $b \leq a$. Note that $T$ and $TR$ will always be adjacent. This is because $T$ would occupy cells $\{C_{x,y} : 1 \leq x \leq 2a+1, y = b\}$ and $TR$ would occupy cells $\{C_{x,y} : x = n-c+1 = 2a+2, 1 \leq y \leq 2a+1\}$. This implies that the four poloyominoes will create five disjoint empty regions, four along the outside of the board and one on the interior of the board. The empty interior region will be a subset of a square with side length $n - 2b = 2a + 1 - b \leq 2a$ and therefore we can not fit another $T$ polyomino in the region. Each empty region on the outside of the board will be subsets of a $(b-1) \times 2a+1$ rectangle and therefore cannot fit another $T$ polyomino. Thus, we have a clumsy packing with four $T$ polyominoes. See Figure 13.

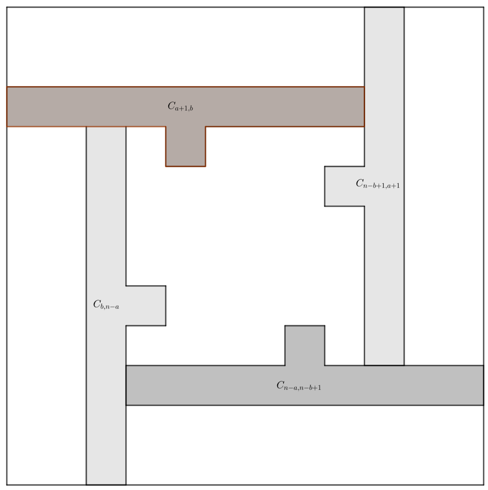


Figure 13: Pattern for $cp^{free}(T_{a,b})$ when $b \leq a$.

□

Note that these are not always optimal packings. For example, Theorem 3.16 shows that $cp^{free}(T_{a,a}) = 2$. Also, consider $a = 4$ and $b = 3$. This can be packed using only three $T_{4,3}$ polyominoes, specifically two $T_{4,3}$ polyominoes with centers $C_{5,4}$ and $C_{5,9}$ as well as $TR_{4,3}$ with center $C_{10,8}$. See Figure 14 for more details.

### 3.4 Plus Polyominoes

Our last polyomino is a *plus* polyomino.

**Definition 3.18** *A plus polyomino, $P_a$, is a polyomino such that $P_a = \{C_{i,a+1} : 1 \leq i \leq 2a+1\} \cup \{C_{a+1,j} : 1 \leq j \leq 2a+1\}$. The $|P_a| = 4a+1$. We will define the anchor of the polyomino to be the cell $C_{a+1,a+1}$.*

A *plus polyomino* when rotated by an integer multiple of 90 deg remains the same polyomino. Thus, a *free,* and *fixed,* polyominoes are indistinguishable. Therefore, we will use $cp(P_a)$ to denote the clumsy packing number, since $cp^{fix}(P_a) = cp^{free}(P_a)$

**Theorem 3.19** *For a plus polyomino, $P_a$, $cp(P_a) = 1$.*

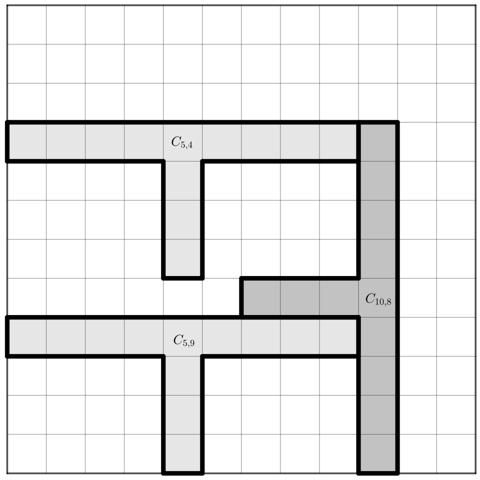


Figure 14: Shows $cp^{free}(T_{4,3}) \leq 3$.

**Proof.** Let $P_a$ denote a *plus* polyomino. Note that no polyomino can have an anchor in the left $a$ columns, right $a$ columns, top $a$ rows, or bottom $a$ rows. Place the polyomino $P_a$ on the board with anchor at cell $C_{2a+1,2a+1}$ and call this polyomino $P$. To place another *plus* polyomino anchor at the board we must have at least $2a+1$ consecutive empty cells in a single column. This only happens in the left $a$ columns and the right $a$ columns that cannot contain an anchor. Therefore $cp(P_a) = 1$.

□

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