Isoperimetric 3- and 4-Bubble Results on $\mathbb{R}$ With Density $|x|$

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Abstract - We study the isoperimetric problem on $\mathbb{R}^1$ with a prescribed density function $f(x) = |x|$. Under these conditions, we find that isoperimetric 3-bubble and 4-bubble results satisfy a regular structure. As our regions increase in size, the intervals that form them alternate back-and-forth across the origin, with the smaller regions closer to the origin. This expands on previously known observations about the single- and double-bubble results on $\mathbb{R}$ with density $|x|^p$.

Keywords : isoperimetric problem; density; geometric optimization

Mathematics Subject Classification (2020) **:** 49Q10

1 Introduction

The classic isoperimetric (or equivalent iso-area) problem asks one to find, out of all simple closed curves in the plane with fixed perimeter P, the curve that encloses the maximal amount of area A (or equivalently, find the curve that minimizes perimeter out of al curves that encloses fixed area). These specific problems, as well as their solution (a circle of an appropriate area or perimeter), have been known since antiquity. Related problems (such as the corresponding isoperimetric problem in $\mathbb{R}^3$) have been studied more recently, in which solutions to an isoperimeteric problem are often called an isoperimetric region or a "bubble" (the word is due to how soap films naturally form isoperimetric surfaces in $\mathbb{R}^3$).

Isoperimetric problems can be generalized in a number of ways: we can look at closed hypersurfaces (as analogues of simple closed curves) within $\mathbb{R}^n$ or other manifolds; we can look to optimize the total perimeter of multiple regions simultaneously; or we can introduce density functions on the ambient space, affecting how perimeter and area are measured.

Our work will examine the multibubble isoperimetric problem on $\mathbb{R}^1$ with a prescribed density function of $|x|$. Under such a framework, individual intervals (of the form $I = [a, b]$) on the real number line have weighted area (which we will call mass) M and weighted perimeter P affected by the density function so that $M(I) = \int_a^b |x|\, dx$ and $P(I) = |a| + |b|$. Multiple regions will have their masses prescribed, and their weighted perimeter calculated so that regions sharing an endpoint are only counted once: in this way, multiple regions can have lower total perimeter if they share an endpoint. Using the framework above, our question is this: suppose we are given masses $0 < M_1 \leq \cdots \leq M_n$, where n is 3 or 4.

Can we find a configuration of n regions, each comprised of one or more intervals, with the prescribed masses and with a minimal amount of total perimeter?

In this note, we are able to positively answer this question and provide a least-perimeter solution. Our results are reported in Theorem 5.8 and Theorem 6.5. We will also record the results for $n = 1, 2$ first identified in [3] (and included in this work for completeness). The results for the first four isoperimetric problems are displayed in Figure 1. Although we will wait to formally state the result, we note that the regions alternate their position across the origin as they increase in weighted mass. This leads to a natural conjecture that the pattern will continue for larger n-bubbles; this conjecture is addressed and positively proven in forthcoming work [6].

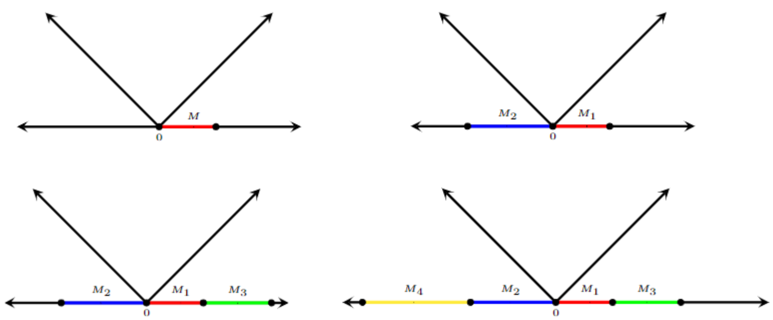


Figure 1: Solutions for the 1-, 2-, 3-, and 4-bubble isoperimetric problem on $\mathbb{R}$ with density $|x|$. For these masses, note that $M_1 \leq M_2 \leq M_3 \leq M_4$

Although studying isoperimetry with density functions is a more recent development, much has already been said. By an argument in [4], Morgan and Pratelli showed that a perimeter-minimizing solution will exist on $\mathbb{R}^m$ when using a density function that radially increases to infinity. Bayle, Canete, Morgan, and Rosales found early isoperimetric results for a single bubble on $\mathbb{R}^1$ in [5]. Additionally, a number of teams have found single- and double-bubble solutions on $\mathbb{R}^1$ with density (see [1], [3]). In [3], the single- and double-bubble solutions were found for density $|x|^p$, extending the single-bubble result from [2] in the 1-dimensional case. In [1], single- and double-bubble results were found for density functions that were log-convex. In recent work, Sothanaphan [7] extended these results to identify possible triple-bubbles.

Although working in such a low dimension reduces the complexity of our regions, there are still complications we must deal with. For example, a priori, each region may consist of multiple intervals which may or may not share endpoints with other regions. In [1], it was found that the 2-bubble had either two or three intervals, and this could change depending on the scale of the actual sizes (not just relative sizes) of the two regions in question. An early step in our argument will be to show that we can shrink our total perimeter by consolidating each region into, at most, two intervals. This will reduce our n-bubble candidates to, at most, $2n$ total intervals. From there, we examine a number of tools we can use to further consolidate our number of intervals. These tools are enough to fully prove the 3-bubble. For the 4-bubble, those tools still leave us with 5 intervals, and so an explicit comparison is needed. A crucial element of this is a set of "optimal arrangment" propositions that tell us the perimeter-minimizing way to order 4, 5, or 6 adjacent regions. These propositions require many cases and are included as an appendix.

2 Definitions

Definition 2.1 *A* ***weighted number line*** *is the pair* $(\mathbb{R}, f)$*, where* $f : \mathbb{R} \to [0, \infty)$ *is called the density function.*

Definition 2.2 *Given* $(\mathbb{R}, f)$ *and an interval* $[a, b]$*, the* ***weighted mass*** *of the interval with respect to the density is* $\int_a^b f$*. The* ***weighted perimeter*** *with respect to the density is defined to be* $f(a) + f(b)$*.*

Note that weighted mass will often simply be called mass (and occasionally, by abuse of language, be called area or volume), and weighted perimeter will often be called perimeter. When looking at an interval $[a, b]$, we will exclusively use the word **length** to refer to the quantity $b - a$.

Definition 2.3 *A* ***region*** *on* $(\mathbb{R}, f)$ *is a collection of disjoint intervals* $[a_i, b_i]$ *with total mass* M *and perimeter* P *calculated as*

$$M = \sum_i \int_{a_i}^{b_i} f \tag{1}$$

$$P = \sum_i f(a_i) + f(b_i). \tag{2}$$

We remark that, under density functions that have a positive lower bound, regions with finite mass or finite perimeter must necessarily be made up of only finitely many intervals. With other weighted number lines (such as $(\mathbb{R}, |x|)$, for example), it is possible to have a region of finite mass consisting of infinitely many intervals. In such a scenario, a point of zero density will act as an accumulation point for the interval endpoints.

Definition 2.4 *A region* R *with mass* M *and perimeter* P *is said to be* ***isoperimetric in*** $(\mathbb{R}, f)$ *(also referred to as a* ***bubble****) if, out of all possible regions with mass* M*,* R *has the least perimeter.*

We can also consider multiple regions on the same number line.

Definition 2.5 *A* ***configuration of regions*** $R_1, \ldots, R_n$ *is a set of* n *regions such that each region is comprised of disjoint intervals, and two intervals from different regions are either completely disjoint or meet at a single endpoint.*

On $(\mathbb{R}, f)$*, we can measure the mass or weighted perimeter of a single region as before. Additionally, we can measure the* ***total weighted perimeter*** *of the configuration by* *summing the perimeter of each of the regions, doing so in such a way that each endpoint where two intervals meet is only counted once.*

Definition 2.6 *A configuration of n regions $R_1, \dots, R_n$ with masses $M_1, \dots, M_n$ is said to be* ***isoperimetric on*** $(\mathbb{R}, f)$ *(also referred to as an n-**bubble**) if, out of all possible configurations of n regions with these same masses, the configuration of $\{R_1, \dots, R_n\}$ has the least total perimeter.*

Remark 2.7 2-bubbles and 3-bubbles are sometimes called a **double bubble** and **triple bubble**.

The major aim of this paper is to prove a particular configuration of regions is isoperimetric on $(\mathbb{R}, |x|)$ in the case of the 3-bubble and 4-bubble.

3 Preliminaries

To start exploring isoperimetry in earnest, we begin by reducing the total number of intervals that will appear in a perimeter-minimizing configuration of n regions. A priori, each region could consist of one or more intervals, and that there might be "empty" intervals that do not correspond to any particular region. However, we show that this cannot produce an isoperimetric result by showing that perimeter is lowered by condensing our intervals near the origin.

Proposition 3.1 *Let f be an increasing density function defined on $[0, \infty)$. Suppose we have a configuration of n regions $\{R_i\}$ on the nonnegative real number line, each with prescribed masses M_i. Then there exists a configuration of n regions on $[0, \infty)$ with the same masses M_i, arranged so that each region R_i consists of a single interval $[a_i, b_i]$ and such that the regions are adjacent to each other with no empty intervals. Furthermore, this new configuration has total perimeter less than or equal to the original configuration, which equality only if the configurations are identical.*

Proof.

Suppose we have an arbitrary configuration on n regions, each consisting of (possibly multiple) intervals on the non-negative axis $x \geq 0$. We identify R_1 to have a maximum value (and rightmost endpoint) of b_1, R_2 to have a maximum value of b_2, and so on, renaming if necessary so that $b_1 < b_2 < \cdots < b_n$. Then we create a new configuration of regions R_i with mass M_i, each region a single interval, created adjacent to each other and ordered so that region R_i sits to the left of region R_j if $i < j$. (See Figure 2 for an example of this.)

Figure 2: We see how an arbitrary arrangement (on the left) can be consolidated (shown the right) in a manner that lowers perimeter.

Because of how we have constructed these regions, it is clear that the new rightmost endpoint b'_j of the region R_j satisfies $0 < b'_j \leq b_j$. Since f is assumed to be increasing, we know $f(b'_j) \leq f(b_j)$. This gives us the following inequalities for the new and original weighted perimeter:

$$P_{new} = \sum_{j=1}^{n} f(b'_j) \leq \sum_{j=1}^{n} f(b_j) \leq P_{orig}. \tag{3}$$

Furthermore, equality will only occur if $f(b'_j) = f(b_j)$ for each j. By the increasing nature of f, this will only occur if the new and old configuration were identical. □

Corollary 3.2 *Let f be a radially increasing density function defined on $\mathbb{R}$. If a configuration of n regions is isoperimetric on $(\mathbb{R}, f)$, then the configuration consists of at most $2n$ adjacent intervals, with the origin contained in at least one interval (either in the interior or as an endpoint). Furthermore, each region will consist of at most 2 intervals, and those consisting of two intervals will have one interval on the positive side and one interval on the negative side of the origin.*

Proof. Proposition 3.1 says that perimeter can be lowered by reconfiguring regions on each side of the origin, so that there are at most n intervals on the positive and n on the negative side. Furthermore, as our consolidation from before occurs at the origin, it is clear that the origin will either be between two different regions (in which case it's an endpoint of each), or in a single region (with some mass to be found on both the positive and negative side). □

Remark 3.3 We will collectively use the word **reconfiguration** to describe exercises in rearranging and consolidating intervals, when done in such a way that preserves weighted masses.

Definition 3.4 *A configuration of n regions that consists of at most $2n$ adjacent intervals (with at most 1 interval on the positive and 1 interval on the negative side of the origin), with the origin contained either in an interior of an interval or at the endpoint of (at least one) interval, is said to be a* ***condensed configuration of n regions****.*

Beyond simply knowing that our regions accumulate near the origin, we also learn that we can reorder them so that they "grow" from smallest to largest mass as we move further out from the origin.

Proposition 3.5 *On $[0, \infty)$ with an increasing density function: Consider intervals $R_i = [a_i, b_i]$ of mass M_i for $i = \{1, 2\}$. Suppose $M_2 < M_1$. Finally, suppose $a_2 = b_1$. Then we would reduce total perimeter by switching the relative positions of R_1 and R_2.*

Proof. Keeping the leftmost and rightmost endpoints fixed, and transposing the position of the two regions, we see that the perimeter is lowered when the interval with less mass is placed closer to the origin (as this configuration places the regions' shared endpoint closer to the origin). See Figure 3. $\square$

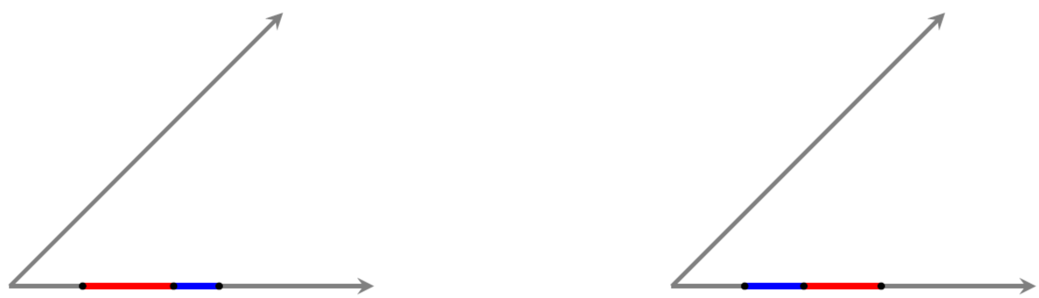


Figure 3: Transposition of adjacent intervals: If we transpose R_1 and R_2 without changing their global location, the lower perimeter results from placing the smaller interval closer to the origin (as the only endpoint to move is the internal endpoint). Iterating the transposition lemma guarantees that, on one side of the interval, regions will be ordered (according to weighted area) from smallest to largest as we move away from the origin.

We conclude this section with an important comparison proposition. Although the proposition is stated for the positive real number line, it is clear (due to the symmetry of $|x|$) that there are equivalent statements for comparing intervals on either side of the number line; in such a comparison, all that matters is which interval's "inner" endpoint is positioned closer to the origin.

Proposition 3.6 *Let f be an radially increasing density function on $[0, \infty)$. Consider two intervals $R_1 = [a_1, b_1]$, $R_2 = [a_2, b_2]$. Suppose that each of these intervals contains the same weighted mass (so $M_1 = M_2$), and suppose that $0 \leq a_1 < a_2$. Then we get the* *following two inequalities:*

$$b_1 < b_2 \tag{4}$$
$$b_1 - a_1 > b_2 - a_2. \tag{5}$$

Proof. The first inequality is an immediate consequence of the two intervals enclosing the same mass. For the second inequality, note that since our density function is increasing as we move away from the origin, the average density of $[a_2, b_2]$ will be larger than that of $[a_1, b_1]$: since they each contain the same amount of mass, this means the length of the interval $[a_2, b_2]$ will be smaller. □

Corollary 3.7 *The first inequality above continues to hold true in the cases where $M_2 \geq M_1$. The second inequality also holds true in the cases where $M_2 \leq M_1$.*

4 Modifying Regions on $(\mathbb{R}, |x|)$

As the previous section shows, when our density function is radially increasing an n-bubble must be a condensed configuration of (at most) $2n$ intervals. In this section, we will develop strategies for lowering perimeter by reconfigurations that consolidate two intervals (one on each side of the origin) into a single interval. As a corollary to our work in this section, we will provide concise proofs for the single and double bubble results for $(\mathbb{R}, |x|)$, previously proven in [3].

Proposition 4.1 *On $(\mathbb{R}, |x|)$, suppose we have a condensed configuration of n regions. Suppose the outermost interval on the negative side of the origin comes from a region that also has an interval on the positive side of the origin. Then there exists a modified configuration of the same regions, but with less perimeter, achieved by moving all mass from this region to the negative side.*

Proof. Suppose our n regions are arranged as adjacent intervals with endpoints ordered sequentially as $-L_1, -L_2, ..., -L_l, 0, R_1, R_2, ..., R_m$. Suppose the interval $[-L_l, -L_{l-1}]$ is part of a region that has additional mass in another interval on the right-hand-side (RHS).

The interval we move from the RHS is of the form $[R_{i-1}, R_i]$ for some i between 1 and m. (Here, we identify R_0 with the origin.) Our strategy will be to move the mass from the RHS to adjoin it to $[-L_l, -L_{l-1}]$ on the LHS, while sliding the outer intervals on the right closer to the origin to fill the empty space.

The initial configuration will have a total perimeter of

$$TP = L_l + L_{l-1} + ... + L_2 + L_1 + R_1 + R_2 + ... + R_{m-1} + R_m.$$

The new configuration will have total perimeter of

$$TP = L_l' + L_{l-1} + ... + L_2 + L_1 + R_1 + R_2 ... + R_{i-1} + R_{i+1}' + ... + R_m'.$$

In this calculation, the endpoint L_l has been shifted to the left, resulting in L'_l; for $j > i$, the endpoints R_j have shifted to the left, resulting in R'_j; and the endpoint R_i has disappeared.

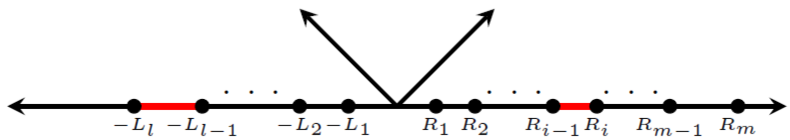


Figure 4: Configuration before joining $[R_{i-1}, R_i]$ with $[-L_l, -L_{l-1}]$

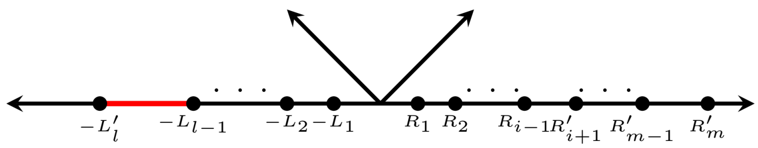


Figure 5: Configuration after joining results in new interval $[-L'_l, -L_{l-1}]$

There are two possibilities:

- **Case 1:** $R_{i-1} \leq L_l$. In this case, we see that that $R_i - R_{i-1} \geq L'_l - L_l$ by Proposition 3.6. This means that $R_i + L_l > L'_l$. Furthermore, for each $j > i$, we have $R'_j < R_j$. This is enough to tell us that our new perimeter is smaller than our original perimeter.

- **Case 2:** $R_{i-1} > L_l$. In this case, since $L_l < R_{i-1}$, we get $L'_l < R_i$ (again by Proposition 3.6). Using reasoning similar to above, we see that the new perimeter will again be smaller than our original perimeter.

□

We remark that the result does not depend on the relative size of the intervals involved (although, by Proposition 3.5, it can be made so that the outer interval contains the largest mass on its side of the origin). We also remark that choosing the negative side to move our mass to was arbitrary, and without loss of generality an equivalent argument could be made to enlarge the outermost region on the right side of the origin.

Proposition 4.1 immediately gives us the result for the single and double bubbles. Previously proven in [3], we include the results here for completeness.

Theorem 4.2 (1- and 2-Bubble Theorem with Density $|x|$) *On* $(\mathbb{R}, |x|)$*: The n-bubble solution to the isoperimetric problem for $n = 1$ is a single interval with one endpoint at the origin. For $n = 2$, the solution is two intervals, each with an endpoint at the origin. These isoperimetric solutions are unique up to symmetry.*

Proof. **For the single bubble**: If we start with any arbitrary configuration for one region with mass M, we are starting with an arbitrary (finite or countably infinite) union of intervals (with the only possible accumulation point for the endpoints being the origin). We can first consolidate near the origin using Corollary 3.2, resulting in at most two intervals: one from $[-a, 0]$ and one from $[0, b]$. Then, using Proposition 4.1, this can be converted into a single interval at $[-c, 0]$. Since each action we applied reduced perimeter or kept it the same, we can conclude that an interval of the form $[-c, 0]$ is isoperimetric.

For the double bubble: If we start with an arbitrary configuration for two regions with masses M_1 and M_2, we can first consolidate to the origin, resulting in at most two intervals on each side. We can then apply Proposition 4.1 twice (first to the negative side of the origin, then to the positive side) to convert each region to a single interval. Doing so cannot make the perimeter larger, and the resulting configuration places each region entirely to one side of the origin. Thus, we can conclude that a pair of intervals of the form $[-d, 0]$ and $[0, e]$ form an isoperimetric double-bubble. □

We conclude this section by remarking that, for configurations of n regions with $n > 2$, we can still apply Proposition 4.1 twice (first to the negative side, then to the positive side) to reduce the total number of intervals in our configuration. This gives us the following:

Corollary 4.3 *An isoperimetric n-bubble is in condensed form, and has at most $2n - 2$ adjacent intervals.*

5 The First Variation Formula

Up until now, we have reconfigured our regions by moving masses around in discrete steps and as complete intervals. In this section, we introduce the concept of continuously varying configurations via the first variation formula. We will formulate this for a differentiable density function f. Consider an interval of $[a, x]$ with one fixed endpoint. We have already seen how the weighted mass M and perimeter P are measured as $\int_a^x f$ and $f(a) + f(x)$, respectively. An immediate consequence of this is that

$$\frac{dM}{dx} = f(x) \tag{6}$$

$$\frac{dP}{dx} = f'(x). \tag{7}$$

Next, suppose that we let $x = x(t)$ vary as a function of time, and with a specified velocity $x'(t)$. This will allow us to calculate

$$\frac{dM}{dt} = f(x(t))x'(t) \tag{8}$$

$$\frac{dP}{dt} = f'(x(t))x'(t). \tag{9}$$

If we choose our velocity to equal $1/f(x(t))$, these become

$$\frac{dM}{dt} = 1 \tag{10}$$

$$\frac{dP}{dt} = f'(x(t))/f(x(t)) = \frac{d}{dx}\left[\log(f(x(t)))\right]. \tag{11}$$

We record this observation as follows:

Proposition 5.1 (First Variation Formula) *On $(\mathbb{R}, f)$: Suppose we take an interval $[a, b]$, and suppose f is differentiable in a neighborhood of both a and b. Drag both endpoints to the right at velocity $1/f(x)$. Then the weighted mass of the interval will not change, while the perimeter's instantaneous rate of change will be $(\log(f))'(a) + (\log(f))'(b)$.*

Corollary 5.2 *On $(\mathbb{R}, f)$: Suppose we have an isoperimetric n-bubble, comprised of a number of intervals with all endpoints of all intervals listed as $x_1, x_2, \ldots, x_m$. Furthermore, assume $\log(f)$ is differentiable at each of the x_i. Then*

$$\sum_{i=1}^{m} \log(f)'(x_i) = 0. \tag{12}$$

Proof. If our region is isoperimetric, it has minimal perimeter out of all other configurations that have regions with the same prescribed masses. This means, as we vary our configuration by dragging every endpoint to the right at velocity $1/f$, our masses will stay the same and our perimeter must have been at a local/global minimum. Thus, its instantaneous rate of change should be 0. $\square$

We will use the first variation formula to show that configurations of intervals that exhibit certain patterns cannot be isoperimertic. We begin by defining the patterns in question.

Definition 5.3 *Suppose we have a condensed configuration of n regions, and suppose that there are two regions (call them A and B) that are each composed of two intervals. Call these intervals A^-, A^+, B^-, and B^+ depending on their relative positions (so that A^- and B^- will sit to the left of A^+ and B^+, respectively). Suppose, up to renaming, that A^- falls to the left of B^-. Then there are three possibilities for the relative ordering of these intervals:*

- *If their relative positions, ordered left to right, are A^-, B^-, A^+, B^+, then we say A, B are* ***alternating****.*
- *If their relative ordering is A^-, B^-, B^+, A^+, then we say A, B are* ***nested****.*
- *If their relative ordering is A^-, A^+, B^-, B^+, then we say that A, B are* ***ordered****.*

Our earlier work (specifically Corollary 3.2) guarantees that an isoperimetric configuration is necessarily a condensed configuration, and therefore each side of the origin has at most one interval from each region. Not only does this guarantee that an ordered A and B will not occur in an isoperimetric configuration, it also guarantees that an isoperimetric configuration will havea A^- and B^- to the left of the origin, and B^+, A^+ to the right.

Our next lemma will guarantee that an isoperimetric region cannot have an alternating pattern either: if such a configuration exists, we can use the first variation formula to create a continuous movement that preserves the masses of our n regions while reducing total perimeter.

Remark 5.4 The core idea present in the next lemma – that of simultaneous siphoning of area from each of the inner intervals in the alternating pattern – was used in [1] to reduce the number of possible intervals in a 2-bubble.

Proposition 5.5 *On* $(\mathbb{R}, |x|)$*: Suppose there exists a condensed configuration of* n *regions* R_i*, each with mass* M_i*. Furthermore, suppose there are two specific regions (identified as regions* A *and* B*) that are of an alternating pattern (with ordering* A^-, B^-, A^+, B^+*). Then we can create a new configuration of* n *regions, with identical masses* M_i *and with lower total perimeter, by eliminating one of the interior intervals (*B^- *or* A^+*).*

Proof. With an initial configuration described as above: Identify the right endpoint of A^-, the left endpoint of B^-, and each endpoint of an interval that falls these two. Note that each of these endpoints will sit to the left of the origin, and we collectively call these endpoints $\{r_j\}$. Similarly, identify the right endpoint of A^+, the left endpoint of B^+, and all interval endpoints between these two. These endpoints will all sit to the right of the origin, and we collectively call them $\{r_j\}$.

We create a variation by moving each of the endpoints $\{l_j\}$ in the positive direction at velocity $1/|x|$, while simultaneously moving each of the $\{r_j\}$ to the left at velocity $1/|x|$. (This process is illustrated in Figure 6.) We claim that this variation does not change the weighted masses of any of our regions. Indeed, regions for which both endpoints are moving will not change mass by Proposition 5.1. Additionally, the mass contained in A and B are not changing: the rate of change for the mass being lost by A^+ is identical to the rate of change of the mass being gained by A^- (similarly for B^-, B^+). Because our density function is $|x|$ and our endpoints are all moving towards the origin, this variation reduces total weighted perimeter. Since no moving endpoint is in danger of reaching the origin, this process can continue without risk of f becoming nondifferentiable. Therefore, this variation continues until either B^- or A^+ has shrunk entirely to size 0. We see that perimeter was made to decrease throughout this process, and continued to decrease until one of the central intervals in the alternating pattern completely disappears. $\square$

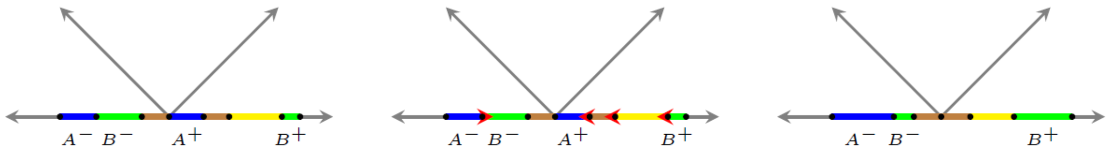


Figure 6: Over these three images, we see the interior intervals B^- and A^+ shrink while the exterior intervals grow. This is done in a manner that reduces total perimeter and leaves total mass of A and B unchanged. In the final image, A^+ has completely vanished. At this point, the sliding is done and the alternating pattern has been eliminated.

Next, using a similar argument as above, we show that two intervals cannot be nested *directly* around the origin in an isoperimetric region.

Proposition 5.6 *On* $(\mathbb{R}, |x|)$*: Suppose there exists a condensed configuration of n regions (not necessarily isoperimetric). Furthermore, suppose two specific regions (identified as regions A and B) are nested such that the origin sits in the interior of region B, and region B is immediately surrounded by region A. Then we can create a reconfiguration with lower perimeter by using a variation that moves region B entirely to the right.*

Proof. Identify our regions of interest as $A^- = [-a, -b]$, $B = B^- \cup B^+ = [-b, 0] \cup [0, c]$, and $A^+ = [c, d]$. We focus on the perimeter contributed by a, b, c, and d, noting that there might be other interval endpoints in our configuration that correspond to regions other than A and B.

Create a variation that moves the endpoints $-b$ and c to the right with velocity $1/|x|$. The weighted mass A^- will grow at exactly the same rate for which A^+ shrinks, leaving the total mass of region A unchanged. Similarly, the weighted mass of region B remains unchanged. We continue this until one of two things happens:

- The A^+ region completely disappears, and the endpoint c collides with the endpoint d. This will occur if, initially, A^+ has less weighted mass than B^-. The result is a new configuration with perimeter points of $-a$, $-b'$, 0, and d (c has been eliminated since it collided with d). Noting that $b' < b$, we can conclude that this new configuration has less total perimeter.

- The B^- interval completely disappears, and the endpoint b collides with the origin. This will occur if the weighted mass of A^+ is initially larger than the weighted mass of B^-. The result is a new configuration with perimeter points of $-a$, 0, c', and d. Although the c endpoint has grown, we can guarantee that c' contributes less total perimeter than $c + b$ did. Indeed, comparing our original region B (with perimeter points $-b$ and c) to our new region B (with perimeter points 0 and c') is equivalent to applying the 1-bubble results of Theorem 4.2 and of [3] to the region B. Thus, this new configuration has less total perimeter.

Regardless, we see that total perimeter has been reduced, the weighted masses in A and B have remained the same, and our nested intervals adjacent to the origin are gone. This completes the proof. $\square$

Our next proposition also works to adjust the position of the regions around the origin. Note that this argument, which allows for all interval endpoints to shift simultaneously, is specific for our $|x|$ density function.

Proposition 5.7 *On* $(\mathbb{R}, |x|)$*: Suppose we have* n *regions in a condensed configuration, and suppose that the origin lands in the interior of one interval. Then, by creating a variation that slides all intervals and regions in one direction (either the left or the right), we can lower perimeter and position the origin as a boundary point between two intervals.*

Proof. Given a condensed configuration of regions, suppose the origin sits in the interior of a region A. By the nature of condensed configurations, this means A can be split into two intervals directly adjacent to the origin, $A = A^- \cup A^+ = [-a_1, 0] \cup [0, a_2]$. Denote all other endpoints of other regions as x_i's.

Consider a variation that takes all endpoints except the origin and slides them to the right at velocity $1/f = 1/|x|$. According to the first variation formula for enclosed mass, this results in movement that keeps the weighted mass of each region constant (with A's mass staying constant because A^- loses mass at the same rate A^+ gains mass). By the first variation formula for perimeter, we know that a twice-differentiable density function will have total perimeter changing at rates of

$$\frac{dP}{dt} = \sum f'(x(t))/f(x(t)) \tag{13}$$

$$\frac{d^2P}{dt^2} = \sum \frac{f(x(t))f''(x(t)) - [f'(x(t))]^2}{[f(x(t))]^3} \tag{14}$$

where f is the density function, and the the summation is taken over all perimeter points that are moving. When $f(x) = |x|$, this simplifies to

$$\frac{dP}{dt} = \sum 1/x \tag{15}$$

$$\frac{d^2P}{dt^2} = \sum -1/|x|^3 \tag{16}$$

provided that the origin is not a perimeter point (so $x \neq 0$ in the equation above). Notably, we can observe that the second derivative of perimeter is negative for $f(x) = |x|$. There are two possibilities:

- If perimeter is decreasing under this variation (so $P' < 0$ initially), it will continue to decrease. We can therefore continue applying this variation until the endpoint a_1 has been dragged to collide with the origin.

- If perimeter is increasing under this variation, we reverse the variation and drag each of our endpoints to the left with velocity $1/|x|$ instead. This will keep the masses the same; will change the sign of P'; and will leave P'' negative. The result is that perimeter will decrease, and will continue to decrease until the endpoint a_2 collides with the origin.

Thus, we see that we can drag endpoints in a manner that decreases perimeter until either A^- or A^+ disappears. Either way, we are left with the entirety of region A on one side of the origin. □

Using Proposition 5.7, we are ready to identify the isoperimetric configuration for three regions.

Theorem 5.8 (3-Bubble Theorem) *On $\mathbb{R}$ with density $|x|$: The n-bubble solution to the isoperimetric problem for $n = 3$ is a condensed configuration of three intervals. If the three regions have masses $M_1 \leq M_2 \leq M_3$, then the regions of mass M_1 and M_2 are adjacent to the origin, with M_3 positioned to be adjacent to M_1.*

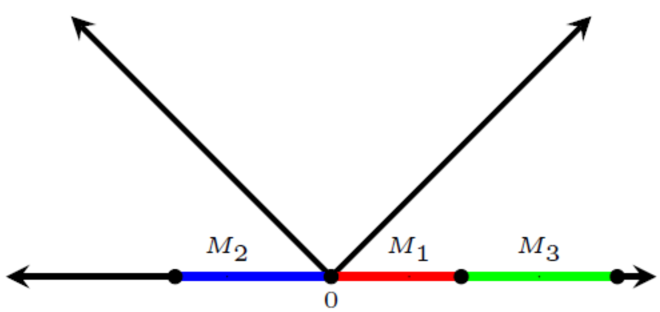


Figure 7: Solution to the triple-bubble problem with density $|x|$

Proof. By condensing our configuration and applying Proposition 4.1 twice, we know our configuration has exactly three adjacent intervals. By applying Proposition 5.7, we know the middle interval cannot contain the origin in its interior. Therefore, we have exactly three intervals, with the origin acting as an endpoint shared by two of them. A direct computation tells us the correct ordering is with M_1 and M_3 on the same side of the origin, with M_2 on the opposite side. Proposition 3.5 confirms the ordering of M_1 and M_3. This completes the proof. □

6 Proof of the 4-Bubble Theorem

In this section, we identify the isoperimetric configuration of 4 distinct regions on the real number line with density $|x|$. From our earlier work, we know that such a configuration is condensed into at most 6 intervals (first by condensing into 8 intervals using Corollary 3.2, and then by applying Proposition 4.1 to condense the outer interval on each side of the origin). Before we state and prove the 4-bubble theorem, we state a lemma which tells us that isoperimetric configurations of 4 regions that also have exactly 4 intervals must have two intervals on each side of the origin.

Theorem 6.1 (4-Bubble Lemma) *On* $(\mathbb{R}, |x|)$*: If a configuration of four regions is isoperimetric, and if each region consists of a single interval, then there must be exactly two intervals on each side of the origin.*

Proof. Using Proposition 5.7, we know that the origin will not appear in the interior of one of the intervals. If 4 intervals are on one side of the origin, we can simply move the outermost interval to the other side of the origin. This will clearly result in a configuration with less perimeter. We are left exploring the configuration with 3 intervals on one side of the origin, and 1 interval on the other. In this case, we identify the interval of smallest mass as S. There are several possibilities:

Case 1: If S is on the side with 1 interval, then it is there alone. Without loss of generality, suppose our four intervals have endpoints given by $-a < -b < -c < 0 < d$ (so that $S = [0, d]$ in this framework). We reconfigure by moving the region defined by $[-a, -b]$ to the other side of the origin. Our original figure had total perimeter $a+b+c+d$, compared to our new perimeter of $b + c + d + e'$. (See Figure 8) Because $-a$ was further from the origin than e', this new configuration will have less total perimeter.

(a) Initial configuration

(b) Ending configuration

Figure 8: Case 1: Moving the outermost interval over.

Case 2: If S is on the side with 3 intervals, we name the lone interval on the other side of the origin L, and name the remaining two intervals X_1 and X_2. (See Figure 9.)

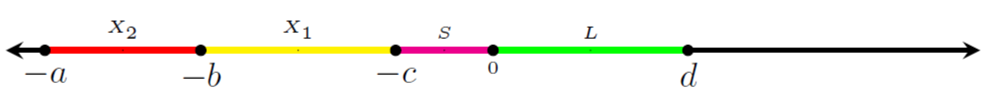
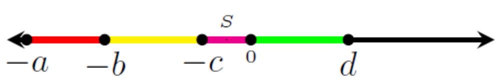


Figure 9: Our naming convention for the rest of this proof.

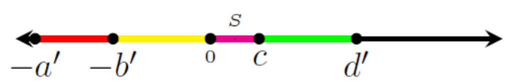
Figure 9: Our naming convention for the rest of this proof.

Proposition 3.5 guarantees that our masses will be ordered as $S \leq X_1 \leq X_2$. However, we do not know the relative size of L. We have two cases to consider:

Case 2.1: If $X_1 \leq L$, then we will want to position our smallest region S across the origin. This will result in endpoints $-a$ and $-b$ moving closer to the origin, with endpoint d moving further away from the origin. The initial and final configurations are shown in Figure 10. Our original perimeter was $a+b+c+d$, while our new perimeter is $a'+b'+c'+d'$. By assumption, $X_1 \leq L$, and so $d \geq b'$. This will mean that $[d, d']$ is narrower than (or the same length as) $[-b, -b']$ by Proposition 3.6. We can conclude that $d'-d \leq b-b'$, and therefore that $d'+b' \leq b+d$. Since $a' < a$, we get that our new configuration has less perimeter than our original one.



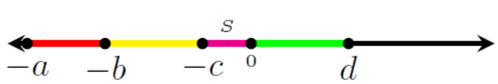
(a) Initial configuration



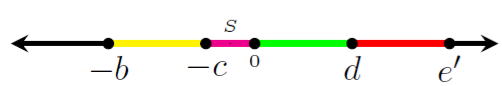
(b) Ending configuration

Figure 10: Case 2: Moving the smallest region across the origin.

Case 2.2: If $L < X_1$, we can move region X_2 over to the other side of the origin. We see this in Figure 11. Our old perimeter was $a+b+c+d$, which we must compare to $b+c+d+e'$. However, since $L < x_1$, it is clear that $-b$ is further from the origin than d is, and therefore we get that $-a$ is further from the origin than e'. Thus, this new configuration will have less total perimeter.



(a) Initial configuration



(b) Ending configuration

Figure 11: Case 3: Moving the largest region over.

This exhausts all of our possible cases. We conclude that, if we know our 4-bubble has exactly four intervals, then it must have two intervals on each side of the origin. □

Next we state several results that display the appropriate way to order intervals of different masses if we know, a priori, that we have a fixed number of intervals on each side of the origin.

Proposition 6.2 (Alternating 4-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have four regions, condensed so that each region consists of a single interval, and organized so that there are two intervals on each side of the origin. Then the least-perimeter way to meet these specifications is to have intervals alternate back and forth across the origin as they increase from smallest mass to largest. This means that the masses can be ordered, up to reflection, as $R_3R_1.R_2R_4$ (where the masses of region R_i satisfy $M_1 \leq M_2 \leq M_3 \leq M_4$).*

Proposition 6.3 (Alternating 5-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have five regions, condensed so that each region consists of a single interval, and organized so that one side of the origin has two intervals while the other side has three. Then the least-perimeter way to meet these specifications is to order the masses, up to reflection, as $R_5R_3R_1.R_2R_4$ (where the masses of region R_i satisfy $M_1 \leq \cdots \leq M_5$).*

Proposition 6.4 (Alternating 6-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have six regions, condensed so that each region consists of a single interval, and organized so that there are three intervals on each side of the origin. Then the least-perimeter way to meet these specifications is to have intervals alternate back and forth across the origin as they increase from smallest mass to largest. This means that the masses can be ordered, up to reflection, as $R_5R_3R_1.R_2R_4R_6$ (where the masses of region R_i satisfy $M_1 \leq \cdots \leq M_6$).*

These propositions are proved by cases, and their proofs are left for the appendix.

Theorem 6.5 (4-Bubble Theorem) *On $\mathbb{R}$ with density $|x|$: The n-bubble solution to the isoperimetric problem for $n = 4$ is a condensed configuration of four regions, with each region consisting of a single interval, and with two intervals on each side of the origin. If the four regions have masses $M_1 \leq M_2 \leq M_3 \leq M_4$, then the regions of mass M_1 and M_2 are adjacent to the origin; M_3 is positioned to be adjacent to M_1; and M_4 is positioned to be adjacent to M_2.*

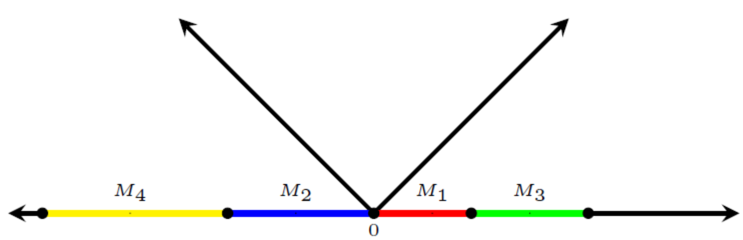


Figure 12: Solution to the 4-bubble problem with density $|x|$

Proof. Based on Corollary 3.2, we know an isoperimetric 4-bubble can be made up of at most 8 intervals. Applying Proposition 4.1 twice, we can reduce this maximum number to 6. Additionally, Proposition 4.1 guarantees that the outermost interval on each side of the origin is the only interval of its respective region. Thus, if we do have 6 total intervals, we have two regions of a single interval (the outermost intervals of the configuration) and two regions of two intervals (positioned on the interior of the configuration).

Using the language of Definition 5.3, there are two possible scenarios when there are six intervals: the innermost intervals are either alternating, or they are nested directly adjacent to the origin. Based on how these inner intervals are configured, we can apply either Proposition 5.5 (for the alternating configuration) or Proposition 5.6 (for the nested configuration) to eliminate one more interval. Finally, we can apply Proposition 5.7 (if necessary) to give us a configuration with at most five distinct intervals, and with the origin between two of these intervals in the interior.

We have shown that a non-isoperimetric configuration of more than 5 intervals can be reconfigured to have 5 intervals, lowering perimeter at every intermediate reconfiguration. Therefore, for a configuration of 4 regions to be isoperimetric, it must have no more than 5 intervals in total. If it has 4 intervals, every region is made up of a single interval and Proposition 7.1 will prove the theorem for us. This leaves us with a five interval scenario. In such a case, we know that a single region (which we identify as $A = A^- \cup A^+$) is split into two intervals, one on each side of the origin. The other regions we will label as R_i, with mass M_i. Throughout this proof, we will abuse notation by conflating regions R_i with their mass size M_i, and will use the understanding that $M_1 \le M_2 \le M_3$. Because of the Proposition 4.1, we know neither A^+ nor A^- is part of an outermost interval on either side. At the same time, Proposition 5.7 allows us to conclude that A^+ and A^- cannot both be directly adjacent to the origin. Therefore, our configuration (up to reflection) must look like Figure 13. Due to Proposition 7.2 in the appendix, we know that the relative sizes of these masses is $R_1 \le A^- \le A^+ \le R_2 \le R_3$.

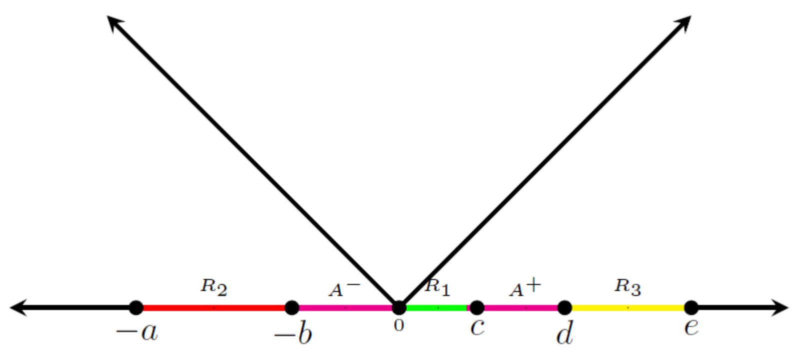
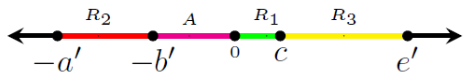


Figure 13: Initial arbitrary 4 bubble configuration of 5 intervals

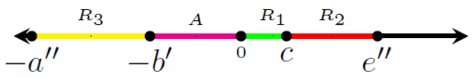
To show this 5-interval configuration is not isoperimetric, we will want to show that we can reduce perimeter by creating a consolidated configuration (with A^- and A^+ joined into a single interval which we identify as A). There are four possible scenarios we have to consider, depending on how the size of A fits into the relative sizes of R_1, R_2, and R_3.

Case 1: $A \le R_1$**.** This is impossible, as it would mean A^+ and A^- were each individually smaller than R_1, and therefore R_1 would not be directly adjacent to the origin.

Case 2: $R_1 < A \le R_2$**.** First, we will consolidate A by bringing A^+ to the left side of the origin and adjoining it to A^- . This intermediate step gives the new arrangement (a) with new endpoints $-a'$, $-b'$, and e'. The next step (b) is to switch the position R_2 and R_3. This gives us our final position with new endpoints $-a''$ and e'' and, therefore, the new total perimeter $a'' + b' + c + e''$. We will show this new perimeter is less than our starting perimeter.



(a) A^- and A^+ consolidated to A



(b) R_2 and R_3's positions are switched

Figure 14: Case 2: Initial rearrangements

We keep track of the changes that have been made with Figure 15, where we introduce $\delta_x = R_3 - R_2$ and $\delta_{x_1} = R_2 - A^+$. The appropriate masses have been identified in the figure.

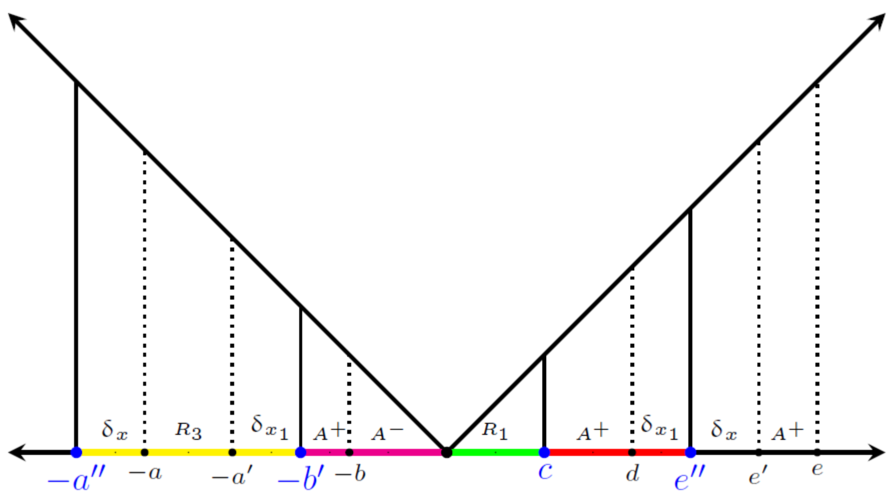


Figure 15: Case 2: Endpoints and masses labeled

We want to show that $a'' + b' + c + e'' \leq a + b + c + d + e$. To start, notice in Figure 15 that $[-b', -b]$ and $[c, d]$ both have mass of A^+. However, because $-b$ is farther away from the origin than c, Proposition 3.6 guarantees that $b' - b < d - c$. By some algebra, we have that $b' < b + d - c < b + d$. Additionally, notice that $[-a'', -a]$ has mass δ_x and $[e'', e]$ has a greater mass of $(\delta_x + A^+)$. Also note that $-a$ is farther from the origin than e'' because $R_3 + \delta_{x_1} + A > \delta_{x_1} + R_1$. Therefore, we are able to say $a'' - a < e - e''$ (again by Proposition 3.6) and thus $a'' + e'' < e + a$. Taking these inequalities together, we conclude that there is less total perimeter in this new arrangement.

Case 3: $R_2 < A \leq R_3$**.** First, we will again consolidate the region A. However, this time we will move A^- to the positive side of the origin, adjoining it to A^+ and shifting R_3 to the right. Then, we move R_3 to the negative side of the origin (further away from the origin than R_2). Note that this will end with our intervals in our hypothesized isoperimetric configuration.

(a) Region A is consolidated (b) Region R_3 is moved

Figure 16: Case 3: Initial rearrangements

To keep track of our old and new endpoints, we have the following figure. Note that the new total perimeter will consist of $a'' + a' + c + d'$. Also, note that we are defining new masses $\delta_{x_2} = R_3 - A^-$ and $\delta_{x_3} = R_2 - A^-$.

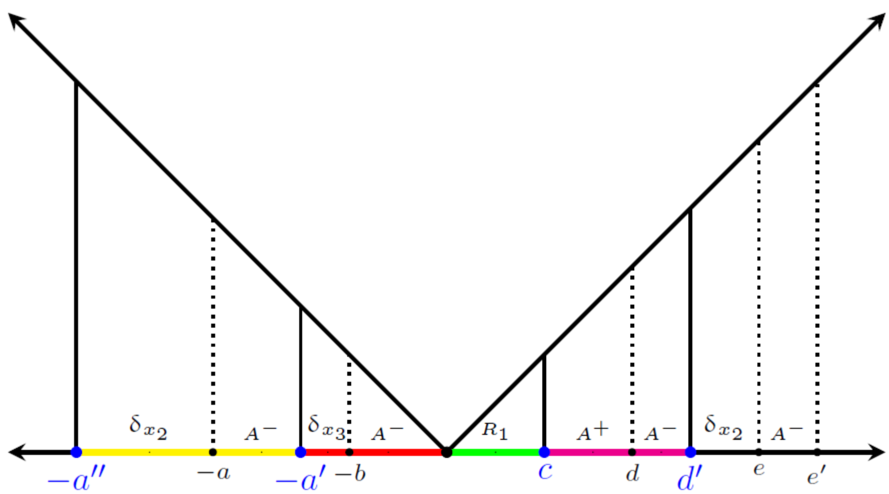


Figure 17: Case 3: Masses and endpoints labeled

We want to examine the relationship between the new and old perimeter points. First, note that because $A \geq R_2$, we have the following two inequalities.

$$R_1 + A \geq R_2 \tag{17}$$
$$R_1 + A^+ \geq R_2 - A^- \tag{18}$$

There are two sub-cases to consider:

1. $R_1 + A^+ \leq R_2$:

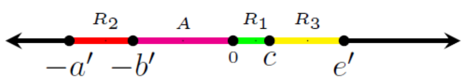
   $[-a'', -a]$ and $[d', e]$ have mass $R_3 - A^-$. However, in this case, $a' > d$, so $a > d'$. This implies that $[-a'', -a]$ is narrower than $[d', e]$ by Proposition 3.6, giving us $a'' - a \leq e - d'$, and therefore $a'' + d' \leq e + a$. By (18), $[0, d]$ contains more mass than $[-a', -b]$. In addition, $[0, d]$ has an inner endpoint that is closer to the origin than $[-a', -b]$, meaning $[-a', -b]$ is narrower than $[0, d]$. This allows us to conclude that $a' - b < d - 0$ and therefore, $a' < b + d$.

2. $R_1 + A^+ > R_2$:

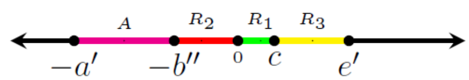
   $[-a, -a']$ and $[d, d']$ both have mass A^-. In this case $d > a'$, so $[-a, -a']$ is not as narrow as $[d, d']$. Again using Proposition 3.6, we can say that $d' - d \leq a - a'$, which gives that $a' + d' \leq a + d$. Additionally, $[a'', -b]$ has mass no greater than $[0, e]$ (which is true since $R_2 - A^- \leq R_1 + A^+$ by (18)), giving us that $[a'', -b]$ is narrower. This allows us to conclude that $a'' - b \leq e - 0$, and therefore that $a'' < b + e$.

In each sub-case, our new endpoints contribute less perimeter than before, yielding an isoperimetric arrangement of four intervals.

Case 4: $A > R_3$. As before, we consolidate A into a single interval. This time, we do so on the negative side of the origin. Then, we transpose the A and the R_2 interval. The final arrangement has a new total perimeter of $a' + b'' + c + e'$, which we compare to the old perimeter of $a + b + c + d + e$.



(a) A^- and A^+ consolidated to A



(b) Transpose A and R_2

Figure 18: Case 4: Initial rearrangements

Like before, we can keep track of all of the changes we have made with Figure 19, which has all endpoints and masses identified. As part of this image, we continue to let $\delta_{x_3} = R_2 - A^-$ introduce mass $\delta_{x_4} = R_3 - A^+$.

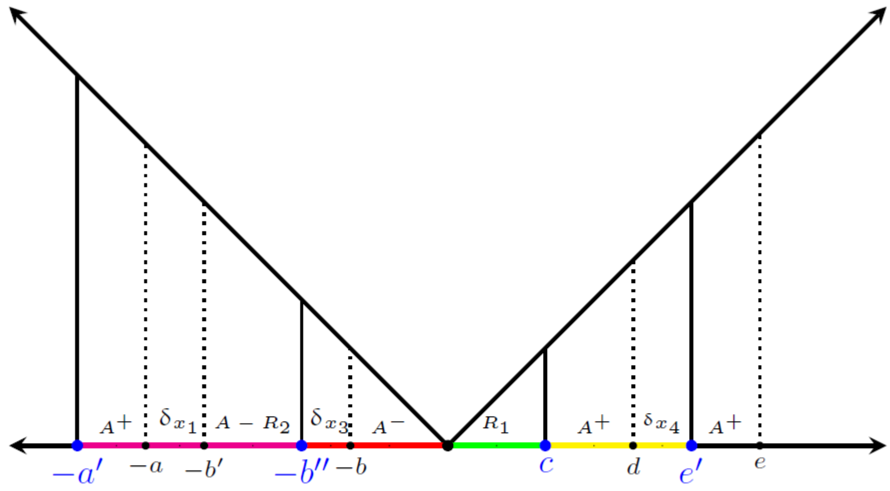


Figure 19: Case 4: Masses and endpoints labeled

Figure 19: Case 4: Masses and endpoints labeled

There are two sub-cases to consider:

1. $R_2 + A^- \geq R_1 + R_3$:

   The interval $[-a', -a]$ has mass A^+ and is separated from the origin by an interval of mass $R_2 + A^-$. $[e', e]$ has mass A^+ as well, but is separated from the origin by an interval of mass $R_1 + R_3$ away from the origin. We conclude $a' - a \leq e - e'$ and therefore that $a' + e' \leq e + a$.

   Next, we want to show that $b'' < b + d$. Now, because it is true that $A \geq R_3 \geq R_2 \geq R_1$, we see that $A^- + A^+ > R_2 - R_1$ is certainly true. Rearranging, we can have $R_1 + A^+ > R_2 - A^-$. This is important because $[-b'', -b]$ has mass $R_2 - A^-$ and $[0, d]$ has mass $R_1 + A^+$. The interval $[-b'', -b]$ has a smaller mass and is farther away from the origin, so by Proposition 3.6 we have that $b'' - b < d - 0$, and therefore $b'' < d + b$.

   Thus, there is less total perimeter in this new configuration.

2. $R_2 + A^- < R_1 + R_3$:

   The interval $[d, e']$ has mass $R_3 - A^+$ and is separated from the origin by an interval of mass $R_1 + A^+$. At the same time, $[-a, -b'']$ has mass A^- and is separated from the origin by an interval of mass R_2. We know that $A \geq R_3 \geq R_2 \geq R_1$, and therefore $A^- \geq R_3 - A^+$. Thus, the interval $[-a, -b'']$ has less mass than $[d, e']$. Additionally, b'' is closer to the origin than d. Applying Proposition 3.6 once more gives us $a < e'$ and that $a - b'' > e' - d$, and therefore that $a + d > b'' + e'$.

Lastly, we want to examine the endpoint a'. We begin by noting that the interval $[-a', -b]$ has a mass of $R_2 + A^+$ while $[0, e]$ has mass $R_1 + R_3 + A^+$. By the relative size of our masses, we know $R_2 + A^+ \leq R_1 + R_3 + A^+$. So, not only does $[0, e]$ have the larger mass than $[-a', -b]$, but it is also closer to the origin. Therefore, we can say that $a' - b \leq e - 0$ and thus $a' \leq e + b$.

Taking these earlier inequalities together, we see that in this case the new configuration has less total perimeter.

We observe that, in any possible case or sub-case, our configuration of 4 intervals uses less perimeter than any possible candidate with five intervals. Therefore, our proposed 4-interval solution is the perimeter-minimizing way to arrange four regions on $\mathbb{R}$ with density $|x|$. $\square$

7 Appendix: The Ordering Propositions

In this appendix, we collect theorems that identify appropriate orderings of intervals. Specifically, we establish the ordering of n-interval frameworks, for $n = 4, 5, 6$, where the masses come from distinct intervals and have masses $M_1 \leq \cdots \leq M_n$. It turns out that the $n = 4$ and $n = 5$ cases can be seen as corollaries of the $n = 6$ case, in the specific situation where one (or two) of the intervals have size 0. Thus, in this appendix we prove the 6-interval framework and state the 4- and 5-interval propositions as corollaries.

Proposition 7.1 (Alternating 4-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have four regions, condensed so that each region consists of a single interval, and organized so that there are two intervals on each side of the origin. Then the least-perimeter way to meet these specifications is to have intervals alternate back and forth across the origin as they increase from smallest mass to largest. This means that the masses can be ordered, up to reflection, as $R_3R_1.R_2R_4$ (where the masses of region R_i satisfy $M_1 \leq M_2 \leq M_3 \leq M_4$).*

Proposition 7.2 (Alternating 5-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have five regions, condensed so that each region consists of a single interval, and organized so that there are two intervals on one side of the origin and three on the other side. Then the least-perimeter way to meet these specifications is to have intervals alternate back and forth across the origin as they increase from smallest mass to largest. This means that the masses can be ordered, up to reflection, as $R_5R_3R_1.R_2R_4$ (where the masses of region R_i satisfy $M_1 \leq M_2 \leq M_3 \leq M_4 \leq M_5$).*

As said above, these are corollaries of the 6-interval framework proposition, which we state and prove below.

Proposition 7.3 (Alternating 6-interval framework) *On $\mathbb{R}$ with density $|x|$: Suppose we have six regions, condensed so that each region consists of a single interval, and* *organized so that there are three intervals on each side of the origin. Then the least-perimeter way to meet these specifications is to have intervals alternate back and forth across the origin as they increase from smallest mass to largest. This means that the masses can be ordered, up to reflection, as $R_5R_3R_1.R_2R_4R_6$ (where the masses of region R_i satisfy $M_1 \leq \cdots \leq M_6$).*

Proof. [Proof of Proposition 6.3]

This is a proof of exhaustion. Proposition 3.5 tells us that each side of the origin will be ordered from smallest to largest mass (as we move away from the origin). We will identify the regions of smallest mass and second-smallest mass as "S" and "M" throughout this proof.

Without loss of generality, we can assume that S, appears directly adjacent to the origin on the negative side. The major dividing line in our cases is whether or not M, appears on the same side of the origin as S. In the case where M appears on the opposite side of the origin, we will label our other regions as having masses L_1, L_2, X_1, and X_2, and consider the relative sizes of the L_i and X_i. We can use Figure 20 as a reference.

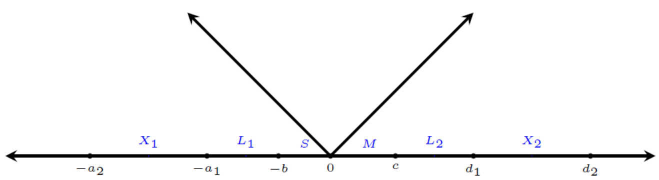
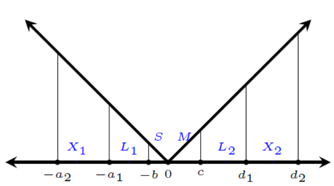


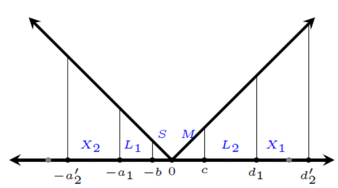
Figure 20: The figure we use as an initial configuration when the second-smallest mass, M, is on the opposite side of the origin as S.

We can assume (by Proposition 3.5) that $L_1 < X_1$ and $L_2 < X_2$, but there are still many possible cases to consider.

- Case 0: $S \leq M \leq L_1 \leq L_2 \leq X_1 \leq X_2$. This is the case we are claiming is optimal.
- Case 1: $S \leq M \leq L_1 \leq L_2 \leq X_2 \leq X_1$. In this case, we can further reduce perimeter by flipping the positions of the X_i's. The process is shown in the image below.



(a) Original configuration



(b) New configuration

Figure 21: Case 1: X_i's have flipped their positions.

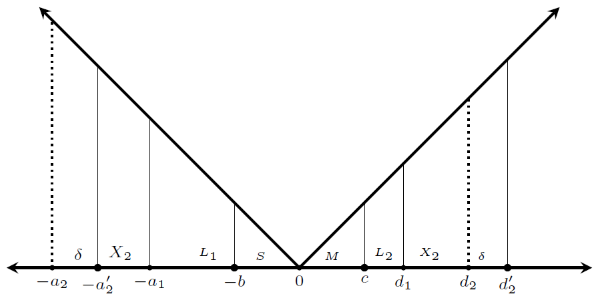
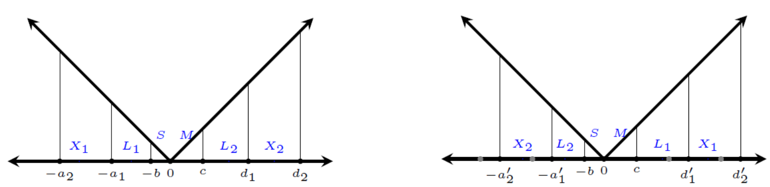


Figure 22: Case 1: all endpoints and masses identified.

Specifically, we can define $\delta = X_1 - X_2$. Note that the original configuration has perimeter $a_2 + a_1 + b + c + d_1 + d_2$, while the modified configuration has perimeter $a_2' + a_1 + b + c + d_1 + d_2'$. Because $S \leq M$ and $L_1 \leq L_2$, see that the mass in $[-a_2', 0]$ is less than the mass in $[0, d_2]$. This tells us that $a_2' < d_2$. This, in turn, implies that the interval $[-a_2, -a_2']$ is not as narrow as the interval $[d_2, d_2']$ since both contain the same mass δ. We conclude that $d_2' - d_2 < a_2 - a_2'$, and therefore that $a_2' + d_2' < a_2 + d_2$. Thus, our new configuration has less total perimeter than our old configuration. This returns us to a framework that is identical to Case 0.

- Case 2: $S \leq M \leq L_2 \leq L_1 \leq X_2 \leq X_1$. In this case, we can further reduce perimeter by flipping the positions of the X's and L's simultaneously. The process is shown in the picture below.



(a) Original configuration

(b) New configuration

Figure 23: Case 2: X's and L's have flipped position.

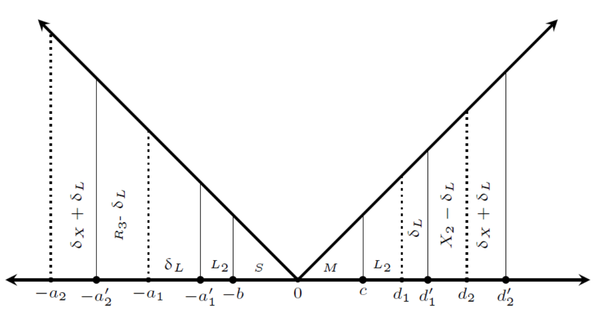
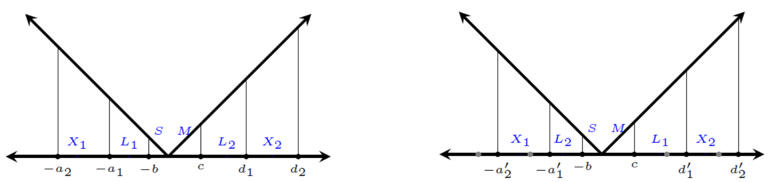


Figure 24: Case 2: all endpoints and masses identified.

Specifically, we can define $\delta_L = L_1 - L_2$ and $\delta_X = X_1 - X_2$. Note that the original configuration has perimeter $a_2 + a_1 + b + c + d_1 + d_2$, while the modified configuration has perimeter $a_2' + a_1' + b + c + d_1' + d_2'$. The second figure shows all endpoints and masses calculated. We can see that, because $S \leq M$, we get that $a_1' \leq d1$ and therefore the interval $[-a_1, -a_1']$ is no narrower than the interval $[d_1, d_1']$. In other words, $d_1' - d_1 \leq a_1 - a_1'$, and so $a_1' + d_1' \leq a_1 + d_1$. A similar argument tells us that $a_2' + d_2' \leq a_2 + d_2$, and we can conclude that this reconfiguration did not increase the total perimeter. This returns us to a framework that is identical to Case 0.

- Case 3: $S \leq M \leq L_2 \leq L_1 \leq X_1 \leq X_2$. In this case, we can switch the roles of the L's, leaving the X's in their current position.The process is shown in the picture below.



(a) Original configuration (b) New configuration

Figure 25: Case 3: Flip the L's, leaving the X's where they are.

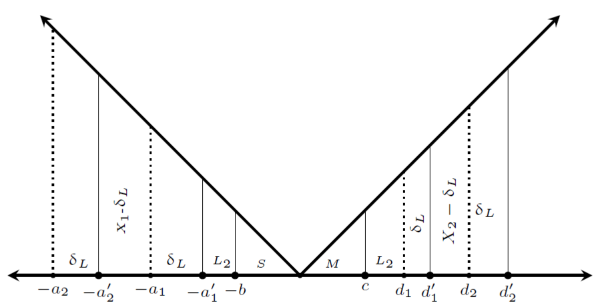
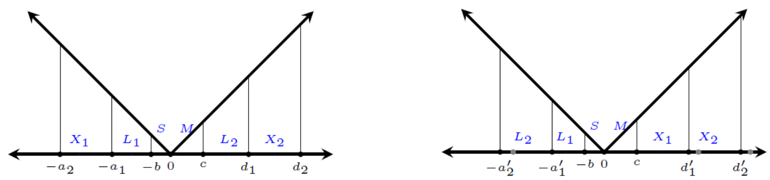


Figure 26: Case 3: All endpoints and masses identified.

Specifically, we can define $\delta_L = L_1 - L_2$. The initial configuration has perimeter $a_2 + a_1 + b + c + d_1 + d_2$, while the modified configuration has perimeter $a_2' + a_1' + b + c + d_1' + d_2'$. In the figure above, we see all endpoints and masses calculated. Note that similar reasoning to that used in Case 2 shows us that $d_1' - d_1 \leq a_1 - a_1'$, and also that $d_2' - d_2 \leq a_2 - a_2'$. Taken together, this gives us that the new modified configuration will never have more perimeter than the old configuration. Applying this rearrangement returns us to a framework that is identical to Case 0.

- Case 4: $S \leq M \leq L_1 \leq X_1 \leq L_2 \leq X_2$. In this case we can transpose the L_2 and the X_1 regions, leaving the others in their current spots. The process is shown in the picture below.



(a) Original configuration (b) New configuration

Figure 27: Case 4: transpose the locations of X_1 and L_2.

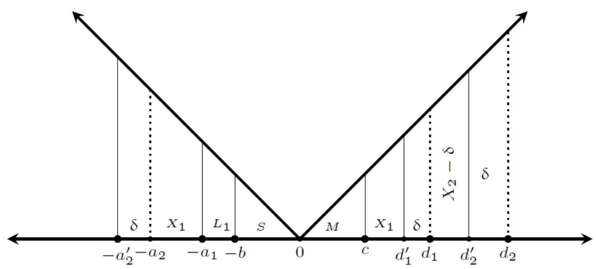
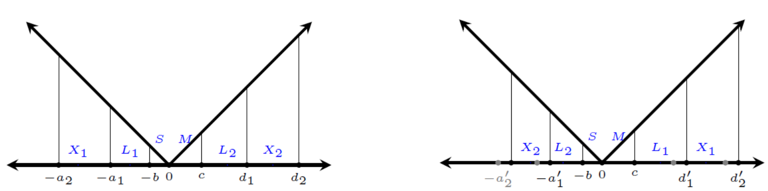


Figure 28: Case 4: all endpoints and masses identified.

Specifically, we can define $\delta = L_2 - X_1$. Note that $X_1 + L_1 + S \geq X_1 + M$, and therefore the interval $[-a_2', -a_2]$ cannot be wider than the interval $[d_1', d_1]$. This tells us that $a_2' - a_2 \leq d_1 - d_1'$, which implies that $a_2' + d_1' \leq a_2 + d_1$. Similarly, we can immediately see that $d_2' \leq d_2$. Taken together, this means that the perimeter of the new configuration is necessarily not greater than the perimeter of the original configuration. Applying this rearrangement returns us to a framework that is identical to Case 0.

- Case 5: $S \leq M \leq L_2 \leq X_2 \leq L_1 \leq X_1$. In this case, we can transpose the positions of L_1 and L_2, and simultaneously transpose the positions of X_1 and X_2. The result is depicted in the picture below.



(a) Initial configuration (b) New configuration

Figure 29: Case 5: transpose the locations of the L_i and the X_i simultaneously.

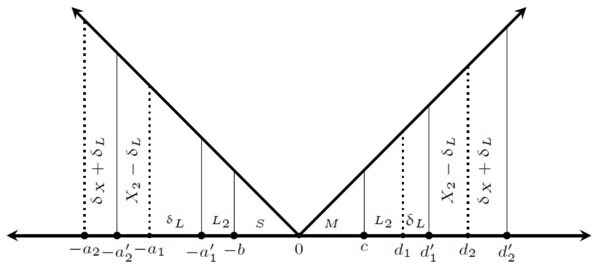


Figure 30: Case 5: all endpoints and masses identified.

By setting $\delta_X = X_1 - X_2$ and $\delta_L = L_1 - L_2$ and reasoning as in previous cases, we can observe that this move will lower overall perimeter. The rearrangment will leave us in a framework that is identical to Case 4.

Next, we must consider what happens if the second-smallest mass is on the same side of the origin as the smallest mass. We claim that this is not optimal, and that we can rearrange our masses to move the second-smallest mass across the perimeter, lowering perimeter in the process. To begin, let's rename our regions as in Figure 31

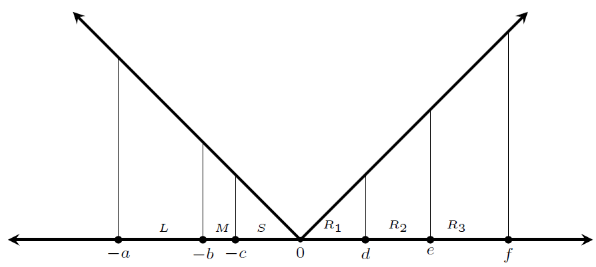
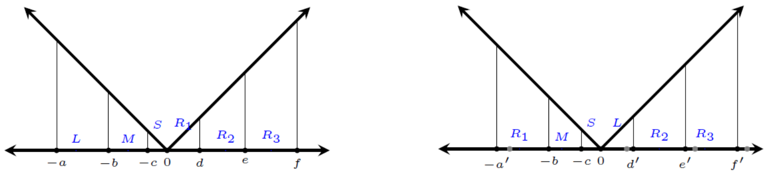


Figure 31: New cases, in which the second smallest mass (M) is on the same side of the origin as the smallest mass (S).

We already know that $S \leq M \leq L$ and that $R_1 \leq R_2 \leq R_3$, and that $M < R_1$. The only question that remains is what the size of L is relative to the R_i. There are four cases:

- Case 6: $L \leq R_1 \leq R_2 \leq R_3$. In this case we can transpose L and R_1, leaving the others in their current spots. The process is shown in the picture below;



(a) Original configuration

(b) New configuration

Figure 32: Case 6: transpose the locations of R_1 and L.

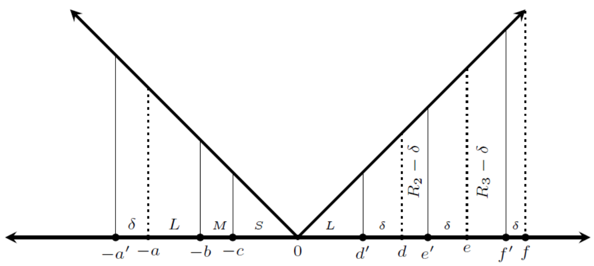
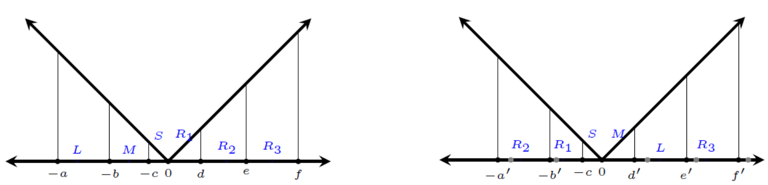


Figure 33: Case 6: all endpoints and masses identified.

Specifically, we can let $\delta = R_1 - L$. The figures above show both the original configuration (with perimeter $a + b + c + d + e + f$) and the new configuration (with perimeter $a' + b + c + d' + e' + f'$). The final figure identifies all endpoints and corresponding masses. We can make the observation that, as endpoints, $a > d'$. This implies that the interval $[-a', -a]$ is narrower than the interval $[d', d]$ which contains the same mass. Therefore, we can conclude that $a' - a < d - d'$ and therefore that $a' + d' < a + d$. Additionally, we can see that $e' < e$ and $f' < f$. Taken together, these inequalities imply that after reconfiguration we have successfully lowered perimeter. The rearrangment will leave us in a framework that is identical to Case 7.

- Case 7: $R_1 \le L \le R_2 \le R_3$. In this case we can transpose M and R_1 and, simultaneously, transpose L and R_2. The process is shown in the picture below.



(a) Original configuration

(b) New configuration

Figure 34: Case 7: transpose R_1 with M, and also R_2 with L.

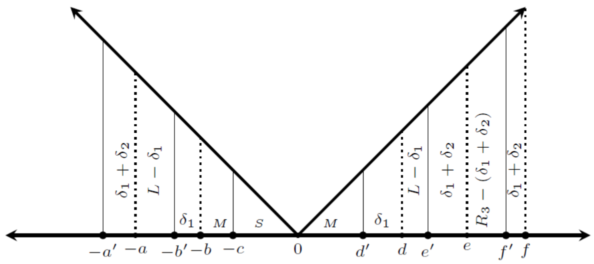
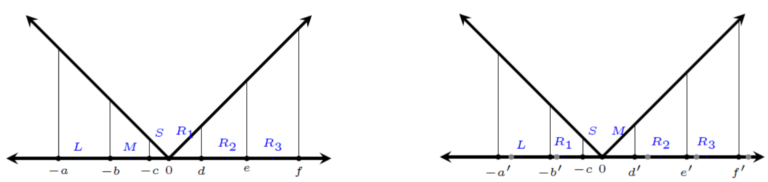


Figure 35: Case 7: all endpoints and masses identified.

Specifically, we can let $\delta_1 = R_1 - M$ and let $\delta_2 = R_2 - L$. In the figures above, we see the original configuration (with perimeter $a + b + c + d + e + f$) as well as the new configuration (with perimeter $a' + b'c + d' + e' + f'$).

Since $b > 0$ and $[-b, -c]$ has the same mass as $[0, d]$, can conclude that $b > d'$. This implies that $b' - b < d - d'$, and therefore that $b' + d' < b + d$. Similarly, we can conclude that $a > e'$, leading to $a' - a < e - e'$ and $a' + e' < a + e$. Finally, we can conclude that $f' < f$ automatically. Taking these inequalities together, we can conclude that our total perimeter has decreased after our reconfiguration. Applying this rearrangement returns us to a framework that is identical to Case 0.

- Case 8: $R_1 \leq R_2 \leq L \leq R_3$. In this case we can transpose M and R_1, leaving the others in their current spots. The process is shown in the picture below.



(a) Original configuration

(b) New configuration

Figure 36: Case 8: transpose R_1 with M.

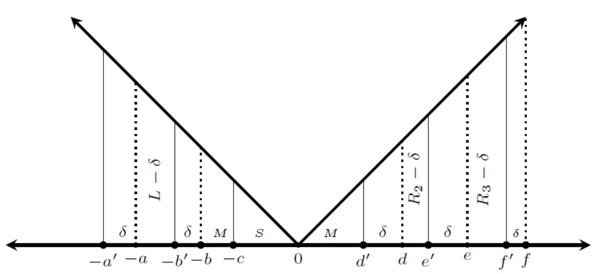


Figure 37: Case 8: all endpoints and masses identified.

Specifically, we can let $\delta = R_1 - M$. The figures above show both the original configuration (with perimeter $a+b+c+d+e+f$) as well as the new configuration (with perimeter $a'+b'+c+d'+e'+f'$). Since $S \geq 0$ and the intervals $[-b,-c]$ and $[0,d']$ have the same mass, we know that $b \geq d'$. This allows us to conclude that the interval $[-b',-b]$ is narrower than (or the same size as) the interval $[d',d]$, as both contain the same mass. We conclude that $b'-b \leq d-d'$, and therefore that $b'+d' \leq b+d$. Using similar reasoning (and the fact that $L > R_2$), we can deduce that $a'+e' < a+e$. Finally, we can identify that $f' < f$. Taken together, we can conclude that the total perimeter is smaller after moving to the new configuration. Applying this rearrangement returns us to a framework that is identical to Case 0.

- Case 9: $R_1 \leq R_2 \leq R_3 \leq L$. In this case we can transpose L and R_1, leaving the others in their current spots. The associated picture and proof are identical to that of Case 8: however, after considering the new ordering of intervals, we can conclude that have reconfigured into Case 1 rather than Case 0.

□

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