Spider Web Graphs

C. Hobbs, K. Martino, and L. Scull*

Abstract - Recent results from Chih-Scull show that the $\times$-homotopy relation on finite graphs can be expressed via a sequence of "spider moves" which shift a single vertex at a time. In this paper we study a "spider web graph" which encodes exactly these spider moves between graph homomorphisms. We show how composition of graph homomorphisms relate to the spider web, study the components of spider webs for bipartite and tree graphs, and finish by giving an explicit description of the spider web for homomorphisms from a bipartite graph to a star graph.

Keywords : graph homomorphism; graph homotopy

Mathematics Subject Classification (2020) **:** 05C60; 05C76; 18A10

1 Introduction

This paper introduces and studies spider web graphs, which arise from looking at all graph homomorphisms from graph G to graph H. These are related to $\times$-homotopy of graphs as studied by [6, 10, 3, 4, 2]. The $\times$-homotopy relation on graph homomorphisms can be defined by looking at looped walks in the exponential graph. However, this exponential graph is a large and complicated object. The spider web graph is a subgraph of the exponential. The main motivation for its definition comes from a result of Chih-Scull [2] that shows that any $\times$-homotopy from a finite graph can be represented by a series of so-called spider moves which change the image of a single vertex at at time. The spider web graph encodes this relationship, with vertices defined by graph homomorphisms and edges defined by spider moves. Thus any $\times$-homotopy can be represented by a walk in the smaller spider web graph. This gives a potentially simpler way to study $\times$-homotopy of graphs. While spider web graphs arise naturally out of ideas in the literature, they have not themselves appeared widely. Spider web graphs for complete graphs are studied in [9], and some ideas in this paper (particularly on bipartite graphs) relate to ideas from [7], although that work did not look at spider webs directly.

In this paper, we define and study the properties of spider web graphs, beginning with how they interact with composition of graph homomorphisms. We show that composition may not preserve spider moves directly, but the composition of maps connected by spider moves does result in composite maps which are connected by a finite sequence of spider moves. We then examine the structure of spider web graphs for bipartite graphs, and

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show that each connected component of the spider web consists of graph homomorphisms which act the same on the bipartition. We use this to enumerate the number of connected components when our bipartite graph is a tree. We end by giving an explicit description of spider web graphs when the codomain graph is a star graph, connecting these to the well-known Hamming graphs.

Our paper is structured as follows. Section 2 is general background and Section 3 is devoted to the definition and examples of our main objects of study, the spider web graphs. Section 4 looks at composition of graph homomorphisms. Section 5 studies spider webs of bipartite graphs, Section 6 specializes to trees, and then Section 7 further specializes to the case when the codomain is a star graph.

2 Background: Graphs

In this section we give definitions and basic properties of graphs which we will be using throughout the paper. We use standard graph theory terminology as found in [11, 6].

Definition 2.1 *[6] A* ***graph*** *consists of a set $V(G)$ of vertices and a set $E(G)$ of edges connecting them, where each edge is given by an unordered set of two vertices.*

Example 2.2 Here are some examples of graphs.

C_3 S_4 K_5 T_6

In this work, we will assume that all graphs have a **finite** set of vertices (and therefore also edges). If there is an edge from v_1 to v_2, we say v_1 is **adjacent** to v_2 and we use notation $v_1 \sim v_2$. If $v_1 \sim v_1$ we call that edge a **loop**. We will assume that every vertex is adjacent to at least one other vertex (possibly itself), so there are no isolated vertices.

Definition 2.3 *A* ***walk*** *is a sequence of vertices $v_1, v_2, ...v_n$ such that $v_i \sim v_{i+1}$. A* ***cycle*** *is a walk in which $v_1 = v_n$ and all other vertices are unique.*

Definition 2.4 *A graph G is* ***connected*** *if for each pair of vertices $x, y \in V(G)$, there exists a walk from x to y in G. The* ***connected components*** *of a graph G are its maximal connected subgraphs. We write $G = H_1 + H_2 + ... + H_n$ where H_i are the connected components of G.*

Definition 2.5 *A graph G is* ***bipartite*** *if there is a way to partition the vertices $V(G) = G_1 \cup G_2$ such that if $x, y \in G_i$ then $x \nsim y$: vertices from the same subset cannot be adjacent. For any bipartite graph G, we will denote the bipartition as $V(G) = G_1 \cup G_2$ unless otherwise stated.*

An intuitive way to look at bipartite graphs is that the vertices of a bipartite graph can be colored in with two different colors, such that no two vertices of the same color share an edge.

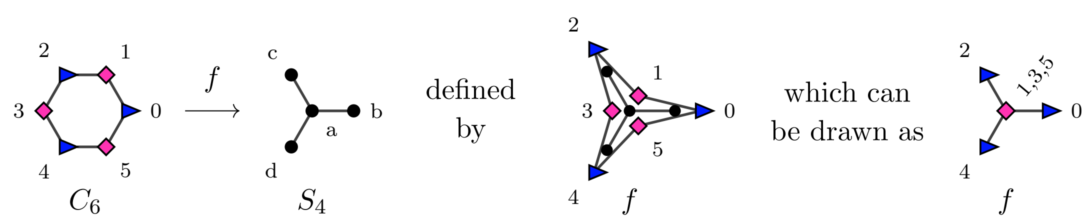
Example 2.6 In Example 2.2, the graphs S_4 and T_6 are bipartite. In each, the round vertices are one bipartition and the square or triangular vertices are the other. The graphs C_3 and K_5 are not bipartite.

Theorem 2.7 *[8, 11, 7] A graph G is bipartite if and only if it has no odd cycles. Moreover, if G is a connected bipartite graph and $v, w \in V(G)$ then there is an even walk from v to w if and only if v, w are in the same partition of G and there is an odd walk from v to w if and only if v, w are in opposite partitions of G. If G is connected and not bipartite, then for each pair of vertices $v, w \in G$ there is both an even walk and an odd walk from v to w.*

Our objects of interest, spider web graphs, are defined using graph homomorphisms.

Definition 2.8 *[6] Let G, H be graphs. A* ***graph homomorphism*** *$f : G \to H$ is a set map $f : V(G) \to V(H)$ such that if $v_1 \sim v_2 \in E(G)$ then $f(v_1) \sim f(v_2) \in E(H)$.*

Example 2.9 Let C_6, S_4 be the graphs shown below. Define $f : C_6 \to S_4$ by $f(0) = b, f(2) = c, f(4) = d$, and $f(1) = f(3) = f(5) = a$. Two representations of f are shown. The rightmost representation style will be used throughout this paper: the codomain graph is drawn, with each vertex labeled by the list of vertices from the domain which are mapped there.



Observation 2.10 A graph G is a bipartite graph if and only if there exists a graph homomorphism $f : G \to K_2$, where K_2 is the graph with two vertices, $\{0, 1\}$ connected by a single edge. The bipartition can be defined by the pre-images of the two vertices.

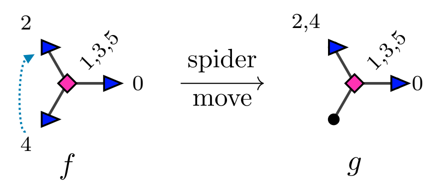
Definition 2.11 *The graphs G and H are* ***isomorphic****, denoted $G \cong H$, if there exists a graph homomorphism $f : V(G) \to V(H)$ which is bijective on vertices and satisfies $u \sim v \in E(G)$ if and only if $f(u) \sim f(v) \in E(H)$ (so f is also bijective on edges). The homomorphism f is called an* ***isomorphism*** *between G and H.*

3 Spider Webs

The object of study throughout this paper is the structure of the **spider web graph** $W(G, H)$ associated with graphs G, H. To define this graph, we begin with the following.

Definition 3.1 *[2] Let $f, g : G \to H$ be graph homomorphisms. We say f and g are a **spider pair** if there exists a single vertex x such that $f(x) \neq g(x)$, so that $f(y) = g(y)$ for all $y \neq x$. If $x \sim x$ we also require that $f(x) \sim g(x)$. When we replace f with g we refer to it as a **spider move**.*

Example 3.2 Recall C_6, S_4, and $f : C_6 \to S_4$ from Example 2.9. Define $g : C_6 \to S_4$ by $g(0) = b$, $g(2) = g(4) = c$, and $g(1) = g(3) = g(5) = a$ (shown below). The graph homomorphisms f, g are a spider pair, since they agree on all vertices except for 4. In the figure below, each graph homomorphism is represented by labelling each vertex in S_4 with names of the vertices in C_6 which are sent there.

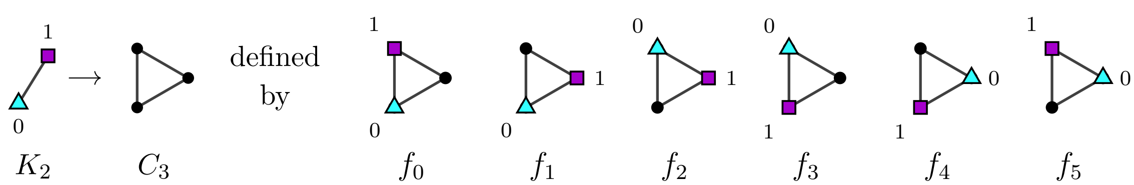


Definition 3.3 *The **spider web** $W(G, H)$ for graphs G, H is defined as follows:*

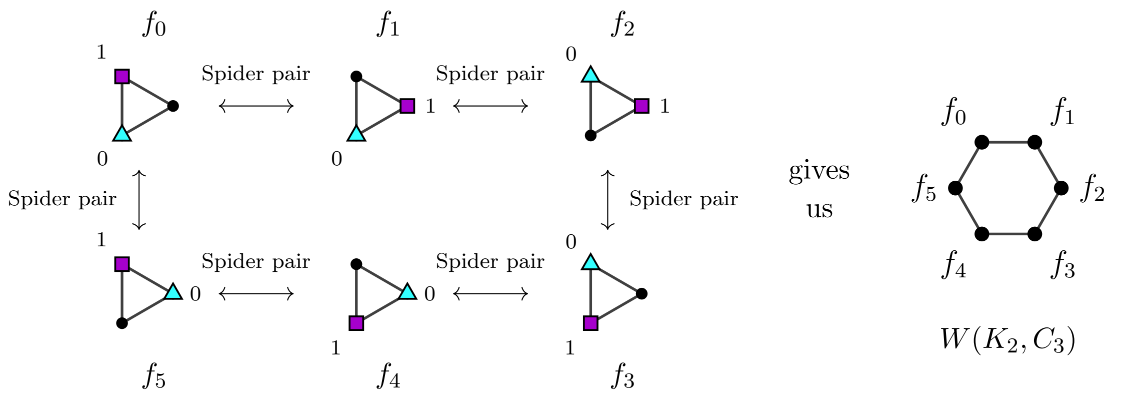
- *The vertex set of $W(G, H)$ is defined by the set of all graph homomorphisms $f : G \to H$*
- *there is an edge $f \sim g$ in $W(G, H)$ if f, g are a spider pair.*

Note that the resulting spider web graph $W(G, H)$ may contain isolated vertices, even if G, H do not.

Example 3.4 We create $W(K_2, C_3)$ with vertices representing graph homomorphisms from the complete graph K_2 to the cycle graph C_3. First, we enumerate the graph homomorphisms from K_2 to C_3:



Next, we identify which graph homomorphisms are spider pairs. Then we create a new graph where each vertex represents a graph homomorphism, and there is an edge between two vertices if and only if they are a spider pair. We see that $W(K_2, C_3)$ forms a 6-cycle.



Definition 3.5 *We define an equivalence relation on graph homomorphisms as follows: A walk in $W(G, H)$ defines a finite sequence of spider moves connecting f and g. If such a walk exists, then f, g are in the same connected component of $W(G, H)$ and we say that $f \simeq g$.*

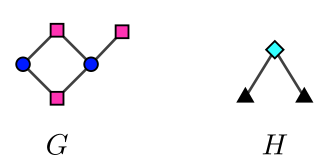
The remainder of this section illustrates the connection between our work and the broader topic of $\times$-homotopy of graphs, for readers familiar with that subject. Readers with no particular interest in $\times$-homotopy of graphs are free to skip to the next section.

Definition 3.6 *[3] For graphs G, H, the* ***exponential graph*** *H^G is defined by:*

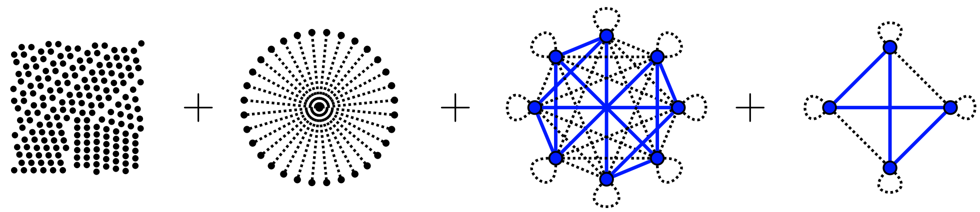
- *The vertex set of H^G is defined by all possible set maps $V(G) \to V(H)$ [not necessarily graph homomorphisms].*
- *There is an edge $f \sim g$ in H^G if $f(v_1) \sim g(v_2)$ in H whenever $v_1 \sim v_2$ in G*

Observation 3.7 A loop connecting f to itself exists in H^G exactly when $f(v_1) \sim f(v_2)$ in H whenever $v_1 \sim v_2$ in G. Therefore a vertex f in H^G represents a graph homomorphism if and only if f is looped.

Example 3.8 Consider the graphs G and H. We will revisit these same graphs and discuss their spider web graphs in example 5.5.



Below we draw the exponential graph H^G. The vertices which represent graph homomorphisms (and hence are a part of the spider web graph) are those with the loops, and the edges of H^G which are included in $W(G, H)$ are indicated by solid lines, while those not included are indicated by dotted lines.



198 isolated vertices $\quad S_{33} \quad$ looped K_8 $\quad$ looped K_4

Definition 3.9 *[3] Given graphs G, H and graph homomorphisms $f, g : G \to H$, we say that f, g are* $\times$***-homotopic*** *if there is walk from f to g in H^G such that each vertex in the walk is looped.*

It is shown in [2] that f, g are $\times$-homotopic if and only if there is a finite sequence of spider moved from f to g. Thus Definition 3.5 gives an alternate definition of the $\times$-homotopy relation. Instead of having to study the much larger exponential graph, we can instead focus on the subgraph given by the spider web graph, containing only the looped vertices and many fewer edges connecting them. Spider web graphs therefore provide a more efficient representation of the $\times$-homotopy relation. Because of this connection, some of the results in this paper follow from results about $\times$-homotopy. We will make note of where this happens, but include proofs based on spider moves to keep this paper self-contained.

4 Spider Moves and Composition

This section examines the relationship between spider moves and composition of graph homomorphisms. We show that if there is a finite sequence of spider moves between two graph homomorphisms f and g, then there will be a finite sequence of spider moves between their compositions with a third graph homomorphism. Results in this section follow indirectly from results about $\times$-homotopy in [2], without the explicit details about the relevant spider moves.

Theorem 4.1 *Suppose the graph homomorphisms $f, g : G \to K$ are a spider pair, and $\theta : K \to H$ is a graph homomorphism. Then either $\theta f = \theta g$ or $\theta f, \theta g$ are a spider pair.*

$$\begin{tikzcd}
G \arrow[rr, bend left, "\theta f"] \arrow[rr, "\theta g"] \arrow[dr, bend left, "f"] \arrow[dr, bend right, "g"'] & & H \\
& K \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Proof. Since $f, g : G \to K$ are a spider pair there is a unique $v \in V(G)$ such that $f(v) \neq g(v)$ and $f(w) = g(w)$ for all $w \neq v$. Then $\theta f(w) = \theta g(w)$ for all $w \neq v$. If $\theta f(v) = \theta g(v)$ then $\theta f = \theta g$. If $\theta f(v) \neq \theta g(v)$ then $\theta f, \theta g$ differ on a single vertex v. If $v \sim v$ then since f, g are a spider pair, $f(v) \sim g(v)$ so $\theta(f(v)) \sim \theta(g(v))$ since θ is a graph homomorphism. Hence, θf and θg are a spider pair. $\square$

Theorem 4.2 *Suppose $f : G \to K$ is a graph homomorphism and $\theta, \psi : K \to H$ are a spider pair. Then there is a sequence of n spider moves from θf to ψf where n is the size of the pre-image $|f^{-1}(x)|$ for the unique vertex x with $\theta(x) \neq \psi(x)$. Moreover, there is no shorter sequence of spider moves from θf to ψf.*

$$\begin{tikzcd} G \arrow[rr, bend left, "\theta f"] \arrow[rr, bend right, "\psi f"'] \arrow[dr, "f"'] & & H \\ & K \arrow[ur, bend left, "\theta"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

Proof. Denote $f^{-1}(x) = \{v_1, v_2, ...v_n\}$. Then for any other vertex $w \neq v_i$, we know that $\theta f(w) = \psi f(w)$. For each $0 \leq k \leq n$ define $h_k : V(G) \to V(H)$ by $h_k(w) = \theta f(w) = \psi f(w)$ if $w \notin f^{-1}(x)$ and:

$$h_k(v_i) = \begin{cases} \theta f(v_i) = \theta(x) & i \leq n-k \\ \psi f(v_i) = \psi(x) & i > n-k \end{cases}$$

Note that $h_0 = \theta f$ and $h_n = \psi f$. We claim that each h_k defines a graph homomorphism from G to H and that each h_k, h_{k+1} is a spider pair.

	v_1	v_2	$\cdots$	v_{n-2}	v_{n-1}	v_n
h_0	$\theta(x)$	$\theta(x)$	$\cdots$	$\theta(x)$	$\theta(x)$	$\theta(x)$
h_1	$\theta(x)$	$\theta(x)$	$\cdots$	$\theta(x)$	$\theta(x)$	$\psi(x)$
h_2	$\theta(x)$	$\theta(x)$	$\cdots$	$\theta(x)$	$\psi(x)$	$\psi(x)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
h_{n-1}	$\theta(x)$	$\psi(x)$	$\cdots$	$\psi(x)$	$\psi(x)$	$\psi(x)$
h_n	$\psi(x)$	$\psi(x)$	$\cdots$	$\psi(x)$	$\psi(x)$	$\psi(x)$

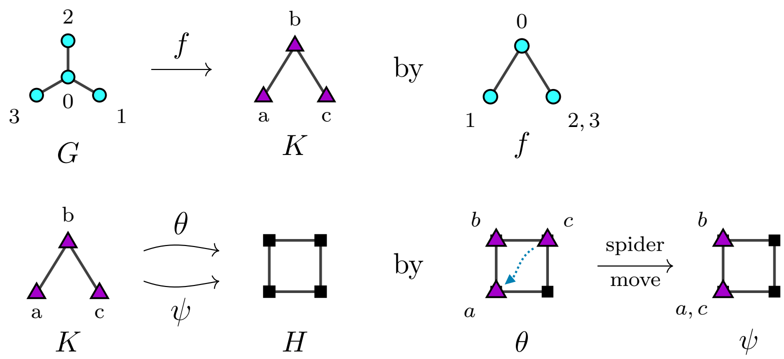
Table 1: Values of $h_k(v_i)$

If $w \sim v_i$ for $w \notin f^{-1}(x)$ and $v_i \in f^{-1}(x)$, then $h_k(v_i)$ is either $\theta f(v_i)$ or $\psi f(v_i)$, and $h_k(w)$ can be taken to be either $\theta f(w)$ or $\psi f(w)$ to match. Both θf and ψf are graph homomorphisms, so taking whichever applies we get $h_k(w) \sim h_k(v_i)$. Lastly, if $v_i \sim v_j$ for $v_i, v_j \in f^{-1}(x)$ then $f(v_i) = f(v_j) = x$ and so x must be looped: $x \sim x$. This means that both $\theta(x)$ and $\psi(x)$ must be looped, so if $h_k(v_i) = h_k(v_j)$ then $h_k(v_i) \sim h_k(v_j)$. On the other hand, if $h_k(v_i) \neq h_k(v_j)$ then one is defined by $\theta(x)$ and the other by $\psi(x)$. Since θ, ψ are a spider pair, we know that $\theta(x) \sim \psi(x)$ and hence $h_k(v_i) \sim h_k(v_j)$.

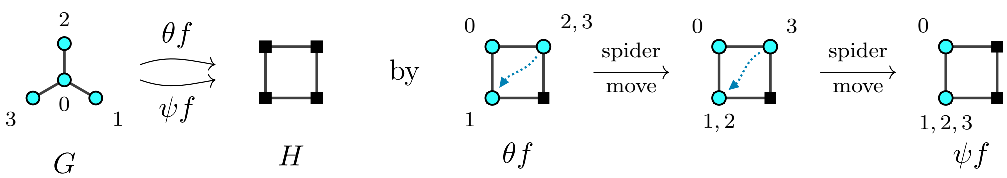
Next we show that each h_k, h_{k+1} are a spider pair. By construction, $h_k(v_{n-k}) = \theta f(v_{n-k}) \neq \psi f(v_{n-k}) = h_{k+1}(v_{n-k})$, and $h_k(w) = h_{k+1}(w)$ for all other vertices. If v_{n-k} is looped, then since f is a graph homomorphism, $f(v_{n-k})$ is also looped, and since θ, ψ are a spider pair, $h_k(v_{n-k}) = \theta(f(v_{n-k})) \sim \psi(f(v_{n-k})) = h_{k+1}(v_{n-k})$. Hence h_k, h_{k+1} are a spider pair, and $\theta f = h_0, h_1, ..., h_{n-1}, h_n = \psi f$ is a sequence of n spider moves from θf to ψf.

Lastly, because $\theta f, \psi f$ differ on the image of exactly n vertices, and each spider move changes the image of exactly one vertex, there cannot fewer than n spider moves from θf to ψf. $\square$

Example 4.3 Consider $f : G \to K$ and $\theta, \psi : K \to H$ as defined below: θ, ψ are a spider pair which disagree on the vertex c, and f maps two vertices to vertex c.



Here is a sequence of two spider moves from θf to ψf:



Corollary 4.4 *Let G, K, H be graphs with $f, g : G \to K$ and $\theta, \psi : K \to H$. If $f \simeq g$ and $\theta \simeq \psi$, then $\theta f \simeq \psi g$.*

Proof. Suppose $f \simeq g$ and $\theta \simeq \psi$. Then there exist $h_0, h_1, ..., h_n$ such that $f = h_0$, $g = h_n$, and each h_i, h_{i+1} are a spider pair, and $\rho_0, \rho_1, ..., \rho_m$ such that $\theta = \rho_0$, $\psi = \rho_m$, and each ρ_j, ρ_{j+1} are a spider pair. By Theorem 4.1, each $\theta h_i, \theta h_{i+1}$ are either equal or a spider pair, so $\theta f \simeq \theta g$. By Theorem 4.2, $\rho_j g \simeq \rho_{j+1} g$ so $\theta g \simeq \psi g$. Hence $\theta f \simeq \psi g$. $\square$

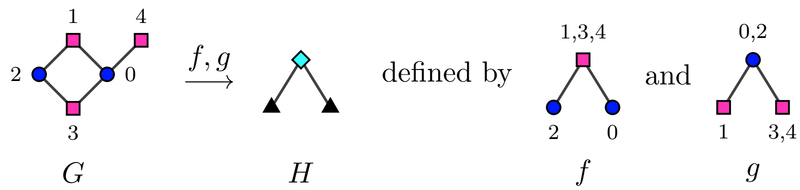
5 Spider Webs of Bipartite Graphs

In this section we will show that spider webs respect the partitions of bipartite graphs: when G, H are bipartite graphs then the spider web $W(G, H)$ consists of two sets of connected components, which represent the graph homomorphisms which take G_1 to H_1 (and hence G_2 to H_2), and the graph homomorphisms which take G_1 to H_2 (and G_2 to H_1). We start with the following result, which follows from [6] Proposition 1.7 and can also be found in [7]. We include a proof here for completeness.

Lemma 5.1 *If G, H are connected bipartite graphs, then all graph homomorphisms from G to H preserve bipartitions.*

Proof. Suppose that $f : G \to H$ maps a vertex in G_1 to a vertex in H_1. We will show that f maps all vertices in G_1 to vertices in H_1 and all vertices in G_2 to vertices in H_2. If there is a walk from v to w in G given by $v, x_1, x_2, ..., x_n, w$, then because graph homomorphisms preserve edges, $f(v), f(x_1), f(x_2), ..., f(x_n), f(w)$ gives a walk of the same length from $f(v)$ to $f(w)$ in H. Now suppose that $v \in G_1$ and let $f : G \to H$ such that $f(v) \in H_1$. If v, w are both in G_1, then there is an even walk between them and hence an even walk between $f(v)$ and $f(w)$, and so by Theorem 2.7, $f(w) \in H_1$. Similarly, if $w \in G_2$, then there is an odd walk from v to w in G, and consequently an odd walk from $f(v)$ to $f(w)$ in H, so by Theorem 2.7, $f(w) \in H_2$. □

Example 5.2 Consider the bipartite graphs G and H below, where the bipartitions are defined by the shape of the vertices. Any graph homomorphism from G to H will either map squares to diamonds and circles to triangles as in the graph homomorphism labeled f, or the other way around as in the graph homomorphism labeled g.



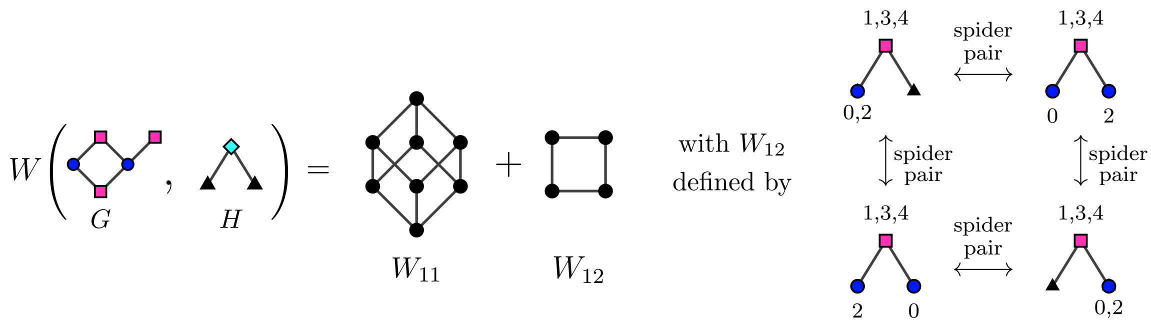
Theorem 5.3 *Spider web components respect bipartitions: If G, H are each connected bipartite graphs, then all vertices in the same connected component of $W(G, H)$ represent maps which have the same behavior on bipartitions.*

Proof. Let $f \simeq g$, so there is a finite sequence of spider moves from f to g, and suppose f maps G_1 to H_1. We proceed by induction on the number of spider moves between f and g. For the base case, if $f = g$ then g maps G_1 to H_1.

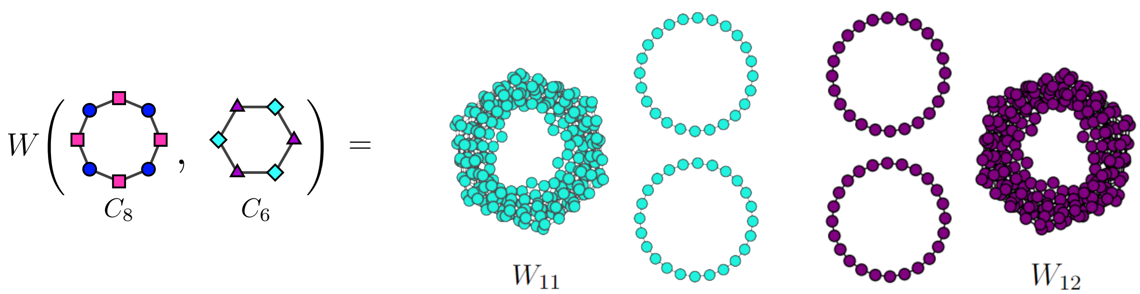
Assume h maps G_1 to H_1 whenever there is a sequence of fewer than n spider moves from f to h, and suppose there is a sequence of n spider moves from f to g. Let $h : G \to H$ be obtained by applying the first $n-1$ spider moves to f, so by the inductive hypothesis, h maps G_1 to H_1. Now g, h are a spider pair so there exists $v \in V(G)$ such that $g(v) = h(v)$. Hence by Lemma 5.1, g and h have the same behavior on bipartitions: g maps G_1 to H_1. □

Notation 5.4 Let G, H be connected bipartite graphs. Let $W_{11}(G, H)$ represent the subgraph on the vertices which map G_1 to H_1 (and G_2 to H_2), and let $W_{12}(G, H)$ denote the subgraph on the vertices which map G_1 to H_2. Since we have shown above that there can be no edges in W between W_{11} and W_{12}, we can say $W(G, H) = W_{11}(G, H) + W_{12}(G, H)$.

Example 5.5 Shown below are the graphs G, H from Example 5.2 and the spider web $W(G, H)$, where W_{11} maps G_1 to H_1 (squares to triangles) and W_{12} maps G_1 to H_2 (squares to diamonds). The four graph homomorphisms that make up W_{12} are shown in detail to the right.



Example 5.6 It is possible for W_{11} and W_{12} to consist of multiple components. The web $W(C_8, C_6)$ of bipartite graphs C_8 and C_6 is shown below. The three connected components on the left are W_{11}, which maps squares to diamonds. The three connected components on the right are W_{12}, which maps squares to triangles.



We conclude by showing that these subwebs of the web graph are non-empty, and so $W(G, H)$ has at least two components when G, H are bipartite.

Theorem 5.7 *Let G, H be connected bipartite graphs. Then $W_{11}(G, H)$ and $W_{12}(G, H)$ are each nonempty.*

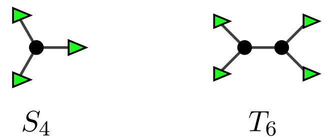
Proof. Let $v \in G_1$ and $w_1 \sim w_2 \in H$ (recall that we disallow the isolated vertex graph). Since H is bipartite, $w_1 \in H_1$ and $w_2 \in H_2$. Recall from Observation 2.10 that there exists $f : G \to K_2$ where G_1 is the preimage of $0 \in K_2$ and G_2 is the preimage of $1 \in K_2$, so $f(v) = 0$. Define $\theta : K_2 \to H$ by $\theta(0) = w_1$, $\theta(1) = w_2$. Then $\theta f(v) = w_1$ so $\theta f \in W_{11}$. Define $\psi : K_2 \to H$ by $\psi(0) = w_2$, $\psi(1) = w_1$. Then $\psi f(v) = w_2$ so $\psi f \in W_{12}$. $\square$

6 Spider Webs of Tree Graphs

In this section, we examine the number of components of $W(G, H)$ when either G or H is a tree graph. We give an explicit count of the number of components when G is a tree.

Definition 6.1 *[8] A* **tree** *is a connected bipartite graph with no cycles. All trees have at least two vertices which are only adjacent to a single other vertex. We call these vertices the* ***leaves*** *of the tree.*

Example 6.2 The S_4 and T_6 below are trees. In each, the triangle vertices are leaves, and the circular vertices are not leaves.



First, we examine the case where a bipartite graph is mapped into a tree. We will need the following lemma, which essentially shows that any tree can be "folded" through a series of spider moves to resemble K_2. Note that this is a spider move version of the known result that any tree is homotopy equivalent to K_2.

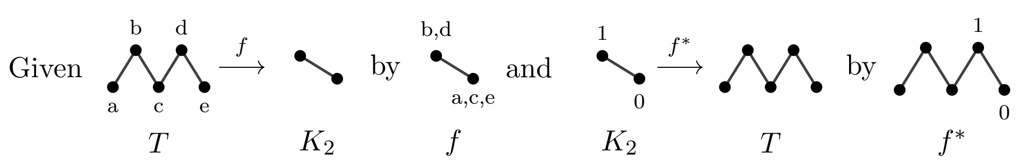
Lemma 6.3 *If T is a tree graph, then there exist graph homomorphisms $f : T \to K_2$, $f^* : K_2 \to T$ such that $f^*f \simeq id_T$.*

Proof. We proceed by induction on n, the number of vertices in the tree. For the base case, if $n = 2$ then $T = K_2$, so we define $f = f^* = id_{K_2} = id_T$.

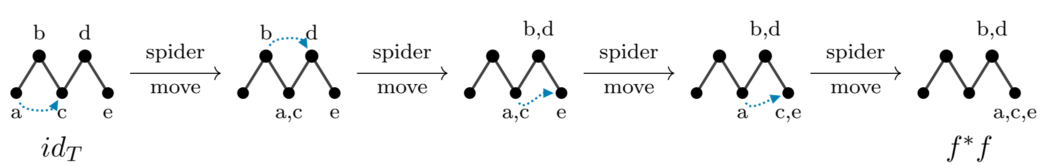
Now for $n \geq 2$ assume that if T_n is a tree with n vertices, then there exist $f : T_n \to K_2$, $f^* : K_2 \to T_n$ such that $f^*f \simeq id_{T_n}$. Let T_{n+1} be a tree with $n+1$ vertices. Let v_1 be a leaf of T_{n+1}, and let v_2 be the unique vertex adjacent to v_1. Then since T_{n+1} is connected and has at least 3 vertices, there exists a third vertex $v_3 \neq v_1$ such that $v_3 \sim v_2$. Now let $T_{n+1} - \{v_1\} = T_n$, a tree with n vertices, and define $\theta : T_{n+1} \to T_n$ by $\theta(v_1) = v_3$ and $\theta(v) = v$ if $v \neq v_1$. Observe that if we define ι to be the inclusion map $T_n \to T_{n+1}$, then the maps $\iota\theta$ and $id_{T_{n+1}}$ are a spider pair.

By the inductive hypothesis there exist $f : T_n \to K_2$, $f^* : K_2 \to T_n$ such that $f^*f \simeq id_{T_n}$. Then we consider the maps $g = f\theta : T_{n+1} \to K_2$ and $g^* = (\iota)f^* : K_2 \to T_{n+1}$. By Corollary 4.4, $g^*g = \iota f^* f\theta \simeq \iota id_{T_n}\theta = \iota\theta \simeq id_{T_{n+1}}$. $\square$

Example 6.4 An example of Lemma 6.3 is shown below.



Here is a sequence of spider moves from id_T to f^*f:



Theorem 6.5 *Let G be a connected bipartite graph and let T be a tree graph. Then $W(G, T)$ has two components, which are exactly W_{11} and W_{12}.*

Proof. We show that W_{11} is connected. Let $\theta, \psi \in W_{11}$. From Lemma 6.3 there exist $f : T \to K_2$, $f^* : K_2 \to T$ such that $f^*f \simeq id_T$. Graph homomorphisms preserve partitions, so f maps the partition T_1 to one vertex, say $0 \in K_2$ and the partition T_2 to the other vertex $1 \in K_2$. Since θ, ψ map G_1 to T_1, the maps $f\theta, f\psi$ must take G_1 to 0 and G_2 to 1, so $f\theta = f\psi$. Hence by Corollary 4.4, $\theta = id_T\theta \simeq f^*f\theta = f^*f\psi \simeq id_T\psi = \psi$.

$$\begin{tikzcd} G \arrow[rr, bend left, "\theta"] \arrow[rr, "\psi"] \arrow[dr, "f\theta = f\psi"'] & & T \arrow[dl, bend right, "f"'] \\ & K_2 \arrow[ur, bend right, "f^*"'] & \end{tikzcd}$$

Similarly, we can show that W_{12} is connected. $\square$

Example 6.6 Example 5.5 shows the spider web of a bipartite domain G and a tree codomain H, with W_{11} and W_{12} connected.

Next we want to consider the spider web $W(T, H)$,with T the domain instead of the codomain. We begin with the following lemma.

Lemma 6.7 *Let H be a graph and let $f, g : K_2 \to H$. If there is an even walk in H from $f(0)$ to $g(0)$ then $f \simeq g$.*

Proof. Suppose there is an even-length walk from $f(0)$ to $g(0)$. We proceed by induction on the length of this walk. For the base case, if $f(0) = g(0)$, then since K_2 has only two vertices either $f = g$ or f, g are a spider pair.

Assume $f \simeq g$ whenever there is a walk of length $2n$ from $f(0)$ to $g(0)$. Suppose there is a walk of length $2(n+1)$ from $f(0)$ to $g(0)$. Denote this walk $f(0)$,$v_1, ..., v_{2n}, v_{2n+1}$,$g(0)$. Define graph homomorphisms $h_1, h_2 : K_2 \to H$ by $h_1(0) = v_{2n}$, $h_1(1) = v_{2n+1}$, and $h_2(0) = g(0)$, $h_2(1) = v_{2n+1}$.

There is a walk of length $2n$ from $f(0)$ to $h_1(0)$, and so by the inductive hypothesis $f \simeq h_1$. By construction, h_1, h_2 are a spider pair, so $f \simeq h_2$. If $g(1) = v_{2n+1}$ then $g = h_2$. If $g(1) \neq v_{2n+1}$ then g, h_2 are a spider pair. In either case, $f \simeq g$. $\square$

We can now give a description of the connected components of $W(T, H)$.

Theorem 6.8 *Let H be a connected graph and let T be a tree. If H is not bipartite then $W(T, H)$ has one connected component, and if H is bipartite then $W(T, H)$ has two connected components, which are exactly W_{11} and W_{12}.*

Proof. Let $\theta, \psi \in W(T, H)$. Since T is a tree, from Lemma 6.3 we know that there exist $f : T \to K_2$, $f^* : K_2 \to T$ such that $f^*f \simeq id_T$. Consider $\theta f^*, \psi f^* : K_2 \to H$. If H is not bipartite, then Theorem 2.7 tells us there is an even walk in H from $\theta f^*(0)$ to $\psi f^*(0)$.

If H is bipartite, we will show that $\theta \simeq \psi$ whenever $\theta, \psi \in W_{11}$ or $\theta, \psi \in W_{12}$. Without loss of generality, suppose $\theta, \psi \in W_{11}$. Then $\theta(f^*(0)), \psi(f^*(0)) \in H_1$, so by Theorem 2.7 there is an even walk from $\theta f^*(0)$ to $\psi f^*(0)$. Then Lemma 6.7 shows that $\theta f^* \simeq \psi f^*$.

$$\begin{tikzcd} T \arrow[rr, bend left, "\theta"] \arrow[rr, "\psi"] \arrow[dr, bend left, "f"] & & H \\ & K_2 \arrow[ul, bend left, "f^*"] \arrow[ur, shift left] \arrow[ur, shift right, "\psi f^* \simeq \theta f^*"'] & \end{tikzcd}$$

Then by Corollary 4.4, $\theta = \theta id_T \simeq \theta f^* f \simeq \psi f^* f \simeq \psi id_T = \psi$. □

Example 6.9 Example 3.4 shows the spider web that results from mapping the tree domain K_2 to the non-bipartite codomain C_3, resulting in a single component.

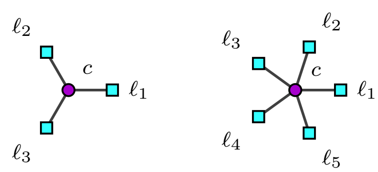
7 Spider Webs with Star Codomains

In this section, we study the structure of the spider web graphs when the codomain is a particular kind of tree called a star graph.

Definition 7.1 *[11] Given a natural number $n \geq 2$, a **star graph** S_n is a tree graph with one **center** vertex adjacent to $n-1$ other vertices, which we will call **leg vertices**.*

Any star graph is bipartite, and we will denote the partition consisting of the single center vertex as $\{c\}$, and the partition consisting of the leg vertices as $L = \{\ell_1, \ell_2 \cdots \ell_{n-1}\}$.

Example 7.2 The star graphs S_4 and S_6 are below



S_4 S_6

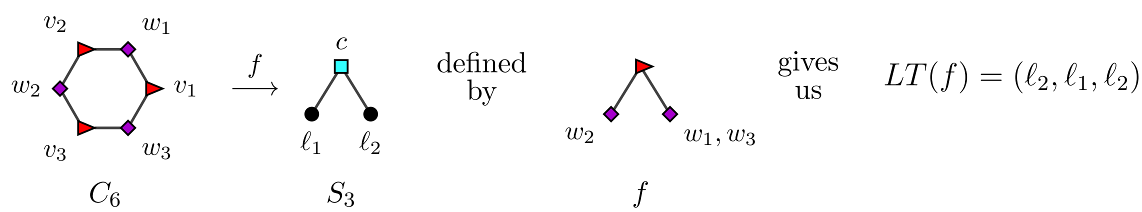
This leads to the following notation.

Notation 7.3 Given a bipartite graph G and a star graph S_n, since S_n is a tree graph, by Theorem 6.5 $W(G, S_n)$ will have two components, which are exactly $W_{11}(G, S_n)$ and $W_{12}(G, S_n)$. To simplify the notation, we will denote these two components by W_{1c} and W_{2c}: the first component contains the maps which take G_1 to the central vertex c and G_2 to the legs L, and the second contains the maps which takes G_2 to c and G_1 to L.

Definition 7.4 *For a star graph S_k, we define a **leg-tuple** LT to be an ordered sequence of vertices chosen from $L \subset V(S_k)$, with the form $LT = (\ell_{n_1}, \ell_{n_2}, \cdots, \ell_{n_i})$.*

Definition 7.5 *For each graph homomorphism f in $W_{1c} \subset W(G, S_k)$, there are $|G_2|$ vertices that get mapped to L. Let the vertices of G_2 be labeled as $G_2 = \{w_1, w_2, \cdots, w_{|G_2|}\}$. We define the **leg-tuple of f**, denoted $LT(f)$, by $LT(f) = (f(w_1), f(w_2), \cdots, f(w_{|G_2|}))$. Similarly, for $g \in W_{2c}$ we obtain a leg-tuple of length $|G_1|$.*

Example 7.6 Let $G = C_6$ which has bipartitions $V(C_6) = G_1 \cup G_2 = \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\}$, and let S_3 be the star graph with center c and leg vertices $L = \{\ell_1, \ell_2\}$. Define $f : C_6 \to S_3$ to be the map that sends $v_i \to c$ and $w_1 \to \ell_2$, $w_2 \to \ell_1$, and $w_3 \to \ell_2$. Then f will have leg-tuple $LT(f) = (\ell_2, \ell_1, \ell_2)$.



Lemma 7.7 *The set $V(W_{2c})$ is in one-to-one correspondence with the set of leg-tuples of length $|G_1|$ and entries in L.*

Proof. Every vertex of W_{2c} corresponds to a graph homomorphism from G to S_k which send $G_2 \to c$ and $G_1 \to L$, and so defines a leg-tuple $LT(f) = (f(v_1), f(v_2), \cdots, f(v_{|G_1|}))$ where each v is in G_1. We show that LT is a bijection. We begin with injectivity. If two maps f, g define the same leg-tuple, then we know that $f(v_i) = g(v_i)$ for all vertices $v_i \in G_1$, and we also know that $f(w_i) = g(w_i) = c$ for all $w_i \in G_2$ since $f, g \in W_{2c}$. Thus $f = g$.

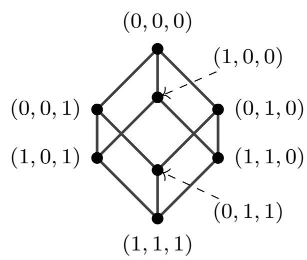
To see that LT is surjective, let $(\ell_{n_1}, \ell_{n_2}, \cdots, \ell_{n_{|G_1|}})$ be a given sequence of leg vertices and define the set map $h : V(G) \to V(S_k)$, shown below to be a graph homomorphism, by $h(w_i) = c$ for all $w_i \in G_2$ and $h(v_i) = \ell_{n_i}$ for all $v_i \in G_1$. Then the leg tuple of this homomorphism is $LT(h) = (\ell_{n_1}, \ell_{n_2}, \cdots, \ell_{n_{|G_1|}})$. We check that h defines a graph homomorphism: if we have two adjacent vertices $x_1 \sim x_2 \in G$, they must be in different partitions, and so $x_1 = v_i$ and $x_2 = w_j$ for $v_i \in G_1$ and $w_j \in G_2$. Then by definition, $h(w_i) = c$ and $h(v_j) = \ell_{n_j}$ for some leg vertex ℓ_{n_j}. Because S_k is a star graph, we know that $\ell_{n_j} \sim c$ and hence $h(x_1) \sim h(x_2)$.

□

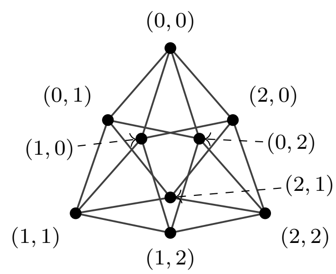
Similarly, the set $V(W_{1c})$ has a one-to-one correspondence with the set of all possible leg-tuples of length $|G_2|$. Now that we have linked $W(G, T)$ to leg-tuples, we show that in fact it has structure given by a well-known graph of tuples called the Hamming Graph.

Definition 7.8 *[1] Given $d, q \in \mathbb{N}$ and a finite q-element set S, the **Hamming graph** $H(d, q)$ is the graph whose vertex set is the set of d-tuples with entries from S, where vertices are adjacent if their tuples differ in a single entry.*

Example 7.9 Shown below are the Hamming graphs $H(3, 2)$ where $S = \{0, 1\}$ and the Hamming graph $H(2, 3)$ where $S = \{0, 1, 2\}$.



$H(3, 2)$



$H(2, 3)$

Lemma 7.10 *If G is bipartite and S_k is a star graph, $W_{2c}(G, S_k) \cong H(|G_1|, k-1)$, where the Hamming Graph's corresponding set of size $k-1$ is $L \subset S_k$.*

Proof. We have already shown in Lemma 7.7 that the vertices of W_{2c} correspond to leg-tuples, which in turn correspond to vertices of the Hamming graph $H(|G_1|, k-1)$. Thus, what remains to be shown is that this bijection from $V(W_{2c}(G, S_k))$ to $V(H(|G_1|, k-1))$ is actually a graph isomorphism: Two vertices in W_{2c} are adjacent when there is a spider move between their maps $f, g : G \to S_k$. This happens exactly when the leg-tuples of f and g differ by a single entry, which in turn happens exactly when the vertices in the Hamming graph are connected.

$\square$

An analogous argument shows that $W_{1c} \cong H(|G_2|, k-1)$. For the rest of this paper, we assume that a Hamming Graph $H(d, q)$ always corresponds to a set L of leg vertices from a star graph S_k. With this we have our final result:

Theorem 7.11 *Let G be a connected bipartite graph. Then for $k \geq 2$,*

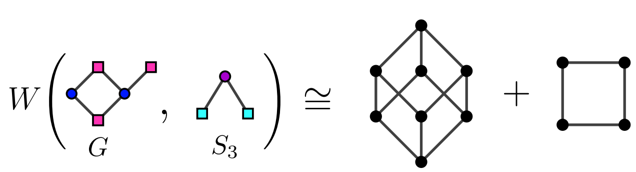
$$W(G, S_k) \cong H(|G_1|, k-1) + H(|G_2|, k-1).$$

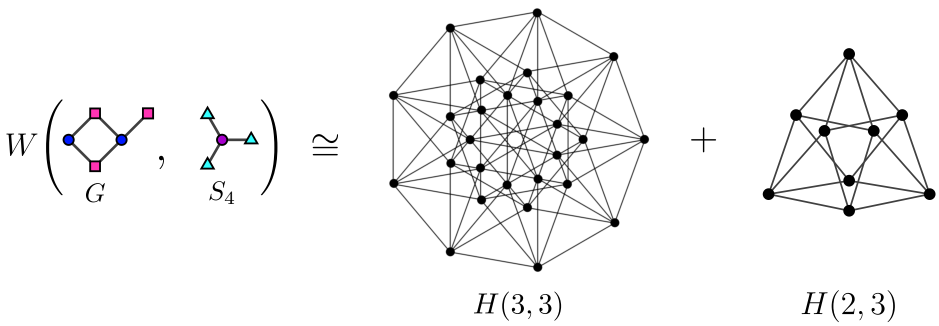
Proof. By Theorem 6.5 we know that $W(G, S_k)$ will have exactly two components, W_{1c} and W_{2c}, and Lemma 7.10 shows that

$$W(G, S_k) \cong H(|G_1|, k-1) + H(|G_2|, k-1).$$

$\square$

Example 7.12 Let G be the domain graph shown below. The webs $W(G, S_3)$ and $W(G, S_4)$ both have two connected components, each corresponding to a Hamming graph.

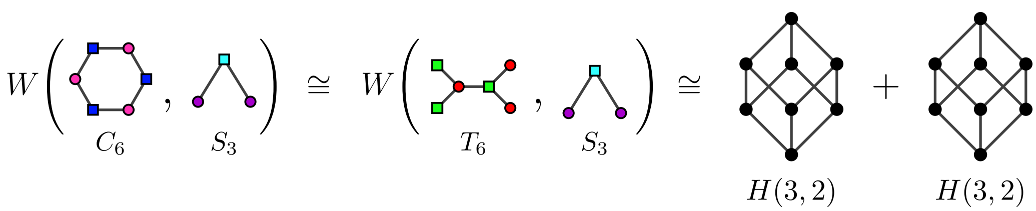




One somewhat surprising result of this theorem is that $W(G, S_k)$ depends only on the size of the partitions of G, and not on any additional structure within the graph.

Corollary 7.13 *If domain graphs G, H are connected bipartite graphs with $|G_1| = |H_1|$ and $|G_2| = |H_2|$, then $W(G, S_k) \cong W(H, S_k)$.*

Example 7.14 The connected bipartite graphs C_6 and T_6 shown below have identical partition sizes: The "squares" partition of C_6 is the same size as the "squares" partition of T_6, and the "circles" partitions of each are likewise identical in size. Therefore, when each are mapped to a star such as S_3, the resulting webs are isomorphic.



We can also consider the case when both domain and codomain are star graphs. In this case, the size of one bipartition of G is just 1 and we have a component of the spider web given by 1-tuples, or single entries. In this case, the Hamming graph gives the following well-known graph:

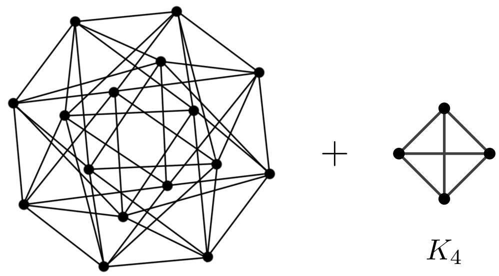
Proposition 7.15 *$H(1, q) \cong K_q$ where K_q is the complete graph with q vertices in which every pair of distinct vertices are adjacent.*

Proof. By definition, $H(1, q)$ has q vertices corresponding to 1-tuples in a size q set. Since each 1-tuple only has a single entry, every pair of distinct vertices will automatically differ by their single entry and thus have an edge between them. $\square$

Corollary 7.16 $W(S_n, S_k) \cong H(n-1, k-1) + K_{k-1}$

Proof. By Theorem 7.11, we have $W(S_n, S_k) \cong H(n-1, k-1) + H(1, k-1)$ and by Proposition 7.15 this can be further simplified to $W(S_n, S_k) \cong H(n-1, k-1) + K_{k-1}$. $\square$

Example 7.17 Let S_3 be the domain and S_5 be the codomain. Then $W(S_3, S_5) \cong H(2,4) + K_4$.



K_4

$H(2,4)$

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