On Prime Labelings of Uniform Cycle Snake Graphs

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Abstract - A *prime labeling* of a graph of order n is an assignment of the integers $1, 2, \dots, n$ to the vertices such that each pair of adjacent vertices has coprime labels. For positive integers m, k, q with $k \geq 3$ and $1 \leq q \leq \lfloor k/2 \rfloor$, the *uniform cycle snake graph* $C^m_{k,q}$ is constructed by taking a path with m edges and replacing each edge by a k-cycle by identifying two vertices at distance q in the cycle with the vertices of the original path edge. We construct prime labelings for $C^m_{k,q}$ for many pairs (k, q) and, in each case, all m. These include: all cases with $k \leq 9$ or $k = 11$; all cases with $q = 2$ when $k \equiv 3 \pmod 4$; all cases with $q = 3$ when k has the form $2^a - 1$, $3^a + 1$ or $3^b + 3$ for all a and all odd b; all cases with $q = 4$ when k is even; and all cases with $q = k/2$ when q is a prime congruent to 1 (mod 3).

Keywords : coprime integers; graph labeling; prime graph; snake graph

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1 Introduction

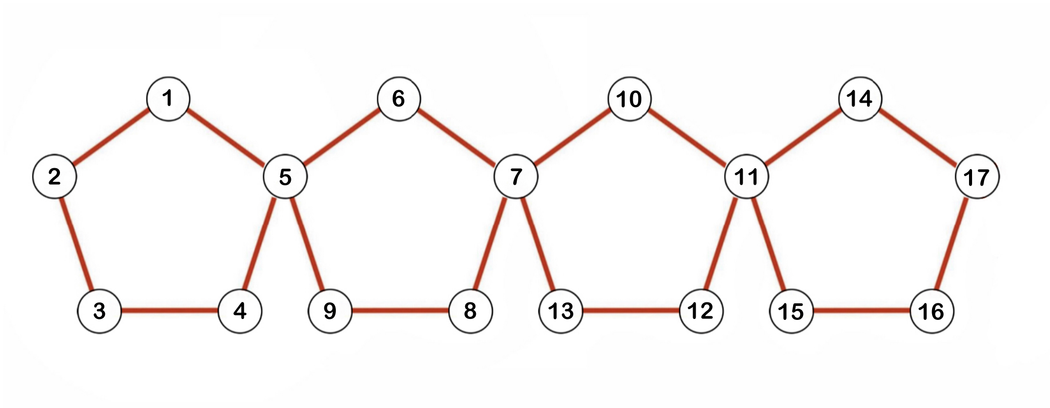
Let G be a graph of order n. A *prime labeling* of G is an assignment of the integers $\{1, 2, \dots, n\}$ to the vertices of G such that the labels of u and v are coprime whenever there is an edge between u and v. If a graph has a prime labeling, it is *prime*.

Prime graphs have been extensively studied. An introduction to the topic may be found in [3, Chapter 4]. It is conjectured that all trees and unicyclic graphs are prime, and this is known to hold for paths, stars, spiders, trees of order at most 50 and trees and unicyclic graphs of sufficiently large order. Other graphs known to be prime include cycles, fans, wheels of even order, books, complete bipartite graphs $K_{2,n}$ for all n and $K_{3,n}$ when $n \notin \{3, 7\}$, and grids $P_m \times P_n$ when m is prime and $1 \leq n \leq p^2$. See Gallian's comprehensive dynamic survey [5] for definitions, references and further results.

The purpose of this paper is to add to the collection of graphs known to be prime. We study the primality of "snake graphs," particularly "uniform cycle snake graphs," natural families of graphs that we define below. This work combines and generalizes several results from the existing literature, providing a more unified way of finding prime labelings for these types of graph and signifcantly expanding the roster of graphs for which a prime labeling is known to exist.

Let P_{m+1} be the path of length m with $m+1$ vertices $p_1, p_2, \dots, p_{m+1}$ in sequence. Let $\mathcal{G} = (G_1, G_2, \dots, G_m)$ be a sequence of graphs and let $\boldsymbol{u} = (u_1, u_2, \dots, u_m)$ and

Figure 1: A prime labeling of $C^4_{5,2}$.



$\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$ each be sequences of vertices, with $u_i, v_i \in V(G_i)$. Then the $\mathcal{G}_{\boldsymbol{u},\boldsymbol{v}}$-*snake* is the graph defined by replacing the edge (p_i, p_{i+1}) in P_{m+1} by the graph G_i, with u_i identified with p_i and v_i identified with p_{i+1}. We call these vertices *fusion points*.

If $G_1 \cong \cdots \cong G_m \cong G$, each of $u_1, u_2, \ldots, u_m$ correspond to the same vertex u of G and each of $v_1, v_2, \ldots v_m$ correspond to the same vertex v of G, then call the corresponding $\mathcal{G}_{\boldsymbol{u},\boldsymbol{v}}$-snake *uniform*. That is, a uniform snake is one in which we replace each edge of the path with the same graph in the same way. If the graph used in a uniform snake is a cycle, then it is a *uniform cycle snake*. As in [1], we denote by $C^m_{k,q}$ the uniform cycle snake with m cycles, each of size k, and with q edges in a path between consecutive fusion points. We may always take $q \leq \lfloor k/2 \rfloor$ and we shall usually do so; however, sometimes it is convenient to describe a general construction in such a way that $q > \lfloor k/2 \rfloor$ for some instances.

A prime labeling of $C^4_{5,2}$, the uniform cycle snake with four 5-cycles joined at distance 2, is given in Figure 1. Note that many pairs of adjacent labels are consecutive. Two consecutive numbers are necessarily coprime and our constructions take advantage of this by using consecutive labels on adjacent vertices as often as possible.

Theorem 1.1 collects what is known about primality of uniform cycle snakes.

Theorem 1.1 *Let $m \geq 1$, $k \geq 3$ and $1 \leq q \leq \lfloor k/2 \rfloor$. If $k < 5$, let $p = 1$; otherwise let p be the smallest prime factor of $k-2$ when k is odd and the smallest prime factor of $k-3$ when k is even. The uniform cycle snake $C^m_{k,q}$ is prime in the following cases:*

- *$m \leq 2$,* [6, reported in [5]; see also [7]],
- *$q = 1$* [8, reported in [5]; see also [2]],
- *$q = 2$, k is odd and $m \leq p$* [1],
- *$q = 3$, $k \equiv 4 \pmod 6$ and $m < p/2$* [1],

Table 1: Existence of prime labelings for uniform cycle snakes with small cycles. An entry in row q and column k indicates that $C^m_{k,q}$ is prime for all m, where the entry indicates the theorem number in the present paper that proves this and/or a reference to a proof elsewhere.

$q\backslash k$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	[4]	2.1	4.1	3.3	6.2	6.1	4.2	3.3	·	3.4	·	3.3	·	·	4.2	3.3
3			[4], 5.2	3.1	5.1	5.1	4.4	5.1	5.2	3.1	·	·	·	5.1	·	3.1
4					[4], 4.3	5.1	4.3	5.1	4.3	3.1	4.3	·	4.3	·	4.3	·
5							·	[1], 6.4	·	·	·	·	[1]	·	·	·
6									·	3.1	·	·	·	·	·	·
7											6.3	3.5	·	·	·	3.1
8													4.1	·	4.3	3.1
9															·	3.1

- $q = 5$ *and* $k \equiv 1 \pmod 5$ [1],
- $q = k/2$ *and* $k = 6$ *or* $k - 1$ *is a Mersenne prime* [4].

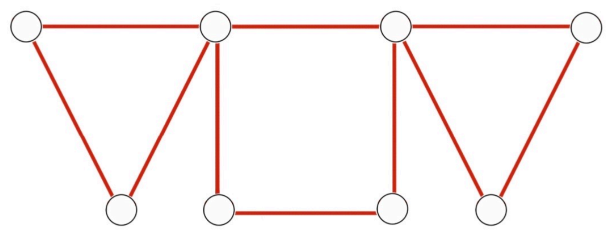
In this paper we extend this result considerably. All of our results are for arbitrary m. Some families we show to be prime are the uniform cycle snakes $C^m_{k,q}$ in the following cases:

- $q = 2$ when $k \equiv 3 \pmod 4$ (see Theorem 3.3),
- $q = 2$ when k has the form $2^a + 2$ or $t^a + t^b + 1$ for $a, b \geq 1$ and t odd (see Theorems 3.4, 4.2),
- $q = 3$ when k has the form $2^t - 1$, $3^a + 1$ or $3^a + 3$ for $a \geq 1$ (see Theorems 3.2, 4.4, 5.2),
- $q = 4$ when k is even (see Theorem 4.3),
- $q = (k - 1)/2$ when q is prime and $q + 1$ has exactly one odd prime factor (see Theorems 2.1 and 6.4),
- $q = k/2$ when q is prime and $q \equiv 1 \pmod 3$ (see Theorem 6.3).

There are many further constructions that are less general, are less concise to state, or are focused on specific small cases. Table 1 shows what is covered for $k \leq 19$ and $2 \leq q \leq 9$, including where to find the construction in the paper. Note that this includes prime labelings for $C^m_{k,q}$ for all q and m when $k \leq 9$ or $k = 11$.

In the next section we give some efficient notation for labelings of cycle snakes and introduce some tools that are used throughout the paper. The constructions add labels one cycle at a time in a repeating pattern. The bulk of the paper, Sections 3–6, gives

Figure 2: A non-prime non-uniform cycle snake.



these constructions and their corresponding results in increasing order of the period of the repeating pattern.

Given the success finding prime labelings for uniform cycle snakes, we propose the following conjecture.

Conjecture 1.2 For all $m \geq 1$, $k \geq 3$ and q with $1 \leq q \leq \lfloor k/2 \rfloor$, the uniform cycle snake $C^m_{k,q}$ is prime.

A natural more general conjecture that all cycle snakes, regardless of uniformity, are prime is not true. To see this, consider the cycle snake with cycles of sizes 3, 4 and 3 respectively shown in Figure 2. There are 8 vertices, 4 of which must have even labels. No pair of even labels may be adjacent in a prime labeling and it is easy to check that there are no four mutually non-adjacent vertices in the graph (that is, there is no "independent set" of size 4).

2 Notation and Tools

Describe a labeled snake by writing the labels for each cycle in turn by listing them in the order they appear starting from adjacent to the fusion point to the previous cycle, and separate the cycles with vertical lines. Mark the label on the fusion point to the next cycle with a circumflex.

For example, here's the labeling of $C^4_{5,2}$ from Figure 1 in this new notation:

$$[1, 2, 3, 4, \widehat{5} \mid 6, \widehat{7}, 8, 9 \mid 10, \widehat{11}, 12, 13 \mid 14, 17, 16, 15]$$

To check for the primality of a labeling in this notation, we need to compare the end vertices of the first cycle list, compare all adjacencies within the list for each cycle, and compare each label with a circumflex to the end vertices of the subsequent cycle list. When checking for coprimality a frequently useful fact is that $\gcd(x, y) = \gcd(x, y - x)$.

The following result, for one of the smallest open cases, illustrates several of the methods we use through the course of the paper.

Theorem 2.1 *Every uniform cycle snake of the form $C^m_{5,2}$ is prime.*

Proof. We add the labels one cycle at a time, and the pattern repeats every three cycles. Initialize the process by labeling the first cycle $\mid 1, 2, 3, 4, \widehat{5} \mid$, where 5 is the vertex to be used as the fusion point.

Suppose we have a prime labeling on a graph with $12x + 5$ vertices. We may fuse a 5-cycle to the vertex labeled $12x + 5$ in such a way that a new vertex at distance 2 from the fusion point has label $12x + 7$ using the pattern

$$\mid 12x + 6, \widehat{12x + 7}, 12x + 8, 12x + 9 \mid .$$

All internal neighbors are consecutive. Where the cycle attaches to the previous one, we have neighbors $12x + 6$, also consecutive, and $12x + 9$. We have $\gcd(12x + 5, 12x + 9) = \gcd(12x + 5, 4) = 1$.

Now suppose we have a prime labeling on a graph with $12x+9$ vertices. We may fuse a 5-cycle to the vertex labeled $12x+7$ in such a way that a new vertex at distance 2 from the fusion point has label $12x + 11$ using the pattern

$$\mid 12x + 10, \widehat{12x + 11}, 12x + 12, 12x + 13 \mid .$$

All internal neighbors are consecutive. Where the cycle attaches to the previous one, we have neighbors $12x+10$ and $12x+13$. We have $\gcd(12x+7, 12x+10) = \gcd(12x+7, 3) = 1$ and $\gcd(12x + 7, 12x + 13) = \gcd(12x + 7, 6) = 1$.

Lastly, suppose we have a prime labeling on a graph with $12x + 13$ vertices. We may fuse a 5-cycle to the vertex labeled $12x+11$ in such a way that a new vertex at distance 2 from the fusion point has label $12x + 17$ using the pattern

$$\mid 12x + 14, \widehat{12x + 17}, 12x + 16, 12x + 15 \mid .$$

There is one pair of non-consecutive internal vertices: $12x + 14$ and $12x + 17$. We have $\gcd(12x + 14, 12x + 17) = \gcd(12x + 14, 3) = 1$. Where the cycle attaches to the previous one, we have neighbors $12x + 14$ and $12x + 15$. We have $\gcd(12x + 11, 12x + 14) = \gcd(12x + 11, 3) = 1$ and $\gcd(12x + 11, 12x + 15) = \gcd(12x + 11, 4) = 1$.

To complete the proof, note that after the third pattern we have $12x+17 = 12(x+1)+5$ vertices and may return to the first pattern. □

In this proof we wrote all labels in the form $12x + y$. This allowed us to easily see that the pattern repeated after three steps and, more importantly, let us ensure that the non-consecutive adjacent labels used were indeed coprime.

In general, we shall use the number of new vertices added by the repeating pattern to play the role of 12 and denote this by α. For any positive integer a, let $\vartheta(a)$ be the set of prime divisors of a. We formalize how to quickly perform the coprimality checks we require in the following lemma.

Lemma 2.2 *Let α, y, z be positive integers with $y < z$. If*

$$\vartheta(z - y) \subseteq \vartheta(\alpha) \quad \text{and} \quad \gcd(y, z) = 1$$

then $\gcd(\alpha x + y, \alpha x + z) = 1$ *for all* $x \geq 0$.

Proof. Suppose p is prime and $p \mid \gcd(\alpha x + y, \alpha x + z)$. Since $\gcd(\alpha x + y, \alpha x + z) = \gcd(\alpha x + y, z - y)$ we have $p \mid z - y$. Therefore, $p \mid \alpha$, as $\vartheta(z - y) \subseteq \vartheta(\alpha)$, and so $p \mid \alpha x$. As we also know that $p \mid \alpha x + y$ it must be that $p \mid y$; similarly $p \mid z$. This contradicts that $\gcd(y, z) = 1$, hence $\gcd(\alpha x + y, \alpha x + z) = 1$. $\square$

In light of this lemma, we can be more effficient in our proofs. Rather than displaying each label as $\alpha x + y$ we shall just write y and obtain the true labeling by adding αx to each element. This reduces the display of the three cycles of the proof of Theorem 2.1 to

$$\mid 6, \widehat{7}, 8, 9 \mid 10, \widehat{11}, 12, 13 \mid 14, \widehat{17}, 16, 15 \mid .$$

As $\vartheta(12) = \{2, 3\}$, the task of checking the necessary coprimalities reduces to ensuring that when we have a pair of non-consective adjacent labels the labels are coprime and their difference has no prime factors other than 2 or 3. The pairs to check here are

$$(5, 9), (7, 10), (7, 13), (14, 17), (11, 14), (11, 15),$$

where the $(5, 9)$ comes from the assumption about how the first cycle is attached.

A graph with n vertices is *r-coprime* if it can be labeled with n consecutive integers $\{r, r + 1, \ldots, r + n - 1\}$ in such a way that every pair of adjacent vertices has coprime labels. Hence 1-coprimality is exactly the same as primality.

The following straightforward result is useful for some of our constructions.

Lemma 2.3 *If a graph is 2-coprime then it is prime.*

Proof. Let G be a 2-coprime graph with n vertices. Take a 2-coprime labeling of G and replace the label $n + 1$ with 1. The labeling now uses the integers $\{1, 2 \ldots, n\}$. All adjacent pairs of vertices are coprime as either they are adjacent in the 2-coprime labeling or one of them is 1, which is coprime with all other labels. $\square$

3 Constructions with Period 1

We start with constructions with period 1; that is, those that use a single pattern repeated on every cycle. In the notation of the last section, this means that when considering $C^m_{k,q}$ we use $\alpha = k - 1$.

Theorem 3.1 covers eight specific (k, q) pairs (for all m in each case, meaning each gives an infinite family of prime graphs) and then Theorems 3.2–3.5 give some infinite families of (k, q) pairs (again for all m in each case).

Theorem 3.1 *The uniform cycle snakes $C^m_{7,3}$, $C^m_{13,3}$, $C^m_{13,4}$, $C^m_{13,6}$, $C^m_{19,3}$, $C^m_{19,7}$, $C^m_{19,8}$ and $C^m_{19,9}$ are prime for all m.*

Proof. Consider $C^m_{7,3}$. Label the first cycle with $1, \ldots, 7$ in consecutive order and take the vertex labeled 5 to be the fusion point. Let $\alpha = 6$ and consider the pattern

$$\mid 8, 9, 10, \widehat{11}, 12, 13 \mid .$$

All internal neighbors are adjacent and so coprime. The pairs to check where this joins the previous pattern are $(5, 8)$ and $(5, 13)$. These are coprime pairs and their differences are 3 and 8, each of whose prime factors are all in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $6x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{7,3}$.

Consider $C^m_{13,3}$. Label the first cycle with $1, \ldots, 13$ in consecutive order and take the vertex labeled 13 to be the fusion point. Let $\alpha = 12$ and consider the pattern

$$| \, 22, 21, \widehat{25}, 24, 23, 20, 19, 18, 17, 16, 15, 14 \, | \, .$$

The internal non-consecutive neighbors are $(21, 25)$ and $(20, 23)$ and the pair to check where this joins the previous pattern is $(13, 22)$ (and there is also the consecutive pair $(13, 14)$). These are coprime pairs and their differences are 4, 3 and 9, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $12x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{13,3}$.

Consider $C^m_{13,4}$. Label the first cycle with $1, \ldots, 13$ in consecutive order and take the vertex labeled 13 to be the fusion point. Let $\alpha = 12$ and consider the pattern

$$| \, 22, 23, 24, \widehat{25}, 21, 20, 19, 18, 17, 16, 15, 14 \, | \, .$$

The internal non-consecutive neighbors are $(25, 21)$ and the pair to check where this joins the previous pattern is $(13, 22)$ (and there is also the consecutive pair $(13, 14)$). These are coprime pairs and their differences are 4 and 9, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $12x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{13,4}$.

Consider $C^m_{13,6}$. Label the first cycle with $1, \ldots, 13$ in consecutive order and take the vertex labeled 13 to be the fusion point. Let $\alpha = 12$ and consider the pattern

$$| \, 16, 15, 14, 23, 24, \widehat{25}, 22, 21, 20, 19, 18, 17 \, | \, .$$

The internal non-consecutive neighbors are $(14, 23)$ and $(22, 25)$ and the pairs to check where this joins the previous pattern are $(13, 16)$ and $(13, 17)$. These are coprime pairs and their differences are 9, 3, 3 and 4, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $12x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{13,6}$.

Consider $C^m_{19,3}$. Label the first cycle with $1, \ldots, 19$ in consecutive order and take the vertex labeled 19 to be the fusion point. Let $\alpha = 18$ and consider the pattern

$$| \, 20, 21, \widehat{37}, 36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22 \, | \, .$$

The internal non-consecutive neighbors are $(21, 37)$ and the pair to check where this joins the previous pattern is $(19, 22)$ (and there is also the consecutive pair $(19, 20)$). These are coprime pairs and their differences are 16 and 3, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $18x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{19,3}$.

Consider $C^m_{19,7}$. Label the first cycle with $1, \ldots, 19$ in consecutive order and take the vertex labeled 19 to be the fusion point. Let $\alpha = 18$ and consider the pattern

$$|\ 23, 24, 25, 26, 27, 28, \widehat{37}, 36, 35, 34, 33, 32, 29, 30, 31, 22, 21, 20\ |\ .$$

The internal non-consecutive neighbors are $(28, 37)$, $(32, 29)$ and $(31, 22)$ and the pair to check where this joins the previous pattern is $(19, 23)$ (and there is also the consecutive pair $(19, 20)$). These are coprime pairs and their differences are 9, 3, 9 and 4, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $18x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{19,7}$.

Consider $C^m_{19,8}$. Label the first cycle with $1, \ldots, 19$ in consecutive order and take the vertex labeled 19 to be the fusion point. Let $\alpha = 18$ and consider the pattern

$$|\ 22, 23, 24, 25, 26, 27, 28, \widehat{37}, 36, 35, 34, 33, 32, 31, 30, 29, 20, 21\ |\ .$$

The internal non-consecutive neighbors are $(20, 29)$ and $(28, 37)$ and the pairs where this joins the previous pattern are $(19, 21)$ and $(19, 22)$. These are coprime pairs and their differences are 9, 9, 2 and 3, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $18x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{19,8}$.

Consider $C^m_{19,9}$. Label the first cycle with $1, \ldots, 19$ in consecutive order and take the vertex labeled 19 to be the fusion point. Let $\alpha = 18$ and consider the pattern

$$|\ 23, 24, 25, 26, 27, 28, \widehat{37}, 36, 35, 34, 33, 32, 29, 30, 31, 22, 21, 20\ |\ .$$

The internal non-consecutive neighbors are $(28, 37)$, $(32, 29)$ and $(31, 22)$ and the pair to check where this joins the previous pattern is $(19, 23)$ (and there is also the consecutive pair $(19, 20)$). These are coprime pairs and their differences are 9, 3, 9 and 4, each of whose prime factors are in $\vartheta(\alpha) = \{2, 3\}$. Hence by taking translates of this pattern by $18x$ for $x \geq 0$ we obtain a prime labeling for $C^m_{19,7}$. $\square$

In the proof of Theorem 3.1, the simplest pattern is perhaps that used for $C^m_{7,3}$, which takes the next $\alpha = 6$ labels and uses them consecutively in the new cycle. Theorem 3.2 gives infinitely many pairs (k, q) for which this approach works.

Theorem 3.2 *Let $t \geq 3$ be odd. Every uniform cycle snake of the form $C^m_{2^t-1,3}$ is prime.*

Proof. Let $\alpha = 2^t - 2$. Consider the pattern

$$|\ 2^{t+1} - 3, 2^{t+1} - 4, \widehat{2^{t+1} - 5}, 2^{t+1} - 6, \ldots, 2^t\ |\ .$$

After adding αx for some x, we shall attach this to the vertex labeled $\alpha x + 2^t - 3$ from the previous cycle.

There are no non-consecutive internal adjacent pairs. From fusion point of the previous cycle we get the pair $(2^t-3, 2^t)$ and the pair $(2^t-3, 2^{t+1}-3)$. Considering the first pair, they are clearly coprime, as $2^t - 3$ is odd, and their difference is 3. As t is odd, $3 \mid 2^t - 2 = \alpha$. Considering the second pair, we have $\gcd(2^t - 3, 2^{t+1} - 3) = \gcd(2^t - 3, 2^t) = 1$. The difference of 2^t has $\vartheta(2^t) = \{2\}$ and $2 \in \vartheta(\alpha)$ as α is even.

Take the first cycle to be $\mid 1, 2, \ldots, \widehat{2^t-3}, 2^t-2, 2^t-1 \mid$ to obtain a prime labeling for $C^m_{2^t-1,3}$. $\square$

For example, to construct a prime labeling for $C^m_{31,3}$ using the proof of Theorem 3.2, take $t = 5$ and the labeling begins

$$[\ 1, 2, \ldots, \widehat{29}, 30, 31 \mid 61, 60, \widehat{59}, 58, \ldots, 32 \mid 91, 90, \widehat{89}, 88, \ldots, 62 \mid \cdots\ ].$$

The next two results again use a long stretch of consecutive elements in the pattern, but remove one of them (leaving two consecutive odd numbers) to use on the short path between fusion points.

Theorem 3.3 *The uniform cycle snakes $C^m_{4t+3,2}$ are prime for all m and t.*

Proof. Let $\alpha = 4t + 2$. Consider the pattern

$$\mid 2t+2, \widehat{4t+3}, 4t+2, 4t+1, \ldots, 2t+3, 2t+1, 2t \ldots, 2 \mid .$$

After adding αx for some x, we shall attach this to the vertex labeled $\alpha x + 1$ from the previous cycle (and note that we use $4t + 3 = \alpha \cdot 1 + 1$ to connect to the next cycle, so the pattern repeats successfully).

The non-consecutive internal adjacent pairs are $(2t+2, 4t+3)$ and $(2t+3, 2t+1)$. We have $\gcd(2t+2, 4t+3) = \gcd(2t+2, 2t+1) = 1$ and $\gcd(2t+3, 2t+1) = \gcd(2, 2t+1) = 1$. Further, the former has difference $2t+1$ and the latter has difference 2 and $\vartheta(2t+1)$ and $\vartheta(2)$ are contained in $\vartheta(\alpha)$ as $2t+1$ and 2 both divide α.

Adjacent to the fusion point of the previous cycle we get the consecutive pair $(1, 2)$ and the pair $(1, 2t+2)$. The latter is clearly coprime and the difference $2t+1$ is acceptable as shown in the previous paragraph.

Consider the first "cycle" to be the single vertex labeled 1, and we may use the pattern from there. Hence $C^m_{4t+3,2}$ is prime. $\square$

For example, when $t = 3$ we get a prime labeling for $C^m_{15,2}$. Here are the first three cycles:

$$[\ 8, \widehat{15}, 14, 13, \ldots, 9, 7, 6, \ldots 2, 1 \mid 22, \widehat{29}, 28, \ldots, 23, 21, 20, \ldots, 16 \mid$$

$$36, \widehat{43}, 42, \ldots, 37, 35, 34, \ldots, 30 \mid\ \cdots\ ].$$

Theorem 3.4 *Let t be odd and take $a, b \geq 1$. The uniform cycle snakes $C^m_{k,2}$ are prime for $k = t^a + t^b + 1$ and all m.*

Proof. Let $\alpha = t^a + t^b$. Consider the following pattern, which is very similar to that of the proof of Theorem 3.3:

$$\mid t^a + 1, \widehat{t^a + t^b + 1}, t^a + t^b, \ldots, t^a + 2, t^a, \ldots, 2 \mid .$$

After adding αx for some x, we shall attach this to the vertex labeled $\alpha x + 1$ from the previous cycle.

The non-consecutive internal adjacent pairs are (t^a+1, t^a+t^b+1) and (t^a+2, t^a). Consider the former first. We have $\gcd(t^a+1, t^a+t^b+1) = \gcd(t^a+1, t^b)$, which is 1 as any prime p that divides t^b must also divide t and $t^a+1 \equiv 1 \pmod{p}$. Also, any prime that divides the difference t^b divides $t^a+t^b = \alpha$. The latter two are coprime, as they are consecutive odd numbers, and have difference 2, which divides α.

Adjacent to the fusion point of the previous cycle we get the consecutive pair $(1, 2)$ and the pair $(1, t^a+1)$. The latter is clearly coprime and the difference t^a is acceptable for the same reasons that t^b is.

Consider the first "cycle" to be the single vertex labeled 1, and we may use the pattern from there. Hence $C^m_{t^a+t^b+1,2}$ is prime. □

The smallest case that falls to Theorem 3.4 but not Theorem 3.3 is $C^m_{13,2}$, using $13 = 3+3^2+1$. In this case the labeling starts

$$[\ 4, \widehat{13}, 12, \ldots, 5, 3, 2, 1 \mid 16, \widehat{25}, 24, \ldots, 17, 15, 14 \mid 28, \widehat{37}, 36, \ldots, 29, 27, 26 \mid \cdots\ ].$$

Other relatively small values of k with $k \equiv 1 \pmod{4}$ for which Theorem 3.4 is successful include $7+7^2+1 = 57$, $3+3^4+1 = 85$ and $11+11^2+1 = 133$.

The last result for this section uses the labeling method of [1], called a "cyclic labeling" there, and we see that it applies to many cases not covered in that paper. It generalizes the pattern used for $C^m_{19,3}$ in Theorem 3.1.

Theorem 3.5 *Let t be odd and a such that $t \mid 2^a-1$. The uniform cycle snakes $C^m_{2^a+t,t}$ are prime for all m. In particular, $C^m_{2^{a+1}-1,2^a-1}$ is prime for all a and m.*

Proof. Let $\alpha = 2^a+t-1$. Consider the pattern

$$\mid 2, 3, \ldots, t, \widehat{2^a+t}, 2^a+t-1, \ldots, t+1 \mid .$$

After adding αx for some x, we shall attach this to the vertex labeled $\alpha x+1$ from the previous cycle. There is one pair of non-adjacent vertices within the cycle: $(2^a+t, t)$. The difference is 2^a and $2 \in \vartheta(\alpha)$ as α is even. We have $\gcd(2^a+t, t) = 1$.

Adjacent to the fusion point of the previous cycle we get the consecutive pair $(1, 2)$ and the pair $(1, t+1)$. The latter is clearly coprime and the difference t divides 2^a-1 and hence also divides $\alpha = 2^a+t-1$. So $C^m_{2^a+t,t}$ is prime for all m.

To see the last part of the statement, set $t = 2^a-1$. □

Some smaller examples covered by Theorem 3.5 are $C^m_{15,7}$, $C^m_{21,5}$, $C^m_{31,15}$, $C^m_{63,31}$ $C^m_{71,7}$ and $C^m_{73,9}$.

4 Constructions with Period 2

The constructions for $C^m_{k,q}$ of the previous section all had k odd. In this section we see our first constructions for even k. We start with a result for $C^m_{6,2}$ and $C^m_{16,8}$. These constructions are generalized in two different ways in Theorem 4.2, which gives more constructions with k even and $q = 2$, and Theorem 4.3, which gives more constructions with q a power of 2. Finally in this section, Theorem 4.4 shows that $C^m_{t^a+1,t}$ is prime when $a \geq 2$ and t is odd.

Theorem 4.1 *The uniform cycle snakes $C_{6,2}^m$ and $C_{16,8}^m$ are prime for all m.*

Proof. Consider $C_{6,2}^m$. We construct a 2-coprime labeling and apply Lemma 2.3. Let $\alpha = 10$. Consider the pair of patterns

$$\mid 8, \widehat{9}, 10, 11, 12 \mid 13, \widehat{17}, 16, 15, 14 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x + 7$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(7, 8), (7, 12), (9, 13), (9, 14), (13, 17).$$

Each pair is coprime; the differences are 1, 5, 4, 5 and 4 respectively and the prime factors of each are in $\vartheta(10) = \{2, 5\}$.

We may attach the above patterns to the first cycle $\mid 2, 3, 4, 5, 6, \widehat{7} \mid$ to see that $C_{6,2}^m$ is 2-coprime and hence prime.

Consider $C_{16,8}^m$. We construct a 2-coprime labeling and apply Lemma 2.3. Let $\alpha = 30$. Consider the pair of patterns

$$\mid 18, 23, 24, 25, 22, 21, 20, \widehat{19}, 28, 27, 26, 29, 30, 31, 32 \mid$$

and

$$\mid 35, 36, 37, 38, 33, 34, 39, \widehat{47}, 42, 41, 40, 43, 46, 45, 44 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x + 17$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(17, 32), (18, 23), (19, 28), (22, 25), (26, 29), (19, 35), (19, 44),$$
$$(33, 38), (34, 39), (39, 47), (40, 43), (42, 47), (43, 46).$$

Each pair is coprime; the differences are 15, 5, 9, 3, 3, 16, 25, 5, 5, 8, 3 and 5 respectively and the prime factors of each are in $\vartheta(30) = \{2, 3, 5\}$.

We may attach the above patterns to first cycle $\mid 2, 3, \ldots, \widehat{17} \mid$ to see that $C_{16,8}^m$ is 2-coprime and hence prime. $\square$

For example, from the proof of Theorem 4.1 we obtain the following prime labeling of $C_{6,2}^6$:

$$[\ 2, 3, 4, 5, 6, \widehat{7} \mid 8, \widehat{9}, 10, 11, 12 \mid 13, \widehat{17}, 16, 15, 14 \mid$$
$$18, \widehat{19}, 20, 21, 22 \mid 23, \widehat{27}, 26, 25, 24 \mid 28, 29, 30, 31, 1\ ].$$

The next two results generalize Theorem 4.1, the first keeping $q = 2$ and the second covering larger powers of 2.

Theorem 4.2 *Suppose $t = 2^a + 1$ for some a. Then the uniform cycle snake $C_{2t,2}^m$ is prime for all m.*

Proof. As in the proof of Theorem 4.1, we first show that the graph is 2-coprime. Let $\alpha = 4t-2$. Consider the pair of patterns

$$\mid 4t, \widehat{4t-1}, 4t-2, \ldots, 2t+2 \mid 6t-2, \widehat{6t-1}, 4t+1, 4t+2, \ldots, 6t-3 \mid .$$

All of the adjacent pairs within the first pattern are consecutive. We join it to the label $2t+1$, which gives the adjacent consecutive pair $(2t+1, 2t+2)$ and also the pair $(2t+1, 4t)$. We have $\gcd(2t+1, 4t) = \gcd(2t+1, 2t-1) = 1$ and the difference $2t-1$ divides $\alpha = 4t-2$.

Internally to the second pattern, there is one non-consecutive adjacent pair: $(6t-1, 4t+1)$. We have $\gcd(6t-1, 4t+1) = \gcd(2t-2, 4t+1) = 1$, as $2t-2 = 2^{a+1}$ and $4t+1$ is odd. The difference $2t-2$ is a power of 2 and 2 divides $\alpha = 4t-2$. Where the second patttern connects to the first, we have the pairs $(4t-1, 6t-2)$ and $(4t-1, 6t-3)$. We have $\gcd(4t-1, 6t-2) = \gcd(4t-1, 2t-1) = \gcd(2t, 2t-1) = 1$ and $\gcd(4t-1, 6t-3) = \gcd(4t-1, 2t-2) = 1$ as we again have a power of 2 and an odd number. The difference of the first pair is $2t-1$ which divides α and the difference of the second pair is $2t-2 = 2^{a+1}$ and $2 \in \vartheta(\alpha)$.

We may attach the above patterns to first cycle $\mid 2, 3, \ldots, \widehat{2t+1} \mid$ to see that $C^m_{2t,2}$ is 2-coprime and so prime by Lemma 2.3. $\square$

Theorem 4.3 *Let s, t be such that $2t > 2^s \geq 4$ and $\gcd(2^s-3, 2t-1) = 1$. Then the uniform cycle snake $C^m_{2t,2^s}$ is prime for all m. In particular, $C^m_{2t,4}$ is prime for all t and m.*

Proof. We show that $C^m_{2t,2^s}$ is 2-coprime and apply Lemma 2.3. Let $\alpha = 4t-2$. There are two patterns that repeat. The first, which shall connect to a vertex labeled $\alpha x + 2t + 1$ for some x, is as follows:

$$\mid 4t, 4t-1 \ldots, 4t \widehat{-2^s} + 1, \ldots, 2t+2 \mid .$$

All internal adjacent labels are consecutive and there is one non-adjacent pair at the fusion point: $2t+1$ and $4t$. We have $\gcd(2t+1, 4t) = \gcd(2t+1, 2t-1) = 1$ and the difference $2t-1$ divides α.

The second pattern is

$$\mid 6t-2^s, 6t-2^s+1, \ldots, \widehat{6t-1}, 6t-2^s-1, \ldots, 4t+1 \mid .$$

Both labels adjacent to the existing fusion point, $6t-2^s$ and $4t+1$, are not consecutive with the fusion point label $4t-2^s+1$. We have $\gcd(4t-2^s+1, 6t-2^s) = \gcd(4t-2^s+1, 2t-1) = \gcd(2t-2^s+2, 2t-1) = \gcd(2^s-3, 2t-1)$, which is 1 by hypothesis. The difference $2t-1$ divides α. For the second point, we have $\gcd(4t-2^s+1, 4t+1) = \gcd(4t+1, 2^s) = 1$ and the difference 2^s has $\vartheta(2^s) = \{2\} \subseteq \vartheta(\alpha)$.

There is one internal non-consecutive pair of adjacent labels: $6t-1$ and $6t-2^s-1$. We have $\gcd(6t-1, 6t-2^s-1) = \gcd(6t-1, 2^s) = 1$ and the difference 2^s is acceptable for the same reasons as the previous paragraph.

If we take the first cycle to be $\mid 2, 3, \ldots \widehat{2t+1} \mid$ we obtain a 2-coprime labeling and hence a prime labeling by Lemma 2.3. $\square$

For example, the prime labeling for $C^4_{12,4}$ given by the proof of Theorem 4.3 is:

$$[\ 2, \ldots, \widehat{13} \mid 24, 23, 22, \widehat{21}, 20, \ldots, 14 \mid 32, 33, 34, \widehat{35}, 31, \ldots, 25 \mid 1, 45, 44, \widehat{43}, 42, \ldots, 36].$$

Lastly for this section we give a period 2 construction with q odd.

Theorem 4.4 *Let t be odd and $a \geq 2$. The uniform cycle snake $C^m_{t^a+1,t}$ is prime for all m.*

Proof. Let $\alpha = 2t^a$. Consider the pair of patterns

$$\mid 2t^a + 1, 2t^a, \ldots, \widehat{2t^a - t +} 2, \ldots, t^a + 2 \mid$$

and

$$\mid 3t^a - t + 2, 3t^a - t + 3, \ldots, \widehat{3t^a + 1}, 3t^a - t + 1, 3t^a - t, \ldots, 2t^a + 2 \mid .$$

The internal neighbors of the former are all consecutive. It will be joined to label t^a+1. This introduces neighbors $t^a + 2$, which is again consecutive, and $2t^a + 1$. We have $\gcd(t^a + 1, 2t^a + 1) = \gcd(t^a + 1, t^a) = 1$ and the difference t^a divides α.

For the latter pattern, we have one instance of non-consective neighbors: $3t^a + 1$ and $3t^a - t + 1$. We have $\gcd(3t^a + 1, 3t^a - t + 1) = \gcd(3t^a + 1, t) = 1$ and the difference t divides α.

Where the second pattern joins the first we introduce two pairs of neighbors that need checking. First consider $2t^a - t + 2$ and $3t^a - t + 2$. We have $\gcd(2t^a - t + 2, 3t^a - t + 2) = \gcd(2t^a - t + 2, t^a)$ and for any necessarily-odd prime p dividing t^a we have that p also divides t and so $p \nmid 2t^a - t + 2$. Hence these neighbors are coprime. The difference t^a divides α. The second pair of neighbors introduced is $2t^a - t + 2$ and $2t^a + 2$. Here, $\gcd(2t^a - t + 2, 2t^a + 2) = \gcd(t, 2t^a - 2) = 1$ as t is odd. The difference t divides α.

Using initial cycle $|1, 2, \ldots, \widehat{t^a + 1}|$, we conclude that $C^m_{t^a+1,t}$ is prime for all m. □

Some smaller examples covered by Theorem 4.4 are $C^m_{10,3}$, $C^m_{26,5}$, $C^m_{28,3}$, $C^m_{50,7}$, $C^m_{82,3}$ and $C^m_{82,9}$. The prime labeling for $C^m_{10,3}$ begins

$$[\ 1, 2, \ldots, \widehat{10} \mid 19, 18, \widehat{17}, 16, \ldots, 11 \mid 26, 27, \widehat{28}, 25, 24, \ldots, 20 \mid \cdots\ ].$$

5 Constructions with Period 3

As the period increases, it seems that more specific cases become possible but that they become harder to generalize. The first result in this section gives six specific cases for inclusion in Table 1. The second is an infinite family for $q = 3$.

Theorem 5.1 *The uniform cycle snakes $C^m_{8,3}$, $C^m_{9,3}$, $C^m_{9,4}$, $C^m_{11,3}$, $C^m_{11,4}$, and $C^m_{17,3}$ are prime for all m.*

Proof. Consider $C^m_{8,3}$. Let $\alpha = 21$. Take the triple of patterns

$$\mid 15, 14, \widehat{13}, 12, 11, 10, 9 \mid 22, 21, \widehat{20}, 19, 18, 17, 16 \mid 27, 28, \widehat{29}, 26, 25, 24, 23 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x+8$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(8,15),(13,16),(13,22),(20,23),(20,27),(26,29).$$

Each pair is coprime; the differences are 7, 3, 9, 3,7 and 3 respectively and the prime factors of each are in $\vartheta(21)=\{3,7\}$.

Consider $C^m_{9,3}$. Let $\alpha=24$. Take the triple of patterns

$$\mid 15,16,\widehat{17},14,\ldots,10 \mid 25,24,\widehat{23},22,\ldots,18 \mid 32,33,\widehat{31},30,\ldots,26 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x+7$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(7,10),(7,15),(14,17),(17,25),(23,26),(23,32),(31,33).$$

Each pair is coprime; the differences are 3, 8, 3, 8, 3, 9 and 2 respectively and the prime factors of each are in $\vartheta(24)=\{2,3\}$.

Consider $C^m_{9,4}$. Let $\alpha=24$. Take the triple of patterns

$$\mid 16,15,14,\widehat{17},13,\ldots,10 \mid 20,23,24,\widehat{25},22,\ldots,18 \mid 26,27,28,\widehat{31},30,29,32,33 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x+7$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(7,10),(7,16),(13,17),(14,17),(17,20),(20,23),(19,21),(22,25),(25,33),(28,31),(29,32).$$

Each pair is coprime; the differences are 3, 9, 4, 3, 3, 3, 2, 3, 8, 3 and 3 respectively and the prime factors of each are in $\vartheta(24)=\{2,3\}$.

Consider $C^m_{11,3}$. Let $\alpha=30$. Take the triple of patterns

$$\mid 17,18,\widehat{19},20,21,16,\ldots,12 \mid 27,28,\widehat{29},30,31,26,\ldots,22 \mid 38,39,\widehat{41},40,37,\ldots,32 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x+11$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(11,17),(16,21),(19,22),(19,27),(26,31),(29,32),(29,38),(37,40),(39,41).$$

Each pair is coprime; the differences are 6, 5, 3, 8, 5, 3, 9, 3 and 2 respectively and the prime factors of each are in $\vartheta(30)=\{2,3,5\}$.

Consider $C^m_{11,4}$. Let $\alpha=30$. Take the triple of patterns

$$\mid 14,15,16,\widehat{19},18,17,20,21,13,12 \mid 28,29,30,\widehat{31},22,\ldots,27 \mid 34,33,32,\widehat{41},40,\ldots,35 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x+11$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(11,14),(13,21),(16,19),(17,20),(19,27),(19,28),(22,31),(31,34),(31,35),(32,41).$$

Each pair is coprime; the differences are 3, 8, 3, 3, 8, 9, 9, 3, 4 and 9 respectively and the prime factors of each are in $\vartheta(30) = \{2, 3, 5\}$.

Consider $C^m_{17,3}$. Let $\alpha = 48$. Take the triple of patterns

$$\mid 33, 32, \widehat{31}, 30, \ldots, 18 \mid 49, 48, \widehat{47}, 46, \ldots, 34 \mid 63, 64, \widehat{65}, 62, \ldots, 50 \mid .$$

After adding αx to each element we attach the first to a vertex labeled $\alpha x + 17$. The adjacent label pairs at fusion points and non-consecutive pairs within a cycle are

$$(17, 33), (31, 34), (31, 49), (47, 50), (47, 63), (62, 65).$$

Each pair is coprime; the differences are 16, 3, 18, 3, 16 and 3 respectively and the prime factors of each are in $\vartheta(48) = \{2, 3\}$. □

Theorem 5.2 *Let $2t = 3^a + 3$ for some $a \geq 1$. The uniform cycle snakes $C^m_{2t,3}$ are prime for all m.*

Proof. Let $\alpha = 3(2t - 1) = 6t - 3$. For the first pattern, take

$$\mid 2t, 2t - 1, \widehat{2t - 2}, 2t - 3, \ldots, 2 \mid$$

attached to a label of 1 from the previous pattern. All internal adjacent labels are consecutive and there is one non-consecutive pair at the join: 1 and $2t$. These are coprime and the difference $2t - 1$ divides α.

For the second pattern take

$$\mid 4t - 3, 4t - 2, \widehat{4t - 1}, 4t - 4, \ldots, 2t + 1 \mid .$$

At the join we have pairs $(2t-2, 4t-3)$ and $(2t-2, 2t+1)$. We have $\gcd(2t-2, 4t-3) = \gcd(2t-2, 2t-1) = 1$ and the difference $2t-1$ divides α. We also have $\gcd(2t-2, 2t+1) = \gcd(2t-2, 3) = 1$, as $2t - 2 \equiv 1 \pmod 3$, and the difference 3 divides α. There is one non-consecutive internal adjacent pair: $(4t - 4, 4t - 1)$. We have $\gcd(4t - 4, 4t - 1) = \gcd(4t - 1, 3) = 1$, as $4t - 1 \equiv 2 \pmod 3$, and the difference 3 divides α.

For the third pattern take

$$\mid 6t - 4, 6t - 3, \widehat{6t - 2}, 6t - 5, \ldots, 4t \mid .$$

There is one non-consective adjacent pair of labels introduced at the join: $4t-1$ and $6t-4$. We have $\gcd(4t - 1, 6t - 4) = \gcd(4t - 1, 2t - 3) = 1$, as $2t - 3 = 3^a$ and $4t - 1 \equiv 2 \pmod 3$, and the difference $2t - 3 = 3^a$ has only the prime factor 3, which is in $\vartheta(\alpha)$.

Considering the first "cycle" to be the single vertex labeled 1, we have a prime labeling of $C^m_{2t,3}$. □

The smallest instances covered by Theorem 5.2 are $C^m_{6,3}$, $C^m_{12,3}$, $C^m_{30,3}$ and $C^m_{84,3}$. For example, the labeling for $C^m_{12,3}$ starts

$$[\ 12, 11, \widehat{10} \ldots, 1 \mid 21, 22, \widehat{23}, 20, \ldots, 13 \mid 32, 33, \widehat{34}, 31, \ldots, 24 \mid 45, 44, \widehat{43} \ldots, 35 \mid \cdots\ ].$$

6 Constructions with Period 4 or More

There are four constructions in this section. The first two are for specific additions to Table 1—$C^m_{9,2}$ and $C^m_{8,2}$—and require repeating patterns of period 15 and 30 respectively. The third has a repeating pattern period that depends on q to give one of the more general results of the paper: $C^m_{2p,p}$ is prime for all m and all odd primes $p \equiv 1 \pmod 3$. The fourth and final construction uses similar ideas to the third to find a prime labeling for $C^m_{2p+1,p}$ for all m when p is an odd prime such that $\vartheta(p+1) = \{2, r\}$ for some odd prime r.

Theorem 6.1 *The uniform cycle snakes $C^m_{9,2}$ are prime for all m.*

Proof. Let $\alpha = 120$. We construct a fifteen pattern sequence; we describe, and check, three at a time. The first attaches to a vertex labeled 9:

$$\mid 10, \widehat{11}, 12, \dots, 17 \mid 20, \widehat{19}, 18, 23, 24, 25, 22, 21 \mid 28, \widehat{33}, 32, \dots, 29, 26, 27 \mid$$

The non-consecutive adjacent label pairs at fusion points and within a cycle are

$$(9,17), (11,20), (11,21), (18,23), (22,25), (19,27), (19,28), (26,29), (28,33).$$

Each pair is coprime and their differences are 8, 9, 10, 5, 3, 8, 9, 3, and 5 respectively, each of whose prime factors are in $\vartheta(120) = \{2,3,5\}$.

The second batch of three patterns:

$$\mid 38, \widehat{41}, 40, 39, 37, \dots, 34 \mid 42, \widehat{43}, 44, \dots, 49 \mid 52, \widehat{57}, 56, 51, 50, 53, 54, 55 \mid$$

The non-consecutive adjacent label pairs at fusion points and within a cycle are

$$(33,38), (37,39), (38,41), (41,49), (43,52), (43,55), (50,53), (51,56), (52,57).$$

Each pair is coprime and their differences are 5, 2, 3, 8, 9, 12, 3, 5 and 5 respectively, each of whose prime factors are in $\vartheta(120) = \{2,3,5\}$.

The third batch of three patterns:

$$\mid 58, \widehat{61}, 60, 59, 62, \dots, 65 \mid 66, \widehat{71}, 72, 73, 70, \dots, 67 \mid 76, \widehat{81}, 80, \dots, 77, 74, 75 \mid$$

The non-consecutive adjacent label pairs at fusion points and within a cycle are

$$(57,65), (58,61), (59,62), (61,66), (61,67), (66,71),$$
$$(70,73), (71,75), (71,76), (74,77), (76,81).$$

Each pair is coprime and their differences are 8, 3, 3, 5, 6, 5, 3, 4, 5, 3 and 5 respectively, each of whose prime factors are in $\vartheta(120) = \{2,3,5\}$.

The fourth batch of three patterns:

$$\mid 86, \widehat{89}, 88, 87, 85, \dots, 82 \mid 92, \widehat{97}, 96, \dots, 93, 91, 90 \mid 98, \widehat{103}, 102, \dots, 99, 104, 105 \mid$$

The non-consecutive adjacent label pairs at fusion points and within a cycle are

$$(81,86),(85,87),(86,89),(89,92),(91,93),(92,97),(97,105),(98,103),(99,104).$$

Each pair is coprime and their differences are 5, 2, 3, 3, 2, 5, 8, 5 and 5 respectively, each of whose prime factors are in $\vartheta(120)=\{2,3,5\}$.

The fifth and final batch of three patterns:

$$| \ 106,\widehat{107},108,109,110,113,112,111 \ | \ 116,\widehat{115},114,119,120,121,118,117 \ | \\ | \ 124,\widehat{129},128,123,122,125,126,127 \ | \ .$$

The non-consecutive adjacent label pairs at fusion points and within a cycle are

$$(103,106),(103,111),(110,113),(107,116),(107,117),(114,119), \\ (118,121),(115,124),(115,127),(122,125),(123,128),(124,129).$$

Each pair is coprime and their differences are 3, 8, 3, 9, 10, 5, 3, 9, 12, 3, 5 and 5 respectively, each of whose prime factors are in $\vartheta(120)=\{2,3,5\}$.

To start the labeling, label the first cycle in consecutive order, using 9 on the fusion point. □

Theorem 6.2 *The uniform cycle snakes $C_{8,2}^m$ are prime for all m.*

Proof. Let $\alpha=210$. We construct a thirty pattern sequence. Unlike the previous proof, we give the full collection of patterns in one go and then describe how three fundamental types of pattern function successfully to cover all of the patterns used.

$$| \ 12,\widehat{13},11,10,9,14,15 \ | \ 16,\widehat{21},20,19,18,17,22 \ | \ 26,\widehat{29},28,27,25,24,23 \ |$$

$$| \ 36,\widehat{35},34,33,32,31,30 \ | \ 38,\widehat{43},42,41,40,39,37 \ | \ 50,\widehat{49},48,47,46,45,44 \ |$$

$$| \ 52,\widehat{57},56,55,54,53,51 \ | \ 64,\widehat{63},62,61,60,59,58 \ | \ 68,\widehat{71},70,69,67,66,65 \ |$$

$$| \ 78,\widehat{77},76,75,74,73,72 \ | \ 82,\widehat{85},84,83,81,80,79 \ | \ 92,\widehat{91},90,89,88,87,86 \ |$$

$$| \ 94,\widehat{99},98,97,96,95,93 \ | \ 100,\widehat{101},102,103,104,105,106 \ |$$

$$| \ 107,\widehat{113},112,111,110,109,108 \ | \ 120,\widehat{119},118,117,116,115,114 \ |$$

$$| \ 122,\widehat{127},126,125,124,123,121 \ | \ 134,\widehat{133},132,131,130,129,128 \ |$$

$$| \ 138,\widehat{141},140,139,137,136,135 \ | \ 148,\widehat{147},146,145,144,143,142 \ |$$

$$| \ 152,\widehat{155},154,153,151,150,149 \ | \ 162,\widehat{161},160,159,158,157,156 \ |$$

$$| \ 166,\widehat{169},168,167,165,164,163 \ | \ 176,\widehat{175},174,173,172,171,170 \ |$$

$$| \ 178,\widehat{183},182,181,180,179,177 \ | \ 190,\widehat{189},188,187,186,185,184 \ |$$

$$| \ 194,\widehat{197},196,195,193,192,191 \ | \ 204,\widehat{203},202,201,200,199,198 \ |$$

$$| \ 206, \widehat{211}, 210, 209, 208, 207, 205 \ | \ 218, \widehat{217}, 216, 215, 214, 213, 212 \ |$$

As far as possible, the patterns put the highest odd label on the fusion point. When this is the highest label, the next cycle is all consecutive; consider, for example, the 6th cycle. The only non-consecutive pair to consider is $(43, 50)$; these labels are coprime and differ by 7, a divisor of α. There are 14 such cycles and they are successful by similar reasoning (including the 14th cycle, which has this structure but lists the vertices in the opposite order, not putting the highest odd label on the fusion point).

When the highest odd label is the second highest label, the next cycle has a structure exemplified by the 7th cycle. The non-consecutive pair at the fusion point is $(49, 52)$ and 52 is also non-consecutive with the next fusion label, giving the pair $(52, 57)$. Both pairs are coprime and the differences 3 and 5 divide α. The remaining adjacent pairs are either consecutive or odd and differ by 2. There are 13 cycles with this structure.

The remaining cycles are the 1st, 2nd and 15th. The non-consecutive pairs from these cycles (including where they attach to the previous cycle, with the 1st attaching to label 7) are

$$(7,12),(7,15),(9,14),(11,13),(13,16),(13,22),$$
$$(16,21),(17,22),(101,107),(101,108),(107,113).$$

Each pair is coprime and the differences 5, 8, 5, 2, 3, 9, 5, 5, 6, 7 and 6 each have their prime factors contained in $\vartheta(\alpha) = \{2, 3, 5, 7\}$.

Attach the first pattern to $| \ 1, 2, \ldots, \widehat{7}, 8 \ |$ and note that the last pattern connects successfully to the first (after adding 210). Hence we have a prime labeling for $C^m_{8,2}$. $\square$

Theorem 6.3 *Let p be a prime with $p \equiv 1 \pmod 3$. The uniform cycle snakes $C^m_{2p,p}$ are prime for all m.*

Proof. For this construction the number of cycles depends on p: the repeating pattern has period p. Let $\alpha = p(2p-1)$.

The ith pattern, for $1 \le i \le p-1$ has the form

$$| \ i(2p-1)+2, \ldots, i(2p-1)+p, \widehat{i(2p-1)+2p}, \ldots, i(2p-1)+p+1 \ | \ .$$

The pth pattern has the form

$$| \ p(2p-1)+2, \ldots, \widehat{p(2p-1)+p+1}, \ldots, p(2p-1)+2p \ | \ .$$

There is one internal adjacent non-consecutive pair in each of the first i patterns, $i(2p-1)+p$ and $i(2p-1)+2p$, and none in the pth pattern. We have

$$\gcd(i(2p-1)+p, i(2p-1)+2p) = \gcd(i(2p-1)+p, p) = \gcd(i(2p-1), p).$$

This is 1 as the prime p does not divide either i or $2p-1$. The difference p divides α.

When $i = 1$ the ith pattern is attached to the label $p+1$. This is non-consecutive with both of the new labels, $(2p-1)+2 = 2p+1$ and $(2p-1)+p+1 = 3p$. For the former $\gcd(p+1, 2p+1) = \gcd(p+1, p) = 1$ and the difference p divides α. For the latter $\gcd(p+1, 3p) = \gcd(p-2, p+1) = \gcd(p-2, 3) = 1$, as $p \equiv 1 \pmod 3$, and the difference $2p-1$ divides α.

When $1 < i < p$ the ith pattern is attached to the label $i(2p-1)+1$. This is non-consecutive with one of the new neighbors, $i(2p-1)+p+1$. We have $\gcd(i(2p-1)+1, i(2p-1)+p+1) = \gcd(i(2p-1)+1, p)$. As $2p-1 \equiv -1 \pmod p$ we have $i(2p-1)+1 \equiv -i+1 \not\equiv 0 \pmod p$ as $1 < i < p$, and hence the labels are coprime. The difference p divides α.

The pth pattern is attached to the label $(p-1)(2p-1)+2p$. This is non-consecutive with one of the new neighbors, $p(2p-1)+2p$. We have

$$\gcd((p-1)(2p-1)+2p, p(2p-1)+2p) = \gcd(2p-1, p(2p-1)+2p) = \gcd(2p-1, 2p) = 1.$$

The difference $2p-1$ divides α.

Take the first cycle to be $|\ \widehat{1}, \ldots, 2p\ |$ to give a prime labeling for $C^m_{2p,2}$ for all m. $\square$

For example, the labeling given by Theorem 6.3 for $C^m_{14,7}$ begins

$$[\ \widehat{1}, \ldots, 14 \mid 15, \ldots 20, \widehat{27}, 26, \ldots 21 \mid 28, \ldots 33, \widehat{40}, 39, \ldots, 34 \mid$$
$$\mid 41, \ldots, 46, \widehat{53}, 52, \ldots, 47 \mid 54, \ldots, 59, \widehat{66}, 65, \ldots, 60 \mid 67, \ldots, 72, \widehat{79}, 78, \ldots, 73 \mid$$
$$\mid 80, \ldots 85, \widehat{92}, 91, \ldots, 86 \mid 93, \ldots, \widehat{99}, \ldots, 105 \mid 106, \ldots, 111, \widehat{118}, 117, \ldots, 112 \mid\ \cdots\ ].$$

Theorem 6.4 *Let p be an odd prime such that $\vartheta(p+1) = \{2, r\}$ for some odd prime r. Then $C^m_{2p+1,p}$ is prime for all m.*

Proof. Although the idea of this proof is similar to the previous one, it has a feature that makes a slightly different approach preferable: each pattern puts the largest element on the fusion point. It thus has much in common with the period 1 constructions of Section 3 and we take $\alpha = 2p$.

Let $y \in \{p, p+1\}$ and consider the pattern

$$|\ 2, 3, \ldots, y, \widehat{2p+1}, 2p, \ldots, y+1\ |\ .$$

The two choices of y each correspond to a shortest path between fusion points of p, one via the first portion and one via the second. We are attaching to a label of 1, so coprimality is immediate at the join with the previous pattern. We still need to consider the difference. The non-consecutive difference here is y. If $y = p$ then y divides α and we have coprime labels. If $y = p+1$ then we might or might not have coprime labels depending on the value of x when adding αx to each element of the pattern. We return to this case shortly.

There is one internal non-consecutive pair: y and $2p+1$. When $y = p+1$ we have $\gcd(y, 2p+1) = \gcd(p+1, p) = 1$. The difference of p divides α, so choosing $y = p+1$ gives coprime labels here. If $y = p$ then, again, the coprimality depends on the value of x when adding αx to each element of the pattern.

In each of the potentially problematic situations we have a difference of $p+1$. The labels in question that are not on fusion points are $\alpha x + p$ and $\alpha x + p + 2$ and we need only one to give rise to a coprime pair. As these differ by 2, at least one is not divisible by r. As the difference in each calculation is $p+1$, the greatest common divisor of the pair must be divisible by 2 or r, and so we can always make a choice in the construction that gives coprime labels.

Initialize the process by taking labeling the first cycle $\mid 1, \ldots, \widehat{2p+1} \mid$. Hence $C^m_{2p+1,p}$ is prime for all m. $\square$

The primes $p < 100$ that satisfy the hypothesis of Theorem 6.4 that $\vartheta(p+1) = \{2, r\}$ for some prime r are

$$5, 11, 13, 17, 19, 23, 37, 43, 47, 53, 61, 67, 71, 73, 79, 97.$$

For example, when $p = 5$, the construction might give the following four cycles to start the labeling:

$$[\ 1, \ldots, \widehat{11} \mid 12, \ldots, 16, \widehat{21}, 20, \ldots, 17 \mid$$

$$\mid 22, \ldots, 25, \widehat{31}, 30, \ldots, 26 \mid 32, \ldots, 35, \widehat{41}, 40, \ldots, 36 \mid \cdots\ ].$$

Observe that the label of 21 on the fusion point connecting the second and third cycles forces the choice of these two cycles as 3 (which is r) divdes 21. However, either choice is possible for the fourth cycle and so it is permissible to use $\mid 32, \ldots, 36, \widehat{41}, 40, \ldots, 37 \mid$ in place of the given one.

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References

[1] A. Bigham, E.A. Donovan, J. Pack, J. Turley, L.W. Wiglesworth. Prime labelings of snake graphs, *PUMP J. Undergrad. Res.*, **2** (2019), 1083–1089.

[2] K. Carlson, Generalized books and C_m-snakes are prime graphs, *Ars Combin.*, **80** (2006), 215–221.

[3] G. Chartrand, C. Egan, P. Zhang, *How to Label a Graph*, Springer International Publishing (2019).

[4] N. Diefenderfer, D.C. Ernst, M.G. Hastings, L.N. Heath, H. Prawzinsky, B. Preston, J. Rushall, E. White, A. Whittemore, Prime vertex labelings of several families of graphs, *Involve*, **9** (2016), 667–688.

[5] J.A. Gallian. A dynamic survey of graph labeling, *Electron. J. Combin.*, **DS6** (2000, updated 2021), 576pp.

[6] S.M. Lee, I. Wui, J. Yeh, On the amalgamation of prime graphs, *Bull. Malaysian Math. Soc. (Second Series)*, **11** (1988), 59–67.

[7] S.K. Patel and J. Vasava, On prime labeling of some union graphs, *Kragujevac J. Math.*, **42** (2018), 441–452.

[8] M.A. Seoud and M.Z. Youssef, On prime labelings of graphs, *Congr. Numer.*, **141** (1999), 203–215.

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