Runs of Consecutive Integers Having the Same Number of Divisors

V.-T. Spătaru

Abstract - Our objective is to provide an upper bound for the length ℓ_N of the longest run of consecutive integers smaller than N which have the same number of divisors. We prove in an elementary way that $\log \ell_N \ll (\log N \log\log N)^\lambda$, where $\lambda = 1/2$. Using estimates for the Jacobsthal function, we then improve the result to $\lambda = 1/3$.

Keywords : divisor counting function; consecutive equidivisible integers

Mathematics Subject Classification (2020) **:** 11A25; 11N37

1 Introduction

Let $d(n)$ denote the number of positive divisors of n. The equation $d(n) = d(n+k)$ has been studied extensively. Spiro [1] showed that it has infinitely many solutions for $k = 5040$. Subsequently, Heath-Brown [2] established the case $k = 1$, and Pinner [3] ultimately proved that all values of k yield infinitely many solutions.

As $d(n)$ is equal to $d(n+1)$ infinitely often, one naturally wonders *how many* consecutive integers can there be, having the same number of divisors. Erdős and Mirsky [4] conjectured that there are arbitrarily long such runs of integers. They were not able to provide any estimates for the length of such sequences: *"A related problem consists in the estimation of the longest run of consecutive integers $\leqslant x$ all of which have the same number of divisors. This problem seems to be one of exceptional difficulty, and we* [Erdős & Mirsky] *have not been able to make any progress with it."*

Our principal objective is to provide an upper bound for the length ℓ_N of the runs in question. In Section 2, we estimate the order of magnitude of the $d(n)$ and $\omega(n)$ functions and obtain the following estimate in an elementary manner:

Theorem 1.1 *Let ℓ_N denote the length of the longest run of consecutive integers smaller than N, having the same number of divisors. Then,*

$$\log \ell_N \ll \sqrt{\log N \cdot \log\log N}.$$

Subsequently, in Section 3, we provide a quicker proof of Theorem 1.1 based on the Prime Number Theorem. We also get the following explicit form of Theorem 1.1:

$$\log \ell_N \leqslant \sqrt{(1/2 + o(1)) \log N \log\log N}.$$

Finally, in Section 4 we prove a stronger version of Theorem 1.1 using estimates of the Jacobsthal function deduced from Brun's sieve method. We obtain the following result:

Theorem 1.2 *Let ℓ_N denote the length of the longest run of consecutive integers smaller than N, having the same number of divisors. Then,*

$$\log \ell_N \ll \sqrt[3]{\log N \cdot \log\log N}.$$

2 An Elementary Proof of Theorem 1.1

In proving Theorem 1.1, we will make use of the following lemmas, the first being proven in an elementary manner in [5] and the second being Mertens' bound.

Lemma 2.1 *Let n be a positive integer. Then,* $\operatorname{lcm}(1, 2, \ldots, n+1) \geqslant 2^n$.

Lemma 2.2 *The sum of the reciprocals of the prime numbers not exceeding n satisfies*

$$\sum_{p \leqslant n} \frac{1}{p} = \log\log n + M + O\left(\frac{1}{\log n}\right) \ll \log\log n.$$

Note that it suffices to prove that Theorem 1.1 holds for large enough N. Assume that there exist $k > 2$ consecutive numbers smaller than N, having the same number of divisors. Let them be $n+1$, $n+2$, …, $n+k$ and write

$$d(n+1) = d(n+2) = \cdots = d(n+k) = D.$$

We will firstly provide an estimate for D, in terms of k. For simplicity, let $K = \lfloor \log_2 k \rfloor$.

As $k \geqslant 2^K$, all residues modulo 2^K are among $n+1$, $n+2$, …, $n+k$. Therefore, for all $1 \leqslant i \leqslant K-1$, there exists some $1 \leqslant t_i \leqslant k$ such that $n + t_i \equiv 2^i \bmod 2^K$. Consequently, $\nu_2(n+t_i) = i$, so $i+1$ divides $d(n+t_i) = D$.

Hence, D is divisible by $\operatorname{lcm}(1, 2, \ldots, K)$. Using Lemma 2.1, we infer that

$$D \geqslant \operatorname{lcm}(1, 2, \ldots, K) \geqslant 2^{K-1}.$$

Recall that $K = \lfloor \log_2 k \rfloor \geqslant \log_2 k - 1$, so $D \geqslant k/4$.

Let $\omega(n)$ denote the number of distinct prime factors of n. Choose $1 \leqslant l \leqslant k$ arbitrarily. As $n + l \leqslant N$, it follows that $\nu_p(n+l) \leqslant \log_p N \leqslant \log_2 N$ for all prime numbers p. Therefore,

$$D = d(n+l) = \prod_p (\nu_p(n+l)+1) \leqslant \prod_{p \mid n+l} (\log_2 N + 1) = (\log_2 N + 1)^{\omega(n+l)}.$$

Taking logarithms, it follows that $\omega(n+l) \geqslant \log D / \log(\log_2 N + 1)$. Moreover, note that a prime number p can divide at most $\lceil k/p \rceil \leqslant k/p + 1$ numbers among $n+1$, …, $n+k$. Therefore, using Lemma 2.2, it follows that

$$\omega((n+1)\cdots(n+k)) \geqslant \sum_{i=1}^{k} \omega(n+i) - \sum_{p \leqslant k} \frac{k}{p} \geqslant \frac{k \log D}{\log(\log_2 N + 1)} - C_1 k \log\log k,$$

for a suitable constant C_1. Further, we will write $\log(\log_2 N+1) \leqslant C_2 \log\log N$, for some constant C_2. Recall that $D \geqslant k/4$, so we have

$$\omega((n+1)\cdots(n+k)) \geqslant \frac{k\log(k/4)}{C_2 \log\log N} - C_1 k \log\log k. \tag{1}$$

Write the right-hand side of equation 1 as $k \cdot f_N(k)$. Clearly, if $\omega(a) \geqslant b$ then $a \geqslant b!$. Using this remark on equation 1, we get $(n+1)\cdots(n+k) \geqslant \lceil k \cdot f_N(k)\rceil !$. Moreover, because $N^k \geqslant (n+1)\cdots(n+k)$, by applying the well-known inequality $\log t! \geqslant t\log t - t$, we have

$$\begin{aligned} k\log N &\geqslant \log\left((n+1)\cdots(n+k)\right) \geqslant \log\left(\lceil k\cdot f_N(k)\rceil !\right) \\ &\geqslant k\cdot f_N(k)\cdot \log(k\cdot f_N(k)) - k\cdot f_N(k). \end{aligned} \tag{2}$$

Finally, dividing equation 2 by k we obtain

$$\log N \geqslant f_N(k)\cdot\log(k\cdot f_N(k)) - f_N(k). \tag{3}$$

Define the interval $I_N = [\exp\left(C_1\cdot C_2\cdot \log\log N\right), \infty)$. Using standard arguments, one may infer that f_N is increasing on I_N.

Let us suppose, for the sake of contradiction, that $k > \exp\left(C\sqrt{\log N \log\log N}\right)$, where $C > \max(\sqrt{C_2}, C_1\cdot C_2)$. Firstly, note that since $\log N > \log\log N$ and $C > C_1 \cdot C_2$ then $\exp\left(C\sqrt{\log N\log\log N}\right)$ and k are in I_N. Therefore, we have

$$\begin{aligned} f_N(k) &> f_N\left(\exp\left(C\sqrt{\log N\log\log N}\right)\right) \\ &= \frac{C}{C_2}\sqrt{\frac{\log N}{\log\log N}} - \frac{\log 4}{C_2\log\log N} - C_1\log\left(C\sqrt{\log N\log\log N}\right). \end{aligned} \tag{4}$$

Viewing equation 4 as a function in N, it is evident that for large enough N (greater than some N_1) we also have $f_N(k) > e$. In what follows, we will assume that $N > N_1$.

As $f_N(k) > e$, it follows from equation 3 that $\log N \geqslant f_N(k)\cdot\log k$. Further, applying equation 4 and the estimate for k and isolating the term $\log N$, we get

$$\frac{C\log 4}{C_2}\sqrt{\frac{\log N}{\log\log N}} + C_1 C\sqrt{\log N\log\log N}\log\left(C\sqrt{\log N\log\log N}\right) \geqslant \left(\frac{C^2}{C_2} - 1\right)\log N.$$

Recall that $C > \sqrt{C_2}$, so the latter inequality is absurd for large enough N (greater than some N_2), as the left-hand side is asymptotically much smaller than $\log N$. Therefore, Theorem 1.1 holds for $N > \max(N_1, N_2)$ and $C > \max(\sqrt{C_2}, C_1\cdot C_2)$.

3 An Explicit Form of Theorem 1.1

Let $\Omega(n)$ denote the number of prime factors of n, counting multiplicities. Note that $2^{\omega(n)} \leqslant d(n) \leqslant 2^{\Omega(n)}$ for all positive integers n. Further, we have the following estimates:

Lemma 3.1 *The product $n\#$ of the prime numbers not exceeding n satisfies $\log n\# \sim n$.*

Lemma 3.2 *The sum of $1/\log p$ taken over the prime numbers not exceeding n satisfies*

$$\sum_{p\leqslant n}\frac{1}{\log p}=\frac{n}{(\log n)^2}+O\left(\frac{n\log\log n}{(\log n)^3}\right)=o(1)\cdot\frac{n}{\log n}.$$

Let M be the greatest positive integer satisfying $M\#\leqslant k$. Then, there exists $1\leqslant m\leqslant k$ so that $M\#$ divides $n+m$. Hence, for any $1\leqslant i\leqslant k$ we have

$$2^{\Omega(n+i)}\geqslant d(n+i)=d(n+m)\geqslant d(M\#)\geqslant 2^{\omega(M\#)}=2^{\pi(M)},$$

so $\Omega(n+i)\geqslant\pi(M)$. It follows that $\Omega(n+1)+\Omega(n+2)+\cdots+\Omega(n+k)\geqslant k\pi(M)$.

Fix a prime number $p\leqslant k$. Then, for every exponent $t\leqslant\log_p N$ there are at most $\lceil k/p^t\rceil\leqslant k/p^t+1$ numbers divisible by p^t among $n+1$, $n+2$, …, $n+k$. Furthermore, if $t>\log_p N$, then $p^t>N$ so none of $n+1$, $n+2$, …, $n+k$ are divisible by p^t.

Following the same double-counting technique as in Legendre's Theorem, we may infer

$$\sum_{i=1}^{k}\nu_p(n+i)\leqslant\sum_{t=1}^{\lfloor\log_p N\rfloor}\left(1+\frac{k}{p^t}\right)<\frac{\log N}{\log p}+\frac{k}{p-1}. \tag{5}$$

Notice that $n+i$ has at most $\log_k N$ prime factors greater than k, including multiplicities. Combining this observation with equation 5 and Lemmas 2.2 and 3.2, we get

$$\begin{aligned}k\pi(M)&\leqslant\sum_{i=1}^{k}\Omega(n+i)=\sum_{p>k}\sum_{i=1}^{k}\nu_p(n+i)+\sum_{p\leqslant k}\sum_{i=1}^{k}\nu_p(n+i)\\&\leqslant\frac{k\log N}{\log k}+\sum_{p\leqslant k}\left(\frac{\log N}{\log p}+\frac{k}{p-1}\right)=(1+o(1))\frac{k\log N}{\log k}+O(k\log\log k).\end{aligned} \tag{6}$$

It follows from Lemma 3.1 and the Prime Number Theorem that $\pi(M)\sim\log k/\log\log k$. Further, note that $k\log\log k=o(1)\cdot k\log k/\log\log k$. Using these observations in equation 6 we get

$$\frac{k\log k}{\log\log k}\sim k\pi(M)\leqslant(1+o(1))\frac{k\log N}{\log k}+o(1)\frac{k\log k}{\log\log k}.$$

Dividing through k and isolating the remaining functions in k on the left-hand side, we have $(\log k)^2/\log\log k\leqslant(1+o(1))\log N$. Using standard arguments, we finally get

$$\log k\leqslant\sqrt{(1/2+o(1))\log N\log\log N}.$$

4 The Proof of Theorem 1.2

We will keep the notation used in Section 2. We will require the following estimates:

Lemma 4.1 *The sum of the prime numbers not exceeding n satisfies*

$$\sum_{p \leqslant n} p = \frac{n^2}{2\log n} + O\left(\frac{n^2}{(\log n)^2}\right) \sim \frac{n^2}{2\log n}.$$

Lemma 4.2 *Let n be a positive integer and p_{min} be its smallest prime divisor. Then,*

$$\frac{\log n}{\log p_{min}} \geqslant \sum_p (p-1)\nu_p(d(n)).$$

Proof. Throughout the rest of the proof, the letters p and q will refer strictly to prime numbers. Note that since $p_{\min}$ is the smallest prime factor of n, by taking logarithms we get

$$\log n = \sum_q \nu_q(n) \log q \geqslant \log p_{\min} \sum_q \nu_q(n). \tag{7}$$

Further, using the fact that $m^n - 1 \geqslant n(m-1)$ for all positive integers, we infer that

$$k = \prod_p p^{\nu_p(k+1)} - 1 \geqslant \sum_p \left(p^{\nu_p(k+1)} - 1\right) \geqslant \sum_p (p-1)\nu_p(k+1) \tag{8}$$

for any positive integer k. Using inequality 8 on $\nu_q(n)$ in equation 7, we further have

$$\frac{\log n}{\log p_{\min}} \geqslant \sum_q \nu_q(n) \geqslant \sum_q \sum_p (p-1)\nu_p(\nu_q(n)+1) = \sum_p \left((p-1)\sum_q \nu_p(\nu_q(n)+1)\right)$$
$$= \sum_p (p-1)\nu_p\left(\prod_q (\nu_q(n)+1)\right) = \sum_p (p-1)\nu_p(d(n)),$$

giving us the desired result. □

The proof now hinges on finding an index i for which $n+i$ has a large minimal prime factor. Jacobsthal [6] defines the function $j(n)$ to be the least integer so that amongst any $j(n)$ consecutive integers there exists at least one relatively prime to n.

Therefore, if a positive integer M satisfies $j(M\#) \leqslant k$, then some $n+i$ has the minimal prime factor larger than M. As pointed out by Erdős [7], it follows directly from Brun's sieve that $j(n) \ll \omega(n)^C$ for a suitable constant C, hence $\log j(n) \ll \log \omega(n)$.

It follows that $\log j(M\#) \ll \log \omega(M\#) = \log \pi(M) \ll \log M$, so among our k consecutive integers we may find one, $n+i$, whose minimal prime factor $p_{\min}$ satisfies $\log p_{\min} \gg \log k$. Further, recall that every prime number not exceeding $\log_2 k$ divides $D = d(n+i)$. Applying Lemmas 4.1 and 4.2 we then get

$$\frac{\log N}{\log k} \gg \frac{\log(n+i)}{\log p_{\min}} \geqslant \sum_p (p-1)\nu_p(D) \gg \sum_{p \leqslant \log_2 k} p \sim \frac{(\log_2 k)^2}{2\log\log_2 k}.$$

Consequently, $\log N \gg (\log k)^3/\log\log k$, from which Theorem 1.2 easily follows.

Acknowledgments

I would like to express my deep gratitude to Andrew Granville, for his support and for his contribution [8] with the material presented throughout Sections 3 and 4. Additionally, I would like to thank Alexandru Gica for his constant guidance, his insightful comments, and for proofreading my paper.

References

[1] C.A. Spiro, *The Frequency with Which an Integral-Valued, Prime-Independent, Multiplicative or Additive Function of n Divides a Polynomial Function of n*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1981.

[2] D.R. Heath-Brown, The Divisor Function at Consecutive Integers, *Mathematika*, **31** (1984), 141–149.

[3] C.G. Pinner, Repeated Values of the Divisor Function, *Q. J. Math.*, **48** (1997), 499–502.

[4] P. Erdős, L. Mirsky, The Distribution of Values of the Divisor Function $d(n)$, *Proc. London Math. Soc. (2)*, **3** (1952), 257–271.

[5] B. Farhi, An Identity Involving the Least Common Multiple of Binomial Coefficients and Its Application, *Amer. Math. Monthly*, **116** (2009), 836–839.

[6] E.E. Jacobsthal, Über Sequenzen ganzer Zahlen: von denen keine zu n teilerfremd ist, *Norske Vid. Selsk. Forhdl.*, **33** (1960), 117–124.

[7] P. Erdős, On the Integers Relatively Prime to n and on a Number-Theoretic Function Considered by Jacobshtal, *Math. Scand.*, **10** (1962), 163–170.

[8] A.J. Granville, private communication, 2023.

Vlad-Titus Spătaru
Liceul Teoretic Internaţional de Informatică Bucureşti
Str. Balta Albina nr. 9, Bucharest, Romania
E-mail: vtspataru@gmail.com

Received: December 18, 2022 **Accepted**: February 2, 2023
Communicated by Kevin J. McGown