A Quadratic Relation in the Bernoulli Numbers

M. Beals-Reid

Abstract - We present a proof of an identity involving the Bernoulli numbers. This identity has been proved, over the centuries, by many different methods. Our method uses a technique from over a millennium ago for finding polynomial expressions for sums of powers of integers.

Keywords : Bernoulli numbers; Bernoulli polynomials

Mathematics Subject Classification (2020) **:** 11B68

1 Introduction

The Bernoulli numbers are usually defined, following Euler, in terms of an exponential generating function:

$$B(z) = \frac{ze^z}{e^z - 1} = \sum_{m \geq 0} B_m \frac{z^m}{m!}, \tag{1}$$

see [8, Equation 6.81, 34th printing, January 2022]. Note that this definition yields $B_1 = 1/2$, and that most sources, including earlier printings of [8], prefer a slightly different definition involving $z/(e^z - 1)$ which yields $B_1 = -1/2$ (the other values are not affected). Donald Knuth explains in [10] that he decided to make the change after being convinced by the arguments given by Peter Luschny in The Bernoulli Manifesto [11].

If we express $B'(z)$ in terms of $B(z)$ we get (see, e.g., [5, 3])

$$B(z)^2 = (z+1)B(z) - zB'(z),$$

and, using the definition of multiplication of power series, we get

Theorem 1.1 *The Bernoulli numbers B_m satisfy, for integers $m \geq 1$,*

$$\sum_{j=0}^{m} \binom{m}{j} B_{m-j} B_j = mB_{m-1} - (m-1)B_m.$$

For even indices greater than 2, this can be written as a recurrence:

Corollary 1.2 *For $m > 1$, we have:*

$$B_{2m} = -\left(\frac{1}{2m+1}\right) \sum_{j=1}^{m-1} \binom{2m}{2j} B_{2m-2j} B_{2j}.$$

These are well known; see [6, Equation 24.14.2], and [12, Chapter III, Section XII, Equation 13]. The earliest discovery seems to be Euler [4, Part 2, Chapter V, paragraph 123], and over the years the result has been proved many times in many different ways [14, 15, 13, 5, 3]. The reference [13] discusses previous methods of proof:

> The methods of these authors are varied—partial fraction decomposition of rational functions, Laurent expansions of meromorphic functions, rearrangement of absolutely convergent double series, Fourier expansions of Eisenstein series and Apostol's extension of a transformation formula due to Lehmer and Newman.[13, p. 1176]

Subsequent authors [5, 3] have derived higher-order generalizations of Theorem 1.1) by considering other differential equations satisfied by the function $B(z) = ze^z/(e^z - 1)$.

Bernoulli numbers were initially defined in a different way. Jakob Bernoulli (1654–1705), and, independently, Takakazu Seki (?–1708) (see [9]), in their study of formulas for sums of powers of integers, made a remarkable discovery. There is a sequence of numbers, $B_0, B_1, \ldots$ with the property that they can be used to express the sum of the first n mth powers as a polynomial in n of degree $m + 1$:

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\left(\sum_{j=0}^{m}\binom{m+1}{j} B_j n^{m+1-j}\right), \tag{2}$$

for all $m, n \geq 0$ [8, Equation 6.78, 34th printing, January 2022]. Plugging in $n = 1$ and starting with $B_0 = 1$ allows one to express B_m in terms of $B_0, \ldots, B_{m-1}$ [8, Equation 6.79, 34th printing, January 2022], yielding :

$$\begin{aligned} \sum_{j=0}^{m}\binom{m+1}{j} B_j &= m+1 \\ (m+1)B_m &= (m+1) - \sum_{j=0}^{m-1}\binom{m+1}{j} B_j \\ B_m &= 1 - \left(\frac{1}{m+1}\right)\sum_{j=0}^{m-1}\binom{m+1}{j} B_j \end{aligned} \tag{3}$$

Note that this definition also yields $B_1 = 1/2$, and that the one that yields $B_1 = -1/2$ is obtained similarly, starting with $\sum_{k=0}^{n-1} k^m$ on the left hand side of (2).

The goal of this paper is to give another proof of Theorem 1.1, using the definition (3) for Bernoulli numbers. We begin with an old geometric argument, Lemma 2.1, which dates back to Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham (ca. 965-1039), known also as Alhazen (see [7, 2]). For nonnegative integers m and n, let $S_m(n)$ denote the sum of the first n mth powers, $\sum_{j=1}^{n} j^m$. Lemma 2.1, in Section 2 below, is a derivation of an expression for $S_m(n)$ involving products of coefficients of $S_{m-1}(n)$ by the polynomials $S_k(n)$ for $k \leq m$. When the coefficients of the $S_k(n)$ are expressed in terms of the Bernoulli numbers, via Lemma 3.2 in Section 3 below, we obtain Theorem 1.1.

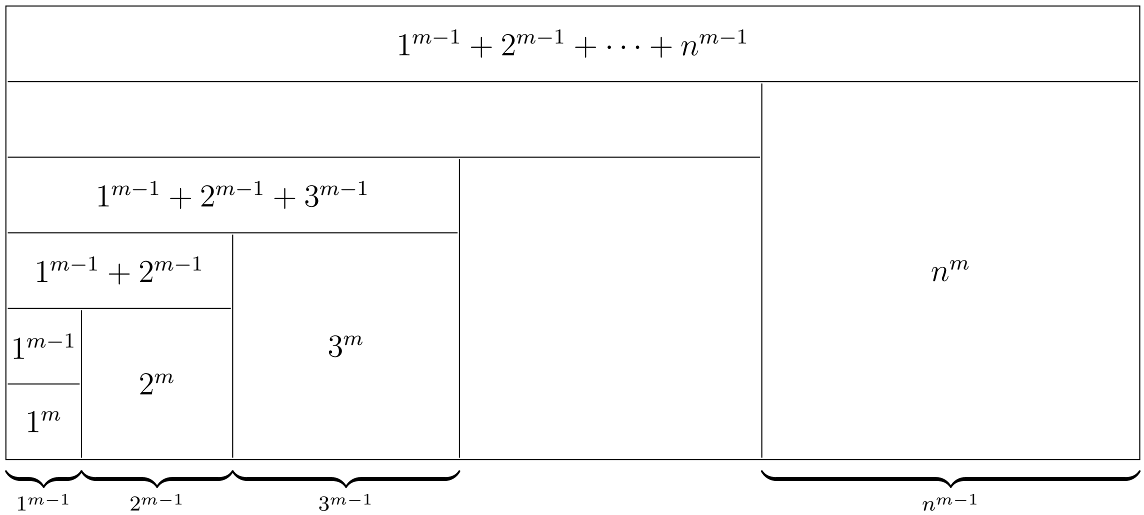

Figure 1: Proof of (4) [7, 2]

Our proof is most similar to that of Underwood [14] from 1928, only it uses Alhazen's geometric argument instead of Underwood's algebraic manipulations.

2 Sums of Powers

Let m and n be nonnegative integers, with $m > 0$. We are interested in finding an expression for $S_m(n)$, assuming inductively that we have expressions for $S_k(n)$ for $0 \le k < m$, and that $S_k(n)$ is a polynomial in n of degree $k+1$:

$$S_k(n) = \sum_{j=1}^{k+1} s_{k,j} n^j.$$

We start with $S_0(n) = n$.

Lemma 2.1 *For all $m \ge 1$ we have*

$$S_m(n) = (n+1)S_{m-1}(n) - \sum_{j=1}^{m} s_{m-1,j} S_j(n).$$

Proof. We prove first that for all integers $m \ge 1$, the functions S_m and S_{m-1} satisfy

$$S_m(n) + \sum_{k=1}^{n} S_{m-1}(k) = (n+1)S_{m-1}(n) \tag{4}$$

for all $n \ge 1$, which is the Alhazen identity [7, p. 84], [2] .

In Figure 1 we see that a rectangle of width $S_{m-1}(n)$ and height $n+1$ can be partitioned into:

- Rectangles of width k^{m-1} and height k, for $k = 1 \dots n$, with total area $S_m(n)$.
- Rectangles of width $1^{m-1} + \cdots + k^{m-1} = S_{m-1}(k)$ and height 1, for $k = 1 \dots n$, with total area $\sum_{k=1}^{n} S_{m-1}(k)$.

Since the area of the large rectangle is $(n+1)S_{m-1}(n)$, equating areas yields (4).

Note that the rectangles in Figure 1 have the correct proportions for the $m = 2$ case. But the diagram holds true for arbitrary $m > 0$.

As we said before, we denote by $s_{i,j}$ the n^j coefficient of $S_i(n)$. We have $s_{i,0} = S_i(0) = 0$, so

$$S_{m-1}(k) = \sum_{j=1}^{m} s_{m-1,j} k^j.$$

Rewriting (4) and expressing the $S_{m-1}(k)$ in terms of the $s_{m-1,j}$ yields:

$$\begin{aligned} S_m(n) &= (n+1)S_{m-1}(n) - \sum_{k=1}^{n} S_{m-1}(k) \\ &= (n+1)S_{m-1}(n) - \sum_{k=1}^{n} \sum_{j=1}^{m} s_{m-1,j} k^j \\ &= (n+1)S_{m-1}(n) - \sum_{j=1}^{m} \sum_{k=1}^{n} s_{m-1,j} k^j \\ &= (n+1)S_{m-1}(n) - \sum_{j=1}^{m} s_{m-1,j} S_j(n). \end{aligned}$$

□

Note that in Lemma 2.1, $S_m(n)$ occurs on the right hand side (with a coefficient of $-s_{m-1,m}$, which is clearly negative). Thus we can solve for $S_m(n)$, obtaining a polynomial of degree $m+1$. But more interesting is the quadratic relation we obtain in the coefficients, leading to Theorem 1.1, see (7) in Section 4.

3 Bernoulli Numbers

The Bernoulli numbers may be defined easily in our notation: $B_m = s_{m,1}$. Bernoulli realized that for all i, j, the coefficient $s_{i,j}$ could be easily described in terms of B_{i+1-j}, as in equation (2). For completeness, we give here a short derivation of this result.

We have seen by induction on m that for any $m \geq 0$, there is a polynomial expression in n of degree $m + 1$ which is equal to $S_m(n)$ for all natural numbers n. This expression may be evaluated at any real number, so we may speak of the polynomial function $S_m(x)$ where x is a real variable.

Lemma 3.1 *For all nonnegative integers m, the polynomial $S_m(x)$ is characterized by the following:*

- $S_m(x) - S_m(x-1) = x^m$.
- $S_m(0) = 0$.

Proof. Let m be an arbitrary nonnegative integer. Clearly the condition $S_m(0) = 0$ holds. And for any natural number $n > 0$, $S_m(n) - S_m(n-1) = n^m$. So for any m, the polynomial $S_m(x) - S_m(x-1) - x^m$ is zero at $x = n$ for all natural numbers $n > 0$; it is the zero polynomial.

Conversely, suppose the polynomial $f(x)$ is such that $f(0) = 0$ and $f(x) - f(x-1) = x^m$. Then by induction on n we see that $f(n) = \sum_{j=1}^{n} j^m = S_m(n)$. □

Let m be an arbitrary nonnegative integer. By differentiating both sides of the polynomial identity

$$S_{m+1}(x) - S_{m+1}(x-1) = x^{m+1},$$

we see that

$$\begin{aligned} S'_{m+1}(x) - S'_{m+1}(x-1) &= (m+1)x^m \\ (1/(m+1))S'_{m+1}(x) - (1/(m+1))S'_{m+1}(x-1) &= x^m \end{aligned}$$

In other words, the polynomial $(1/(m+1))S'_{m+1}(x)$ satisfies the first of the criteria of Lemma 3.1. Subtracting the constant term from this polynomial results in the second criterion being satisfied as well. We have:

$$S_m(n) = (1/(m+1))(S'_{m+1}(n) - S'_{m+1}(0)). \tag{5}$$

For $m \geq 0$ and $j > 1$, considering the n^{j-1} coefficient of both sides of (5) yields an equation relating $s_{m+1,j}$ and $s_{m,j-1}$. Solving for $s_{m+1,j}$, we obtain:

$$s_{m+1,j} = ((m+1)/j)s_{m,j-1}.$$

Expressing $s_{m,j}$ in terms of $s_{m-1,j-1}$, this becomes

$$s_{m,j} = (m/j)s_{m-1,j-1}. \tag{6}$$

Our goal is to prove Bernoulli's expression for $s_{m,j}$ in terms of B_{m+1-j} [8, Equation 6.78]:

Lemma 3.2 *For all $m \geq 0$ and $1 \leq j \leq m+1$*

$$s_{m,j} = \left(\frac{1}{m+1}\right)\binom{m+1}{j}B_{m+1-j}.$$

Proof. We proceed by induction on j. From the definition $B_m = s_{m,1}$ we obtain the base case:

$$s_{m,1} = \left(\frac{1}{m+1}\right)\binom{m+1}{1}B_{m+1-1},$$

for all m. Now suppose we have established the $j-1$ case; the j case follows from (6):

$$\begin{aligned} s_{m,j} &= (m/j)s_{m-1,j-1} \\ &= (m/j)\left(\frac{1}{m}\right)\binom{m}{j-1}B_{m+1-j} \\ &= \left(\frac{1}{m+1}\right)\binom{m+1}{j}B_{m+1-j}. \end{aligned}$$

□

Note that Lemma 3.2 is the same as (2), after equating coefficients. This leads to the definition $B_m = s_{m,1}$ at the beginning of Section 3 being equivalent to (3), since $B_m = s_{m,1}$ implies Lemma 3.2, which is equivalent to (2), which implies (3).

4 A Quadratic Relation

We are ready to prove Theorem 1.1:

Proof. Equating the n^k coefficients of both sides of the equality in Lemma 2.1 yields:

$$s_{m,k} = s_{m-1,k-1} + s_{m-1,k} - \sum_{j=1}^{m} s_{m-1,j}s_{j,k}. \tag{7}$$

Substituting the expression in Lemma 3.2 for the $s_{i,j}$ in (7), we obtain, in the $k=1$ case,

$$\sum_{j=0}^{m}\binom{m}{j}B_{m-j}B_j = mB_{m-1} - (m-1)B_m.$$

Thus Theorem 1.1 is seen to hold. □

Before proving Corollary 1.2 we need the following:

Lemma 4.1 *For odd $m > 1$, the Bernoulli number B_m is zero.*

Proof. Starting with $B_0 = 1$ and applying Theorem 1.1 repeatedly with $m = 1$, $m = 2$, $m = 3$, we compute $B_1 = 1/2$, $B_2 = 1/6$, and $B_3 = 0$. Since the quadratic terms in Theorem 1.1 involve indices which sum to m, for odd m we must have exactly one odd index in each summand, so for odd m greater than 1 we can see by induction that the Bernoulli number B_m is zero (the terms involving B_{m-1} cancel out). □

Now we are ready to prove Corollary 1.2:

Proof. By Lemma 4.1, for even $m > 2$, we can ignore the B_{m-1} term in Theorem 1.1. We have B_m on both sides of the equation (it occurs as B_j for $j = m$ and B_{m-j} for $j = 0$); solving for B_m yields the following:

$$B_m = -\left(\frac{1}{m+1}\right)\sum_{j=2}^{m-2}\binom{m}{j}B_{m-j}B_j,$$

valid for even $m > 2$. Since the terms on the right hand side are zero for odd j, the proof is complete. □

5 Bernoulli Polynomials

The Bernoulli polynomials are usually defined in terms of a generating function:

$$B(x,z) = \frac{ze^{zx}}{e^z - 1} = \sum_{m\geq 0} B_m(x)\frac{z^n}{n!},$$

[6, Equation 24.2.3], but in our notation we may equivalently define them as

$$B_m(x) = S'_m(x-1) = \sum_{j=0}^{m} \binom{m}{j} B_j (x-1)^{m-j} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} B_j x^{m-j}$$

[8, Equation 7.80]. Note that the Bernoulli number B_m is equal to $B_m(1)$. In sources which define B_1 to be $-1/2$, the Bernoulli polynomials are the same as the ones defined here, but there the Bernoulli number B_m equals $B_m(0)$, and the x^{m-j} coefficient of $B_m(x)$ is $\binom{m}{j}B_j$. Also note that (5) is [1, Lemma 2.3, p.35] or [8, Equation 7.79], showing that the definitions (1) and (2) of Bernoulli numbers are equivalent.

In this Section we prove two identities regarding the Bernoulli polynomials, successively generalizing Theorem 1.1. At the level of coefficients, these are quadratic relations in the Bernoulli numbers; indeed, we shall see that they are the same quadratic relations which we obtained via Lemma 2.1. So these identities may be seen as alternative presentations of our main result.

If we denote the partial derivative of $B(x,z)$ with respect to z by $B_z(x,z)$, we get

$$B(z)B(x,z) = (1+zx)B(x,z) - zB_z(x,z),$$

and identifying coefficients yields the first identity:

Theorem 5.1 *The Bernoulli polynomials satisfy*

$$\sum_{j=0}^{m} \binom{m}{j} B_{m-j} B_m(x) = mxB_{m-1}(x) - (m-1)B_m(x)$$

for $m > 0$.

Proof. We provide a proof similar to the one of Theorem 1.1. We take the polynomial identity in Lemma 2.1, substitute $x-1$ for n, and differentiate, obtaining:

$$B_m(x) = xB_{m-1}(x) + S_{m-1}(x-1) - \sum_{j=1}^{m} s_{m-1,j} B_j(x).$$

Using (5) we can substitute $(1/(m))(S'_m(x-1) - S'_m(0)) = (1/m)(B_m(x) - B_m)$ for $S_{m-1}(x-1)$. And using Lemma 3.2 we substitute $(1/m)\binom{m}{j}B_{m-j}$ for $s_{m-1,j}$, obtaining:

$$B_m(x) = xB_{m-1}(x) + (1/m)(B_m(x) - B_m) - \sum_{j=1}^{m} (1/m)\binom{m}{j} B_{m-j} B_j(x).$$

Note that $B_m = \binom{m}{0} B_m B_0(x)$, and so it may be absorbed into the sum by including a term corresponding to $j = 0$. Multiplying through by m and rearranging terms completes the proof. $\square$

This can be generalized further. Subtracting $mB_{m-1}(x+y)$ from both sides of Theorem 5.1 for x replaced by $x + y$ produces [6, Equation 24.14.1]:

Theorem 5.2 *The Bernoulli polynomials satisfy*

$$\sum_{j=0}^{m} \binom{m}{j} B_{m-j}(y) B_j(x) = m(x + y - 1)B_{m-1}(x + y) - (m - 1)B_m(x + y)$$

for $m > 0$.

Proof. We are providing again a proof based on identifying coefficients.

The following equation, satisfied for all for $m \geq 0$, is [6, Equation 24.4.12].

$$B_m(x + y) = \sum_{\ell=0}^{m} \binom{m}{\ell} y^{m-\ell} B_\ell(x) \tag{8}$$

To check this, consider,for any nonnegative integers n, p, $S_m(n + p)$. We have:

$$\begin{aligned} S_m(n+p) &= \sum_{k=1}^{p} k^m + \sum_{k=1}^{n} (k+p)^m \\ &= S_m(p) + \sum_{k=1}^{n} \sum_{\ell=0}^{m} \binom{m}{\ell} k^\ell p^{m-\ell} \\ &= S_m(p) + \sum_{\ell=0}^{m} \binom{m}{\ell} p^{m-\ell} \sum_{k=1}^{n} k^\ell \\ &= S_m(p) + \sum_{\ell=0}^{m} \binom{m}{\ell} p^{m-\ell} S_\ell(n) \end{aligned}$$

This is a polynomial identity. Substituting $x - 1$ for n and y for p, we obtain:

$$S_m(x + y - 1) = S_m(y) + \sum_{\ell=0}^{m} \binom{m}{\ell} y^{m-\ell} S_\ell(x - 1).$$

Taking the partial derivative with respect to x completes the proof of (8).

Now working first with the left hand side in the statement of the theorem, expanding $B_{m-j}(y)$ yields:

$$\sum_{j=0}^{m} \binom{m}{j} B_{m-j}(y) B_j(x) = \sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{m}{j} (-1)^k \binom{m-j}{k} B_k y^{m-j-k} B_j(x).$$

Parametrizing by $\ell = j + k$ and using the identity

$$\binom{m}{j}\binom{m-j}{\ell-j} = \binom{m}{\ell}\binom{\ell}{j},$$

we get:

$$\sum_{j=0}^{m}\binom{m}{j}B_{m-j}(y)B_j(x) = \sum_{\ell=0}^{m}\binom{m}{\ell}y^{m-\ell}\left(\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}B_{\ell-j}B_j(x)\right) \tag{9}$$

Note that since the only nonzero B_k with k odd is $B_1 = 1/2$, the factor of $(-1)^{\ell-j}$ in (9) only affects the $j = \ell - 1$ term. If we remove that factor from the sum, we can compensate by subtracting $\ell B_{\ell-1}(x)$:

$$\begin{aligned}\sum_{j=0}^{m}\binom{m}{j}B_{m-j}(y)B_j(x) &= \sum_{\ell=0}^{m}\binom{m}{\ell}y^{m-\ell}\left(-\ell B_{\ell-1}(x) + \sum_{j=0}^{\ell}\binom{\ell}{j}B_{\ell-j}B_j(x)\right)\\ &= \sum_{\ell=0}^{m}\binom{m}{\ell}y^{m-\ell}\left(\ell(x-1)B_{\ell-1}(x) - (\ell-1)B_\ell(x)\right) \qquad (10)\end{aligned}$$

where (10) is obtained by applying Theorem 5.1.

For each ℓ, (10) gives us an expression for the terms on the left hand side involving $y^{m-\ell}$. Working similarly with the right hand side, we are interested in the $y^{m-\ell}$ terms of

$$m(x+y-1)B_{m-1}(x+y) - (m-1)B_m(x+y).$$

That is,

- The $y^{(m-1)-(\ell-1)}$ terms of $m(x-1)B_{m-1}(x+y)$.
- The $y^{(m-1)-\ell)}$ terms of $mB_{m-1}(x+y)$.
- The $y^{m-\ell}$ terms of $-(m-1)B_m(x+y)$.

Applying (8) to all three of these summands, we obtain:

$$m(x-1)\binom{m-1}{\ell-1}B_{\ell-1}(x) + m\binom{m-1}{\ell}B_\ell(x) - (m-1)\binom{m}{\ell}B_\ell(x),$$

which, after some rearrangement, becomes

$$\binom{m}{\ell}\left(\ell(x-1)B_{\ell-1}(x) - (\ell-1)B_\ell(x)\right),$$

matching the $y^{m-\ell}$ terms from (10). $\square$

Acknowledgments

The author wishes to thank Robert Beals for help with the manuscript and Richard Beals for his reading of the initial draft and calling our attention to [6, Equation 24.14.2]. The author also thanks Oren Patashnik for a careful reading of the manuscript with much constructive feedback, Della Dumbaugh and the editors of the American Math Monthly for numerous comments, and Serban Raianu for explaining the connections to Bernoulli polynomials in Section 5.

References

[1] M. Beck, S. Robins, *Computing the continuous discretely. Integer-point enumeration in polyhedra*, Second edition, Undergraduate Texts in Mathematics, Springer, New York, 2015.

[2] J. Beery, Sums of powers of positive integers, *Convergence* (2010), Available online at the URL: `https://www.maa.org/press/periodicals/convergence/sums-of-powers-of-positive-integers`.

[3] J. Choi, Y.-H. Kim, A note on high order Bernoulli numbers and polynomials using differential equations, *Appl. Math. Comput.*, **249** (2014), 480–486. Available online at the URL: `https://www.sciencedirect.com/science/article/pii/S0096300314014490`.

[4] L. Euler, *Institutiones calculi differentialis*, Academiae Imperialis Scientiarum, St Petersburg, 1755, Available online at the URL: `https://www.e-rara.ch/zut/doi/10.3931/e-rara-3788`.

[5] K. Dilcher, Sums of products of Bernoulli numbers, *J. Number Theory*, **60** (1996), 23–41. Available online at the URL: `https://www.sciencedirect.com/science/article/pii/S0022314X96901109`.

[6] K. Dilcher, *Bernoulli and Euler Polynomials*, Chapter 24 in: F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark, *The NIST Handbook of Mathematical Functions*, 587–600, Cambridge Univ. Press, 2010. The companion is available online at the URL: `https://dlmf.nist.gov/24`

[7] C.H. Edwards, Jr., *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.

[8] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, second ed., Addison-Wesley, Reading, MA, 1994, Updated pages related to Bernoulli numbers available online at the URL: `https://www-cs-faculty.stanford.edu/~knuth/gkp34.pdf`.

[9] T.L. Kitagawa, The Origin of the Bernoulli Numbers: Mathematics in Basel and Edo in the Early Eighteenth Century, *Math. Intelligencer*, **44** (2022), 46–56. Available online at the URL: `https://link.springer.com/article/10.1007/s00283-021-10072-y`.

[10] D.E. Knuth, Concrete Mathematics and Bernoulli, available online at the URL: https://www-cs-faculty.stanford.edu/~knuth/news22.html

[11] P. Luschny, The Bernoulli Manifesto, available online at the URL: http://luschny.de/math/zeta/The-Bernoulli-Manifesto.html

[12] N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, Gauthier-Villars, 1923. Available online at the URL: https://books.google.com/books?id=ylltAAAAMAAJ.

[13] R. Sitaramachandrarao, B. Davis, Some identities involving the Riemann zeta function II, *Indian J. Pure Appl. Math.*, **17** (1986), 1175–1186. Available online at the URL: https://insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a50_1175.pdf.

[14] R.S. Underwood, An expression for the summation $\sum_{m=1}^{n} m^p$, *Amer. Math. Monthly*, **35** (1928), 424–428. Available online at the URL: http://www.jstor.org/stable/2299769.

[15] G.T. Williams, A new method of evaluating $\zeta(2n)$, *Amer. Math. Monthly*, **60** (1953), 19–25. Available online at the URL: http://www.jstor.org/stable/2306473.

Madeline Beals-Reid
Ward Melville High School
380 Old Town Road
Setauket-East Setauket, NY 11733
E-mail: madisonbealsreid@gmail.com

Received: November 2, 2022 **Accepted**: December 31, 2022
Communicated by Serban Raianu