

Limiting Spectral Distributions of Families of Block Matrix Ensembles

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Abstract - We introduce a new matrix operation on a pair of matrices, $\text{swirl}(A, X)$, and discuss its implications on the limiting spectral distribution. In a special case, the resultant ensemble converges almost surely to the Rayleigh distribution. In proving this, we provide a novel combinatorial proof that the random matrix ensemble of circulant Hankel matrices converges almost surely to the Rayleigh distribution, using the method of moments.

Keywords : random matrix theory; block matrices; Rayleigh distribution; method of moments; Hankel matrices

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1 Introduction

Random matrix theory was used by Eugene Wigner as a mechanism for modeling the limiting behavior of the energy distribution of heavy nuclei. The states of individual heavy nuclei are difficult to determine using the Schrödinger Equation, so instead one can examine the eigenvalues of random matrices and thereby obtain information about the statistical behavior of the system, as done in [12].

The techniques from nuclear physics were later abstracted to ensembles of random matrices. The motivation for choice of ensemble corresponded to the properties of physical systems. For example, this was the motivation for studying ensembles of real symmetric matrices, self-adjoint matrices, and Gaussian Orthogonal Ensembles. Given the importance of studying eigenvalues to both physics (as in [11, 26]) and to other fields of mathematics such as analytic number theory (as in [15, 16]), the eigenvalue distribution of the ensemble is the focus of study.

In general, it is rare to find a named, closed form limiting distribution of the eigenvalue distributions for a given ensemble of random matrices. For example, in the ensemble of Toeplitz matrices studied in [9] and [13], the distribution seemed to be approaching a Gaussian distribution, but there were Diophantine obstructions with the index combinatorics of the random variable entries in the matrices. These obstructions prevented the

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distribution from being a Gaussian distribution, and a closed form is still not known. Following these difficulties, an attempt to overcome the obstructions and increase symmetry was done by adding palindromicity; this is sufficient to guarantee almost sure convergence to the Gaussian distribution [18]. Many other related ensembles have been thoroughly investigated, for example in [2, 3, 7, 14, 17, 20].

In this paper, we formulate a new matrix operation, “swirl,” based on the symmetry of the concentric even matrix ensemble. An example of a matrix in this ensemble is

$$\begin{pmatrix} x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\ x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\ x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\ x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\ x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\ x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \end{pmatrix}.$$

Notably, the x_i are variables drawn independently from a probability distribution with mean zero, variance one, and finite higher moments.

We chose this ensemble with the hope that, by increasing symmetry, we would be able to obtain a closed form for the limiting spectral distribution.

It is advantageous to understand such matrices in block matrix form, as evidenced by [3]. In this vein, we split the matrices in the concentric even ensemble into blocks or quadrants and defined the swirl operation using two $N \times N$ input matrices, A and X , to create the larger block matrix of size $2N \times 2N$ corresponding to the concentric even matrix, where A is the upper right quadrant and X is the exchange matrix. That is,

$$\text{swirl}(A, X) = \begin{pmatrix} AX & A \\ XAX & XA \end{pmatrix}. \tag{1}$$

In concentric even matrices, A is a circulant Toeplitz matrix and AX is a circulant Hankel matrix. We reduce studying the concentric even ensemble to studying circulant Hankel matrices with several theorems about the behavior of $\text{tr}(\text{swirl}(A, X))$ in Section 3. Hankel matrices arise in a multitude of applications across fields of mathematics and physics: differential equations, functional analysis, statistics, probability theory, control theory, and more (see [4, 21, 22], for example). Their symmetry also makes them a heavily studied family in random matrix theory, as in [4, 8]. Circulant Hankel matrices also happen to be even centrosymmetric matrices, which have additional specialized applications in physics, for example in [10].

In Section 4 we characterize the summands of the trace of these matrices in terms of the number of repeated entries and compute the moments via combinatorial degree of freedom arguments. By these methods, we obtain a novel combinatorial proof showing that the limiting spectral distribution of the random matrix ensemble of circulant Hankel matrices converges almost surely to the symmetrized Rayleigh distribution (for an earlier proof relying on direct computation, see [9]). As we discuss in Appendix C, our methods



are generally applicable to many random matrix ensembles. In particular, we have a new proof of the following theorem.

Theorem 1.1 (Bryc-Dembo-Jiang [9]) *Let $\mu_{A,N}(x)$ be the empirical spectral measure of the $N \times N$ circulant Hankel random matrix ensemble populated by entries from a sequence of random variables A from a distribution p with mean 0, variance 1, and finite higher moments. Then,*

$$\lim_{N \rightarrow \infty} \mu_{A,N}(x) \rightarrow |x|e^{-x^2} \quad (2)$$

almost surely.

Notably, $|x|e^{-x^2}$ is the symmetrized Rayleigh distribution, with many known applications to physics (see [23]).

As a consequence of the results of Section 3, we also characterize the limiting spectral distribution of a particular swirl ensemble.

Corollary 1.2 *Let $G_{2N} = \text{swirl}(A, J)$ for J the $N \times N$ exchange matrix and A a random $N \times N$ circulant Toeplitz matrix. As $N \rightarrow \infty$, the limiting spectral measure of this ensemble converges almost surely to a symmetrized Rayleigh distribution.*

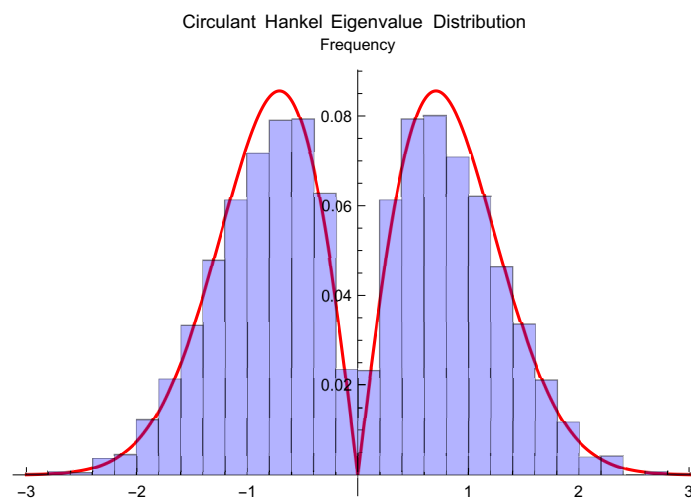


Figure 1: Histogram of eigenvalues for one hundred 40×40 random circulant Hankel matrices. A symmetrized Rayleigh distribution is shown in red.

Many block random matrix ensembles have been investigated in the past (for example, [17]). Some of these have even yielded remarkably similar limiting empirical spectral distributions (see Figure 3 of [20]).

The swirl operation is very rich and lends itself to much further study. In particular, a natural next step is to study matrix ensembles determined by different choices of A and X . We discuss some natural next steps in Section 5.



2 Preliminaries

We characterize the distribution of the eigenvalues of several random matrix ensembles by defining a spectral measure over subfamilies of random matrices from the ensemble. Let A be an element of a family of $N \times N$ random matrices from some ensemble where the entries are drawn from a probability distribution p with mean 0, variance 1, and finite higher moments.

The moment-generating function of a real-valued random variable uniquely specifies its probability distribution. By requiring the probability distributions have finite higher moments, this necessarily implies that there are finite lower moments, and additionally that this moment-generating function exists. We derive the associated ensemble's limiting spectral distribution from these moments.

This is the central idea of the method of moments. We compute the expected value of the moments for random matrices for a fixed size, N , and then take the limit of this as N goes to infinity. This limit yields the moments of the spectral distribution of the given ensemble for infinite dimensions (the limiting spectral distribution). We refer to [13] for additional details on our particular application of the method of moments.

We use the Eigenvalue Trace Lemma to relate the eigenvalues of the random matrices to the matrix elements.

Lemma 2.1 (Eigenvalue Trace Lemma) *Let $\lambda_i(A)$ be the eigenvalues of an $N \times N$ matrix A . Then*

$$\sum_{i=1}^N \lambda_i^k(A) = \text{tr}(A^k). \quad (3)$$

Let c be the number of unique rows in A . For example, if a particular row appears twice in the matrix, the second instance of that row does not contribute to c . A trivially has at most c many nonzero eigenvalues, and we adjust the spectral measure accordingly. Note that c is fixed for a given N and matrix structure.

Then, we define the empirical spectral measure of A_N as the following measure.

Definition 2.2 *Let p be a probability density function with mean 0, variance 1, and finite higher moments. Let A_N be an $N \times N$ random matrix from an ensemble \mathcal{A} with entries drawn independently from p . Then the empirical spectral measure of A_N is defined as*

$$\mu_{A_N}(x)dx := \frac{1}{c} \sum_{i=1}^c \delta \left(x - \frac{\sqrt{c}\lambda_i(A_N)}{N} \right) dx, \quad (4)$$

where $\delta(x)$ is the Dirac-delta functional, the $\lambda_i(A_N)$ are the nonzero eigenvalues of A_N , and c is as defined above.

Remark 2.3 The \sqrt{c}/N scaling factor is derived heuristically from the Central Limit Theorem. By computing the trace of A^2 via the Eigenvalue Trace Lemma, we get

$$\mathbb{E}[\text{tr}(A^2)] = N^2 = \sum_{i=1}^N \mathbb{E}[\lambda_i(A)^2], \quad (5)$$



suggesting that the magnitude of the eigenvalues must be roughly N/\sqrt{c} each in expectation since the expectation of an entry squared is 1, by our definition of p .

Via the method of moments, we will be able to understand the spectral distribution of these eigenvalues. In this instance, the convergence of the moments of the spectral distribution is enough to show the almost sure convergence of the spectral distribution (by the methods of [13]). From the definition of the spectral measure $\mu_{A,N}(x)$ in terms of the Dirac-delta functional, we may compute its moments.

Remark 2.4 The moments of the spectral measure of A are

$$M_k(A, N) := \int_{-\infty}^{\infty} x^k \mu_{A,N}(x) dx = \frac{c^{k/2-1}}{N^k} \sum_{i=1}^c \lambda_i^k(A). \quad (6)$$

Notice that by the Eigenvalue Trace Lemma, $M_k(A, N) = \frac{c^{k/2-1}}{N^k} \text{tr}(A^k)$.

Finally, we are interested in averaging these moments over the entire family of matrices that A belongs to. As is standard, we define the following.

Definition 2.5 Let $M_k(N)$ be the average of $M_k(A, N)$ over all A in our chosen family of matrices.

Our main result is that $\lim_{N \rightarrow \infty} M_k(N)$ exists and that there is a universal limiting distribution for several families of matrices.

In the following, we will calculate M_k as a sum of terms (via the Eigenvalue Trace Lemma). We will show that some terms are negligible by showing that they are $O_k(1)$ (where $f(n) = O_k(g(n))$ if, for $k \in \mathbb{Z}^+$ fixed, there exists n_0 and c such that for all $n > n_0$, $f(n) \leq cg(n)$).

In this paper, we investigate swirl ensembles and circulant Hankel matrices.

Definition 2.6 An $N \times N$ circulant Hankel matrix $H_N = (a_{ij})$ is defined by $a_{ij} = a_{k\ell} \iff i + j \equiv k + \ell \pmod{N}$.

$$H_N = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-3} & b_{N-2} & b_{N-1} \\ b_1 & b_2 & b_3 & \cdots & b_{N-2} & b_{N-1} & b_0 \\ b_2 & b_3 & b_4 & \cdots & b_{N-1} & b_0 & b_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{N-3} & b_{N-2} & b_{N-1} & \cdots & b_{N-6} & b_{N-5} & b_{N-4} \\ b_{N-2} & b_{N-1} & b_0 & \cdots & b_{N-5} & b_{N-4} & b_{N-3} \\ b_{N-1} & b_0 & b_1 & \cdots & b_{N-4} & b_{N-3} & b_{N-2} \end{pmatrix}, a_{ij} = b_{[i+j]_N}.$$

Note that circulant Hankel matrices are the product of circulant Toeplitz matrices and exchange matrices, with the former considered in [9, 13]. In [6], circulant Hankel matrices are referred to as reverse circulant matrices.

The $n \times n$ exchange matrix is the matrix with 1's along the antidiagonal and 0's elsewhere. For example, the 4×4 exchange matrix is as follows:



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In general, Toeplitz matrices are diagonal constant matrices. Circulant Toeplitz matrices are Toeplitz matrices which additionally satisfy $a_{ij} = a_{k\ell} \iff i - j \equiv k - \ell \pmod{N}$. That is, the diagonals wrap around around the edges of the matrix. Below is an example of a 4×4 circulant Toeplitz matrix.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

3 Swirl Matrices

3.1 Motivation

For A an $N \times N$ circulant Toeplitz matrix, and J the $N \times N$ exchange matrix with 1's on the antidiagonal and zeroes elsewhere, the $2N \times 2N$ concentric even matrix is given by the following:

$$\begin{pmatrix} AJ & A \\ JAJ & JA \end{pmatrix}.$$

The swirl operation was inspired by radially symmetric matrices of the following form:

$$\begin{pmatrix} x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\ x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\ x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\ x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\ x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\ x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \end{pmatrix}.$$

We refer to such matrices as “concentric even matrices.” Note that not only are the circles about the center of the matrix composed of equal entries, but also these entries are repeated in later circles such that each matrix entry appears an equal number of times. This was intentional in an effort to increase symmetry and derive a closed form limiting spectral distribution. For a $2N \times 2N$ matrix of this form, each entry appears exactly $4N$ times (N times in each $N \times N$ quadrant). Upon close inspection, it is apparent that the $N \times N$ submatrix in the top right of a $2N \times 2N$ concentric even matrix is an $N \times N$ circulant Toeplitz matrix (which is not necessarily symmetric). Moreover, the other three quadrants of the matrix may be generated from this circulant Toeplitz matrix via a clockwise rotation of the entries.



This block decomposition of the concentric even matrices motivates the following definition and the focus of this section.

Definition 3.1 Let A and X be $N \times N$ matrices. We define $\text{swirl}(A, X)$ as the $2N \times 2N$ matrix where

$$\text{swirl}(A, X) = \begin{pmatrix} AX & A \\ XAX & XA \end{pmatrix}. \quad (7)$$

We aim to characterize the limiting spectral distribution of $\text{swirl}(A, X)$ for A random and X fixed. To do so, we relate $\text{tr}(\text{swirl}(A, X)^k)$ to $\text{tr}((AX)^k)$ via the Eigenvalue Trace Lemma.

Lemma 3.2 Observe that

$$\text{swirl}(A, X) = \begin{pmatrix} AX & A \\ XAX & XA \end{pmatrix} = \begin{pmatrix} AX & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ XAX & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix}. \quad (8)$$

This observation vastly simplifies the computation of $\text{tr}((\text{swirl}(A, X))^k)$.

Notation 3.3 We adopt a convenient shorthand notation for block matrices with four $N \times N$ blocks which are 0 in 3 blocks. For example, a $2N \times 2N$ matrix of the form with zeroes necessarily everywhere except the top right corner will be referred to as a matrix B_{12} . That is, B_{12} is of the form

$$\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$$

for Y an $N \times N$ matrix. Define B_{11} , B_{21} , and B_{22} similarly with the indices corresponding to the block that is not necessarily zero everywhere.

Lemma 3.4 $B_{ij}B_{kl} = 0$ if $j \neq k$ and is of the form B_{il} otherwise.

3.2 Computing $\text{tr}((\text{swirl}(A, X))^k)$

Recall the following facts:

$$\text{tr}(CD) = \text{tr}(DC) \quad (9)$$

and

$$\text{tr}(C + D) = \text{tr}(C) + \text{tr}(D) \quad (10)$$

for $N \times N$ matrices C and D . We are now ready to relate $\text{tr}((\text{swirl}(A, X))^k)$ to $\text{tr}((AX)^k)$.

Theorem 3.5 For A and X both $N \times N$ matrices, $\text{tr}((\text{swirl}(A, X))^k) = 2^k \text{tr}((AX)^k)$.

Proof. Any term in the expansion of

$$\begin{aligned} (\text{swirl}(A, X))^k &= \left(\begin{pmatrix} AX & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ XAX & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix} \right)^k \\ &= \sum B_{i_1 j_1} B_{i_2 j_2} \cdots B_{i_k j_k} \end{aligned}$$



is of the form $B_{i_1 j_k}$. Since trace is additive, we have that a term contributes 0 to the trace of $(\text{swirl}(A, X))^k$ if $i_1 \neq j_k$; if they are not equal, the main diagonal of the matrix is all zeroes.

As such, by Lemma 3.4 and the above, the nonzero summands of $(\text{swirl}(A, X))^k$ correspond to products $B_{i_1 j_1} B_{i_2 j_2} \cdots B_{i_k j_k}$ where $j_\ell = i_{\ell+1}$ for $1 \leq \ell \leq k-1$ and $j_k = i_1$. There are 2^k such summands since one can choose the first indices of the k matrices in the summand in 2^k ways. Then, the second indices are exactly determined by the above requirements.

Observe that the only nonzero block of a $B_{i_1 j_k}$ matrix in the trace expansion of $\text{tr}((\text{swirl}(A, X))^k)$ is a product of matrices. By the construction of swirl, this product begins with A if $i_1 = 1$ and ends with A if $j_k = 2$. Also observe that this product begins with an X if $i_1 = 2$, and ends with X if $j_k = 1$. All such products will start with one of A or X and end with the opposite. These products will also not have consecutive repeated A 's or X 's. These properties follow from Lemma 3.4 and the definition of swirl.

In order for this product of matrices to contribute to the trace, note that the first and last index of such a product must be equal (or else it will not be a diagonal entry). Thus, there must be an equal number of matrices of the form B_{12} and B_{21} in any contributing product. As such, each nonzero summand in the expansion of $\text{tr}((\text{swirl}(A, X))^k)$ (with $\text{swirl}(A, X)$ expressed as $B_{11} + B_{12} + B_{21} + B_{22}$) will be of the form $\text{tr}((XA)^k)$ or $\text{tr}((AX)^k)$. Consequently, there are 2^k such nonzero contributing terms and

$$\text{tr}((\text{swirl}(A, X))^k) = 2^k \text{tr}((AX)^k), \quad (11)$$

by the cyclic property of trace. □

Given that the trace of the k^{th} power of a matrix completely determines the k^{th} moment of its empirical spectral distribution, Theorem 3.5 allows us to reduce characterizing the limiting spectral distribution of $\text{swirl}(A, X)$ ensembles to characterizing the limiting spectral distribution of AX matrices.

In Appendix A and Appendix B, we discuss additional avenues for working with swirl, such as when A is not symmetric or iterating swirl.

4 Circulant Hankel Matrices

In all the ensembles that follow, we assume that the matrices are constructed from a sequence of independently and identically distributed random variables (i.i.d.r.v.) with distribution p having mean 0, variance 1, and finite higher moments. Elements of this sequence correspond to matrix entries via the symmetry of our given ensemble.

4.1 Moments via powers of AX

From Theorem 3.5, studying the trace of the even concentric swirl matrices reduces to studying the trace of powers of $H_N = A_N J_N$, with H_N the $N \times N$ circulant Hankel matrix, A_N the $N \times N$ circulant Toeplitz matrix, and J_N the $N \times N$ exchange matrix. The matrix ensemble of circulant Hankel matrices is exceptional in its own right; its limiting spectral



distribution converges almost surely to a symmetrized Rayleigh distribution (as shown in [9]). In this section, we provide another proof of this remarkable result. We begin by defining the empirical spectral measure for this ensemble of matrices. This measure, for the normalized eigenvalues of our matrix H , is given by the following definition.

Definition 4.1 *The empirical spectral measure of a random $N \times N$ circulant Hankel matrix is*

$$\mu_{H_N}(x) dx := \frac{1}{N} \sum_{i=1}^N \delta \left(x - \frac{\lambda_i(H_N)}{\sqrt{N}} \right) dx. \quad (12)$$

where $\delta(x)$ is the Dirac-delta functional and the λ_i are the non-zero eigenvalues of H_N .

Remark 4.2 The \sqrt{N} scaling factor is derived heuristically. By computing the trace of H_N^2 , we obtain

$$\mathbb{E}[\text{tr}(H_N^2)] = N^2 = \sum_{i=1}^N \lambda_i(H_N)^2, \quad (13)$$

suggesting that the eigenvalues must be roughly \sqrt{N} each in expectation.

In order to use the method of moments, we compute the k^{th} moment for the empirical spectral distribution of a random matrix H_N , $\mu_{H_N}(x)$.

Remark 4.3 The k^{th} moment of the empirical spectral distribution of the random matrix H_N , averaged over an ensemble, is given by

$$M_k(N) := \int_{-\infty}^{\infty} x^k \mu_{H_N}(x) dx = \frac{1}{N^{\frac{k}{2}+1}} \sum_{i=1}^N \mathbb{E}[\lambda_i^k(H_N)] = \frac{1}{N^{\frac{k}{2}+1}} \mathbb{E}[\text{tr}(H_N^k)]. \quad (14)$$

We use M_k to denote $\lim_{N \rightarrow \infty} M_k(N)$.

This standard computation follows from the properties of the Dirac delta functional and the Eigenvalue Trace Lemma.

Proposition 4.4 *We have $M_1 = 0$ and $M_2 = 1$.*

Proof. The first moment is immediate from $\mathbb{E}[\text{tr}(H_N)] = 0$. The second moment follows from substituting $\mathbb{E}[\text{tr}(H_N^2)] = N^2$ into the formula (14). \square

In order to compute M_k for $k \geq 2$ we consider the limiting behavior of the terms in the sum combinatorially. It is useful to note the following fact.

Remark 4.5 In $H_N = A_N J_N$, $h_{ij} = h_{k\ell}$ if and only if $i + j \equiv_N k + \ell$, where we index the matrix beginning at 0.

To build intuition, we now explicitly calculate the third moment combinatorially.

Proposition 4.6 *We have $M_3 = 0$.*



Proof. Note that

$$\operatorname{tr}(H_N^3) = \sum_{i_1=1}^N \sum_{i_2=1}^N \sum_{i_3=1}^N h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_1} \quad (15)$$

where $h_{i_j i_{j+1}}$ is the matrix entry of H_N at the i_j^{th} row and i_{j+1}^{th} column. If any of the random variables in a summand is not equal to any of the others, we can write the expectation of the whole summand as a product of the expectation of the singleton term and the rest of the summand by the independence of our random variables. Since all the random variables have mean 0, such a term contributes zero. As such, there is exactly one option for contributing summands: all three matrix entries are equal.

For this to hold, we must have $h_{i_1 i_2} = h_{i_2 i_3} = h_{i_3 i_1}$. So we must have

$$\begin{aligned} i_1 + i_2 &\equiv_N i_2 + i_3 \\ i_2 + i_3 &\equiv_N i_3 + i_1 \\ i_3 + i_1 &\equiv_N i_1 + i_2 \end{aligned} \quad (16)$$

Thus we have $i_1 \equiv_N i_2 \equiv_N i_3$, leaving only one free variable. Since each of our i.i.d random variables have finite moments by assuming, terms of this kind contribute $O(N)$ to the expectation of $\operatorname{tr}(H_N^4)$. Thus, by Remark 4.3, such terms contribute $\lim_{N \rightarrow \infty} \frac{O(N)}{N^{5/2}} = 0$ to the third moment in the limit. \square

Next, we show $M_{2k} = k!$ for all k and thus $\lim_{N \rightarrow \infty} \mu_{H_N}(x)$, averaged over all H_N converges to the symmetrized Rayleigh distribution.

We begin with a sample calculation showing $M_4 = 2$ to build intuition for the proof.

Proposition 4.7 *We have $M_4 = 2$.*

Proof. Note that

$$\operatorname{tr}(H_N^4) = \sum_{i_1=1}^N \sum_{i_2=1}^N \sum_{i_3=1}^N \sum_{i_4=1}^N h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_1} \quad (17)$$

where $h_{i_j i_{j+1}}$ is the matrix entry of H_N at the i_j^{th} row and i_{j+1}^{th} column. As before, if any of the random variables in a summand is not equal to any of the others, we can write the expectation of the whole summand as a product of the expectation of the singleton term and the rest of the summand by the independence of our random variables. Since all the random variables have mean 0, such a term contributes zero. As such, there are only two options for contributing summands: four equal matrix entries or two pairs of equal matrix entries.

Case 1 In this case, there are four summands that are all matched. That is, $h_{i_1 i_2} = h_{i_2 i_3} = h_{i_3 i_4} = h_{i_4 i_1}$. Up to relabeling, the first case yields the system of equations

$$\begin{aligned} i_1 + i_2 &\equiv_N i_2 + i_3 \\ i_2 + i_3 &\equiv_N i_3 + i_4 \\ i_3 + i_4 &\equiv_N i_4 + i_1 \\ i_4 + i_1 &\equiv_N i_1 + i_2 \end{aligned} \quad (18)$$



This implies $i_1 \equiv_N i_3$ and $i_2 \equiv_N i_4$, leaving only two free variables. Since there are only 2 degrees of freedom in this case and each of our i.i.d. random variables have finite moments by assumption, terms of this kind contributes $O(N^2)$ to the expectation of $\text{tr}(H_N^4)$. Thus, by Remark 4.3, such terms contribute $\lim_{N \rightarrow \infty} \frac{O(N^2)}{N^3} = 0$ to the fourth moment in the limit.

Notably, the system of equations corresponds to the equation matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

which has nullity 2. Thus, since vectors satisfying this system of equations are exactly those in the null space of this matrix, there are $O(N^2)$ valid linear combinations of basis vectors of the null space, and the random variables have finite fourth moments, such terms contribute $O(N^2)$ to the expectation of $\text{tr}(H_N^4)$. This alternate linear algebraic formulation is used in our the proof of Theorem 4.12.

Case 2 In this case, all summands are paired. This case of matching the random variables into pairs has two subcases.

Subcase 2.1 Pair nonadjacent random variables, that is, $h_{i_1 i_2} = h_{i_3 i_4}$, $h_{i_2 i_3} = h_{i_4 i_1}$. This pairing yields the following system of equations:

$$\begin{aligned} i_1 + i_2 &\equiv_N i_3 + i_4 & (19) \\ i_2 + i_3 &\equiv_N i_4 + i_1 \\ i_3 + i_4 &\equiv_N i_1 + i_2 \\ i_4 + i_1 &\equiv_N i_2 + i_3 \end{aligned}$$

This implies $i_2 \equiv_N i_4$ and $i_1 \equiv_N i_3$. Thus, there are only two degrees of freedom in this case, and it does not contribute in the limit.

Note that the equation matrix corresponding to the system of equations has nullity 2, an alternative proof that this case cannot contribute.

Subcase 2.2 Pair adjacent random variables. For example, $h_{i_1 i_2} = h_{i_2 i_3}$ and $h_{i_3 i_4} = h_{i_4 i_1}$. Note that there are two such pairings. Up to relabeling, this pairing yields the following system of equations:

$$\begin{aligned} i_1 + i_2 &\equiv_N i_2 + i_3 & (20) \\ i_2 + i_3 &\equiv_N i_1 + i_2 \\ i_3 + i_4 &\equiv_N i_4 + i_1 \\ i_4 + i_1 &\equiv_N i_3 + i_4 \end{aligned}$$

This implies only $i_1 \equiv_N i_3$, yielding 3 degrees of freedom. Thus, the terms in this case contribute in the limit. Fixing i_1 , there is a unique choice for i_3 and



N choices for both i_2 and i_4 , yielding $2N^3$ choices total after iterating over all i_1 and both choices of pairing orientation.

Substituting into Remark 4.3, we then get that this case contributes precisely 2 to M_4 in the limit and $M_4 = 2$, since this is the only nontrivial contributing case.

□

We see that only a select few of the summands in the computation of moments contribute in the limit. We formalize this observation in the following lemmas.

Lemma 4.8 *For moments M_k , where $k \geq 1$, the only contributing summands $x_{j_1}^{n_1} \cdots x_{j_\ell}^{n_\ell}$ in the trace expansion are those where $n_i = 2$ for all $1 \leq i \leq \ell$.*

Proof. Consider any summand in $\text{tr}(H_N^k)$, $x_{j_1}^{n_1} \cdots x_{j_\ell}^{n_\ell}$, where

$$\sum_{i=1}^{\ell} n_i = k \tag{21}$$

and each $n_i \geq 1$.

Now, if any $n_i = 1$, the expectation of the summand is 0. We can see this by analyzing $\text{tr}(H_N^k)$ via the Eigenvalue Trace Lemma. Observe that since the entries of H_N are independent, if any are to the first power in a summand in the expansion of $\text{tr}(H_N^k)$, the expected value of the entire summand is zero. For example,

$$\mathbb{E}[h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_1}] = \mathbb{E}[x_a x_b x_b] = \mathbb{E}[x_a] \mathbb{E}[x_b^2] = 0 \cdot 1 = 0. \tag{22}$$

Thus, at a minimum, the entries must be matched in pairs. So, we may assume each $n_i \geq 2$.

If there is at least one factor in the summand with $n_r \geq 3$, there are at most $\lceil \frac{k}{2} \rceil$ degrees of freedom of terms with such groupings—there are at most $N^{\lfloor \frac{k-3}{2} \rfloor + 1} = N^{\lceil \frac{k}{2} \rceil - 1}$ ways to choose the x_{j_i} and an additional N ways to fix a matrix index in some term.

There are then a constant in k number of ways to assign each factor in the summand to a particular x_{j_i} , and then the choice of one index of a matrix entry completely determines the remaining matrix indices via Remark 4.5. Note that each grouping of n_i matrix entries equal to x_{j_i} contributes

$$\mathbb{E}[(x_{j_i})^{n_i}] = p_{n_i} = O_k(1) \tag{23}$$

since p has finite higher moments by assumption (where p_{n_i} is the n_i^{th} moment of p). As such, each contributing term contributes $O_k(1)$ to M_k .

Hence, each such term contributes

$$\lim_{N \rightarrow \infty} \frac{O(N^{\lceil \frac{k}{2} \rceil})}{N^{\frac{k}{2} + 1}} = 0 \tag{24}$$

to M_k , as desired.

□



Remark 4.9 The argument in Lemma 4.8 also shows that there are at most $k + 1$ degrees of freedom when assigning x_{j_i} in pairs in the computation of $\text{tr}(H_N^{2k})$.

Remark 4.10 The argument in Lemma 4.8 shows that $M_{2k+1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

For the following arguments, consider an index “even” if its subscript is even. Similarly define odd indices. For example, consider

$$\text{tr}(H_N^4) = \sum_{i_1=1}^N \sum_{i_2=1}^N \sum_{i_3=1}^N \sum_{i_4=1}^N h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_1}. \quad (25)$$

We view i_1, i_3 as the “odd” indices and i_2, i_4 as the “even” indices.

Lemma 4.11 *The pairings of odd indices to even indices contribute $k!$ to M_{2k} .*

Proof. Consider the system of equations resultant in this case. Each relation can be assumed to be of the general form

$$i_j + i_{j+1} \equiv_N i_\ell + i_{\ell+1} \quad (26)$$

for j even and ℓ odd. Note in particular that all even indices arise on the left-hand side of such relations as the first term, and all odds similarly as the first term on the right-hand side. Since in such relations each index is added to the subsequent index, every index appears in a sum exactly once on both sides of the equations.

Interpret these equations as $1 \times 2k$ row vectors with ones in the entries corresponding to the indices on the left-hand side of the relations, negative ones corresponding to those on the right-hand side, and zeroes otherwise, as in Proposition 4.7. Now, from the above observation, the sum of these k row vectors is 0. This implies they are linearly dependent. This means the matrix given by this system of equations has nullity at least $k + 1$. Note that vectors \mathbf{x} in the null space are exactly solutions to

$$E\mathbf{x} = 0 \quad (27)$$

for E the matrix of these row vectors. This implies we have $k + 1$ degrees of freedom in this case. From Remark 4.9, we thus have exactly $k + 1$ degrees of freedom, so these pairings contribute exactly their constant term to M_{2k} in the limit.

To count the number of odd-even pairings, we choose an odd and an even index to pair in k^2 ways. Then we repeat until there are no indices left to pair, yielding $(k!)^2$. However, we introduced an arbitrary ordering of the pairs in this process, so we correct by dividing by $k!$, yielding $k!$ as desired. Note that, given the choice of a single index and pairings, every index is determined uniquely (regardless of the modulo N). \square

Now we complete the proof by showing that the other pairings of indices do not contribute in the limit.

Theorem 4.12 $M_{2k} = k!$.



Proof. From Lemma 4.11, it suffices to show that any arrangement of pairs including an odd-odd or even-even index matching will not contribute. One way to do so is to show that the k row vectors corresponding to the resultant system of equations are all linearly independent and thus the rank of the corresponding matrix is k , implying a nullity of k and less than $k + 1$ degrees of freedom. Note that there being an odd-odd index pairing implies that there must be an even-even index pairing.

Step 1 We will show that if there is an odd-odd index pairing, then the equations corresponding to even-odd index pairings are linearly independent as row vectors.

Fix a relation given by such an even-odd index pairing. Each side of each relation of the form $i_\ell + i_{\ell+1} \equiv_N i_r + i_{r+1}$ can be conceptualized as a “first” index (matched index) plus a “second” index. For the sake of consistency, when converting such relations into row vectors (by moving all the terms to a single side), we negate the side with the odd first index. In order to show linear independence of the even-odd row vectors, it suffices to show that no nonempty linear combination of them sums to 0.

Note that each index appears at most twice amongst the odd-even pair relations. In particular, if we fix

$$i_\ell + i_{\ell+1} \equiv_N i_{\ell+2i+1} + i_{\ell+2i+2} \tag{28}$$

to be in our linear combination, with ℓ even, this yields a row vector of the form

$$(0, 0, \dots, 1, 1, 0, \dots, 0, -1 - 1, 0, \dots, 0)$$

with the 1’s in the ℓ and $(\ell + 1)^{\text{st}}$ positions and the -1 ’s in the $(\ell + 2i + 1)^{\text{st}}$ and $(\ell + 2i + 2)^{\text{th}}$ positions and all other positions 0 (note the indices range from 1 to $2k$).

Crucially, in order to form a linear combination of even-odd vectors summing to 0, we must nullify each index in the sum. Each index appears at most once as a first and a second index. Given our signing convention, even indices are positive as first indices and negative as second indices. Odds indices are negative as first indices and positive as second indices. Note that, since each index occurs at most once as a first index and a second index, the two expressions cannot be exactly equal. In order to cancel out the positive contribution of i_ℓ to the ℓ^{th} column, we need to add the term including i_ℓ as a second index. However, we then must cancel out the contribution of $i_{\ell-1}$ as a first index by including it as a second index. To do that, we must include $i_{\ell-2}$ as a first index. As such, we see that in order to cancel out the contribution of each necessary term, we need to include every term as both a first and second index. However, by assumption, there is an odd-odd index. Therefore, not every index has a row vector corresponding to it as a first and second index. Thus, we cannot cancel out the contribution to every column, and there is no nonempty linear combination of vectors corresponding to the even-odd pair expressions that equals 0. We conclude that these row vectors are linearly independent.



Step 2 We will show that the row vectors corresponding to odd-odd pairs of indices cannot be part of any nonempty linear combination of row vectors summing to zero (the proof follows for even-even pairs as well).

Suppose indices i_r and i_{r+2i} are paired for r odd and $i \geq 1$. The corresponding row vector is of the form

$$(0, 0, \dots, 1, 1, 0, \dots, 0, -1 - 1, 0, \dots, 0)$$

with 1's in the r^{th} and $(r + 1)^{\text{st}}$ indices and -1 's in the $(r + 2i)^{\text{th}}$ and $(r + 2i + 1)^{\text{st}}$ indices. As before, to cancel out the contribution of i_r , we need i_r to appear as a second index and contribute negatively. As a result, i_{r-1} must appear as a first index and contribute negatively. Then we need i_{r-1} to appear as a second index and contribute positively to cancel out that contribution. This requires i_{r-2} to appear as a first index and contribute positively. However, this implies that all first odd indices must contribute positively, and all first even indices must contribute negatively to achieve total cancellation. We know this cannot be the case as there is an odd-odd pair and one of the first odd indices must thus contribute negatively. As such, odd-odd and even-even pairs cannot be a part of linear combinations of the row vectors summing to 0.

We conclude that a linearly dependent family of row vectors must be a subset of the even-odd pair row vectors if it exists. However, from Step 1, this is impossible. So, all of the row vectors are linearly independent. Thus, if there are odd-odd or even-even index pairs, the rank of the matrix is k and the nullity is k . Since the nullity of this matrix is an upper bound on the degrees of freedom in this case, such pairings will not contribute in the limit. The only remaining pairings are all odd-even. From Lemma 4.11 we conclude $M_{2k} = k!$.

□

We are finally able to prove Theorem 1.1. We use in the proof that the moments of the limiting spectral distribution of the Circulant Hankel ensemble are the same as the moments of a symmetrized Rayleigh distribution. More broadly, a Rayleigh distribution is a Weibull distribution with fixed parameters. For our purposes, denote the Weibull distribution with scale parameter λ and shape parameter k by the following:

$$f(x; \lambda, k) = \frac{kx^{k-1}}{x\lambda^k} e^{-(x/\lambda)^k}, \quad (29)$$

for $x \geq 0$ and 0 otherwise. As our eigenvalue distributions are symmetric, we symmetrize the distribution by replacing x with $|x|$ and dividing through by 2 to retain

$$\int_{-\infty}^{\infty} \frac{f(|x|; \lambda, k)}{2} dx = 1. \quad (30)$$

This symmetrization notably has no effect on the even moments of the distribution and zeroes all the odd moments.



The $(2m)^{\text{th}}$ moment of a Weibull distribution $f(x; \lambda, k)$ is given by

$$m_{2m} = \lambda^{2m} \Gamma(2m/k + 1). \quad (31)$$

When this distribution is Rayleigh, i.e., $k = 2$ and $\lambda = 1$, the $2m^{\text{th}}$ moment is then $m!$.

Theorem 1.1 (Bryc-Dembo-Jiang [9]) *Let $\mu_{A,N}(x)$ be the empirical spectral measure of the $N \times N$ circulant Hankel random matrix ensemble populated by entries from a sequence of random variables A from a distribution p with mean 0, variance 1, and finite higher moments. Then,*

$$\lim_{N \rightarrow \infty} \mu_{A,N}(x) \rightarrow |x|e^{-x^2} \quad (2)$$

almost surely.

Proof. This follows by the exact same argumentation as in Section 6 of [13] with plus rather than minus modulo N . \square

We may then conclude that the limiting spectral distribution of swirl ensembles on A circulant Toeplitz and J an exchange matrix also converges almost surely to a symmetrized Rayleigh distribution, the result of Corollary 1.2.

Corollary 1.2 *Let $G_{2N} = \text{swirl}(A, J)$ for J the $N \times N$ exchange matrix and A a random $N \times N$ circulant Toeplitz matrix. As $N \rightarrow \infty$, the limiting spectral measure of this ensemble converges almost surely to a symmetrized Rayleigh distribution.*

Proof. From the observation that $\text{swirl}(A, J)$ trivially has half of its rows repeated, G_{2N} has at most N nonzero eigenvalues. The empirical spectral measure of the $2N \times 2N$ matrix B_{2N} is thus given by the following equation:

$$\mu_{B_{2N}}(x)dx := \frac{1}{N} \sum_{i=1}^N \delta \left(x - \frac{\lambda_i(B_{2N})}{2\sqrt{N}} \right) dx. \quad (32)$$

See Definition 2.2 for the derivation of the scaling factor. From Theorem 3.5, the k^{th} moment of the limiting spectral distribution of this ensemble equals

$$\lim_{N \rightarrow \infty} \frac{1}{N^{k/2+1}} \mathbb{E}[\text{tr}(H_N^k)]. \quad (33)$$

As such, the k^{th} moment in this case is exactly the k^{th} moment of the limiting spectral distribution of H_N . The result then follows from Theorem 1.1. \square

Remark 4.13 For an ensemble such that $\text{swirl}(A, X)$ has no repeated rows, $\text{swirl}(A, X)$ would not have the same limiting spectral distribution as AX . Indeed, its moments would be 2^k times the moments of the limiting spectral distribution of AX . In the case of a Weibull distribution, this would only increase the λ scaling parameter of the Weibull distribution by a factor of $\sqrt{2}$.



5 Future Work

5.1 Other Swirl Ensembles

Another possible next step is to study broader matrix ensembles related to $\text{swirl}(A, X)$. A good starting point is ensembles with $X^2 = I$, due to the following theorem of Tao and Yasuda [24].

Theorem 5.1 (Tao-Yasuda [24]) *Let A and X be real symmetric matrices with $X^2 = I$.*

- $AX = XA$ if and only if the spectrum of A equals the spectrum of XA up to sign.
- $AX = -XA$ if and only if the spectrum of A equals the spectrum of XA multiplied by i .

In particular, if we choose ensembles A and X such that A and X are $N \times N$ real symmetric matrices, $X^2 = I$ and $AX = XA$, then AX has all real eigenvalues.

Another interesting direction is to study the even powers of non-symmetric swirl ensembles. Proposition B.1 provides a useful starting point for such investigations.

Finally, given that circulant Toeplitz and circulant Hankel matrices yield rare named, closed form limiting spectral distributions, it seems likely that they possess some intrinsic, special properties. Inspired by the work of [1, 5, 6, 20] on matrices with patterns governed by link functions, we investigated circulant matrices with link functions along different diagonals, but found the results disappointingly uninteresting. These results are summarized in Appendix C.

A Iterating swirl

Another interesting avenue for swirl is iterating the operation.

Definition A.1 *Let A, X be $N \times N$ matrices. Let X_k be the block matrix with 2^{k-1} X 's on the anti-diagonal and zeroes elsewhere. Note that $X_1 = X$. Then, set*

$$\text{swirl}^k(A, X) := \text{swirl}(\dots \text{swirl}(\text{swirl}(\text{swirl}(A, X_1), X_2), X_3), \dots), X_k) \quad (34)$$

where swirl is repeated k times in the above.

We begin by analyzing the trace of iterated swirl matrices.

Proposition A.2 *Fix A, X both $N \times N$ matrices such that $X^2 = I$ and k a non-negative integer. Then*

$$\text{tr}(\text{swirl}^k(A, X)) = 2^k \text{tr}(AX). \quad (35)$$



Proof. We prove by induction. For $k = 1$, this follows from Theorem 3.5. Now, assume this holds for $r - 1$ for $r \geq 2$. Then,

$$\begin{aligned} \text{swirl}^r(A, X) &= \text{swirl}(\text{swirl}^{r-1}(A, X), X_r) \\ &= \begin{pmatrix} X_r \text{swirl}^{r-1}(A, X) & \text{swirl}^{r-1}(A, X) \\ X_r \text{swirl}^{r-1}(A, X)X_r & \text{swirl}^{r-1}(A, X)X_r \end{pmatrix}. \end{aligned} \quad (36)$$

This implies

$$\begin{aligned} \text{tr}(\text{swirl}^r(A, X)) &= \text{tr}(X_r \text{swirl}^{r-1}(A, X)) + \text{tr}(\text{swirl}^{r-1}(A, X)X_r) \\ &= 2 \text{tr}(X_r \text{swirl}^{r-1}(A, X)). \end{aligned} \quad (37)$$

Now

$$\begin{aligned} X_r \text{swirl}^{r-1}(A, X) &= \begin{pmatrix} 0 & X_{r-1} \\ X_{r-1} & 0 \end{pmatrix} \begin{pmatrix} X_{r-1} \text{swirl}^{r-2}(A, X) & \text{swirl}^{r-2}(A, X) \\ X_{r-1} \text{swirl}^{r-2}(A, X)X_{r-1} & \text{swirl}^{r-2}(A, X)X_{r-1} \end{pmatrix} \\ &= \begin{pmatrix} \text{swirl}^{r-2}(A, X)X_{r-1} & X_{r-1} \text{swirl}^{r-2}(A, X)X_{r-1} \\ \text{swirl}^{r-2}(A, X) & X_{r-1} \text{swirl}^{r-2}(A, X) \end{pmatrix}. \end{aligned} \quad (38)$$

Thus

$$\begin{aligned} \text{tr}(X_r \text{swirl}^{r-1}(A, X)) &= \text{tr}(\text{swirl}(\text{swirl}^{r-2}(A, X), X_{r-1})) \\ &= \text{tr}(\text{swirl}^{r-1}(A, X)) \\ &= 2^{r-1} \text{tr}(AX) \end{aligned} \quad (39)$$

by induction.

Therefore,

$$\text{tr}(\text{swirl}^r(A, X)) = 2^r \text{tr}(AX). \quad (40)$$

The result then follows by induction. \square

Remark A.3 Alternatively, observe that, since $X^2 = I$, $\text{swirl}^\ell(A, X)$ is just the block matrix of $\text{swirl}(A, X)$ repeated $4^{\ell-1}$ times. This means

$$\text{tr}(\text{swirl}^\ell(A, X)) = 2^{\ell-1} \text{tr}(\text{swirl}(A, X)) = 2^\ell \text{tr}(AX)$$

If we wish to study the moments of ensembles of such matrices, we need to understand the trace of powers of the iterated swirl matrices. We reduce this to an analysis of $\text{tr}((AX)^k)$ in the following proposition.

Proposition A.4 Fix A, X to be $N \times N$ matrices such that $X^2 = I$ and k and l non-negative integers. Then

$$\text{tr}((\text{swirl}^\ell(A, X))^k) = 2^{k\ell} \text{tr}((AX)^k). \quad (41)$$



Proof. We prove by induction on ℓ . For $\ell = 1$, this follows from Theorem 3.5. Now, assume

$$\text{tr}((\text{swirl}^\ell(A, X))^k) = 2^{k\ell} \text{tr}((AX)^k) \quad (42)$$

holds for $\ell = r \geq 1$. We show it holds for $\ell = r + 1$. By Definition A.1,

$$\text{swirl}^{\ell+1}(A, X) = \text{swirl}(\text{swirl}^\ell(A, X_\ell), X_{\ell+1}). \quad (43)$$

So,

$$\begin{aligned} \text{tr}((\text{swirl}^{\ell+1}(A, X))^k) &= \text{tr}((\text{swirl}(\text{swirl}^\ell(A, X), X_{\ell+1}))^k) \\ &= 2^k \text{tr}((\text{swirl}^\ell(A, X)X_{\ell+1})^k) \end{aligned}$$

with the last step following from Theorem 3.5.

Let $B = \text{swirl}^{\ell-1}(A, X)$. Then,

$$\begin{aligned} \text{swirl}^\ell(A, X)X_{\ell+1} &= \begin{pmatrix} BX_\ell & B \\ X_\ell BX_\ell & X_\ell B \end{pmatrix} \begin{pmatrix} 0 & X_\ell \\ X_\ell & 0 \end{pmatrix} \\ &= \begin{pmatrix} BX_\ell & B \\ X_\ell BX_\ell & X_\ell B \end{pmatrix} = \text{swirl}^\ell(A, X) \end{aligned} \quad (44)$$

with the last step following from the assumption that $X^2 = I$.

Therefore,

$$\begin{aligned} \text{tr}((\text{swirl}^{\ell+1}(A, X))^k) &= 2^k \text{tr}((\text{swirl}^\ell(A, X)X_{\ell+1})^k) \\ &= 2^k \text{tr}((\text{swirl}^\ell(A, X))^k) \\ &= 2^{k(\ell+1)} \text{tr}((AX)^k) \end{aligned}$$

with the last step from the inductive hypothesis. The result then follows by induction. \square

A.1 Limiting Spectral Distribution of Swirled Matrix Ensembles

From the previous work in this section, we can reduce the analysis of swirl matrix ensembles to the analysis of matrix product ensembles. We consider the empirical spectral measure defined in Definition 2.2. In this case, from Remark A.3, for A and X both $N \times N$ matrices, and $\ell \geq 1$, $\text{swirl}^\ell(A, X)$ has the same number of eigenvalues that are not trivially zero (non-repeated rows), c , as $\text{swirl}^\ell(A, X)$. Let $B_{N2^\ell} = \text{swirl}^\ell(A, X)$. Then the empirical spectral measure of B_{N2^ℓ} is given by

$$\mu_{A, N2^\ell}(x)dx := \frac{1}{c} \sum_{i=1}^c \delta \left(x - \frac{\sqrt{c}\lambda_i(A)}{N2^\ell} \right) dx. \quad (45)$$

From Definition 2.2 and Proposition A.4, the k^{th} moment of the spectral distribution in this case is thus

$$\frac{2^{\ell k} c^{k/2-1}}{2^{\ell k} N^{k+1}} \mathbb{E}[\text{tr}((AX)^k)] = \frac{c^{k/2-1}}{N^{k+1}} \mathbb{E}[\text{tr}((AX)^k)], \quad (46)$$

which does not depend on ℓ . As such, the limiting spectral distribution of swirl is the same for any number of iterations, ℓ .



B The Product of Swirl and its Transpose

If we assume that X is a permutation matrix, then $\text{tr}(\text{swirl}(A, X) \text{swirl}(A, X)^T)$ reduces to understanding $\text{tr}(AA^T)$. This is a useful quantity to understand if A and X are chosen such that $\text{swirl}(A, X)$ does not necessarily have real eigenvalues.

Proposition B.1 Fix A, X to be $N \times N$ matrices with X a permutation matrix. Then

$$\text{tr}((\text{swirl}(A, X) \text{swirl}(A, X)^T)^k) = 2^{2k} \text{tr}((AA^T)^k). \quad (47)$$

Proof. Let $S = \text{swirl}(A, X) \text{swirl}(A, X)^T$. We show by induction that

$$S^k = 2^{2k-1} \begin{pmatrix} (AA^T)^k & (AA^T)^k X^T \\ X(AA^T)^k & X(AA^T)^k X^T \end{pmatrix}. \quad (48)$$

For the base case, consider S^k for $k = 1$. We have

$$S^1 = \begin{pmatrix} AX & A \\ XAX & XA \end{pmatrix} \begin{pmatrix} (AX)^T & (XAX)^T \\ A^T & (XA)^T \end{pmatrix}. \quad (49)$$

This yields

$$= \begin{pmatrix} (AX)(AX)^T + AA^T & (AX)(XAX)^T + A(XA)^T \\ (XAX)(AX)^T + (XA)A^T & (XAX)(XAX)^T + (XA)(XA)^T \end{pmatrix}. \quad (50)$$

Expanding the transpose terms yields

$$= \begin{pmatrix} AXX^T A^T + AA^T & AXX^T A^T X^T + AA^T X^T \\ XAXX^T A^T + XAA^T & XAXX^T A^T X^T + XAA^T X^T \end{pmatrix}. \quad (51)$$

Recall X is a permutation matrix $XX^T = I$. Thus, we have

$$= 2 \begin{pmatrix} AA^T & AA^T X^T \\ XAA^T & XAA^T X^T \end{pmatrix}. \quad (52)$$

Now assume that the inductive hypothesis holds for $k = n$; we will show it holds for $k = n + 1$. Rewrite S^{n+1} as SS^n . Then

$$S^{n+1} = 2 \begin{pmatrix} AA^T & AA^T X^T \\ XAA^T & XAA^T X^T \end{pmatrix} 2^{2n-1} \begin{pmatrix} (AA^T)^n & (AA^T)^n X^T \\ X(AA^T)^n & X(AA^T)^n X^T \end{pmatrix}, \quad (53)$$

by induction. Matrix multiplication yields

$$= 2^{2n} \begin{pmatrix} (AA^T)^{n+1} + (AA^T)^{n+1} & (AA^T)^{n+1} X^T + (AA^T)^{n+1} X^T \\ X(AA^T)^{n+1} + X(AA^T)^{n+1} & X(AA^T)^{n+1} X^T + X(AA^T)^{n+1} X^T \end{pmatrix}. \quad (54)$$

Simplifying, we have



$$= 2^{2n+1} \begin{pmatrix} (AA^T)^{k+1} & (AA^T)^{k+1} X^T \\ X(AA^T)^{k+1} & X(AA^T)^{k+1} X^T \end{pmatrix}. \quad (55)$$

This completes the inductive argument.

Now calculating the trace is trivial. Note by the cyclic property of trace,

$$\text{tr}(S^k) = 2^{2k-1} \text{tr}((AA^T)^k) + 2^{2k-1} \text{tr}(X(AA^T)^k X^T) = 2^{2k} \text{tr}((AA^T)^k). \quad (56)$$

□

Here the limiting spectral distribution reduces to a scaled semicircle distribution, which is handled in [25].

C Generalizing Hankel Matrices

When computing the moments of the limiting empirical spectral measures of our ensembles, we converted our problem of finding degrees of freedom of contributing summands in the trace to a problem of calculating the nullity of a matrix. As a specific example, we can calculate the n^{th} moment of the Hankel ensemble by looking at set partitions of $\{1, 2, \dots, n\}$ and calculating the rank of matrices of the form

$$M_\pi = (I_n - P_\pi)B_n(1, 1), \quad (57)$$

where I_n is the identity matrix, P_π is a permutation matrix, and $B_n(s, t)$ is a matrix with s in the diagonal and t to the right of the diagonal. The matrix $B_4(s, t)$ is written below as an example.

$$B_4(s, t) = \begin{bmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \\ t & 0 & 0 & s \end{bmatrix} \quad (58)$$

The permutation matrix P_π corresponds to the particular matching of indices in the summand corresponding to equal matrix entries. The nullity of the matrix M_π gives the degrees of freedom of assignments of entries to groups that contribute in the case π . Iterating over all set partitions and substituting into the formula in Definition 2.4, we can easily show certain configurations do not contribute in the limit.

We generalize by considering matrices which are constant along certain circulant diagonals. We call these (s, t) -ensembles. Formally, an $N \times N$ matrix is in the (s, t) -ensemble if

$$si + tj \equiv_N sk + t\ell \implies a_{ij} = a_{kl}. \quad (59)$$

An example of a matrix within the (1,2)-ensemble is:

$$A_4(1, 2) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 & x_2 \\ x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 & x_2 \end{bmatrix} \quad (60)$$



Note how the equivalent entries, denoted here by entries of the same value, are spaced apart by $s = 1$ row movements and $t = 2$ column movements. This creates the appearance of a matrix where every s rows, the entries are horizontally permuted by t columns. From this, we obtain the idea of slope as we describe the relationship between equivalent entries in (s, t) -ensemble patterns. This generalization is the idea of a polynomial link function in the literature, except now modulo N (see [1, 5, 6, 20]).

However, this generalization is insufficient if we intend to study symmetric matrices. Thus, we strengthen our condition to

$$a_{ij} = a_{mn} \iff si + tj \equiv_N sm + tn \text{ or } ti + sj \equiv_N tm + sn. \quad (61)$$

This allows us to generalize the special behavior of both circulant Hankel and circulant Toeplitz matrices.

Below we have a 4×4 matrix that belonging to the $(1, 1)$ -ensemble

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_1 & x_2 & x_3 \end{bmatrix}.$$

These represent the circulant Hankel matrices. Moreover, 4×4 matrices from the $(1, -1)$ -ensemble are of the form

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_1 & x_2 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_1 \end{bmatrix}.$$

These represent the circulant Toeplitz matrices.

Inspired by the fact that both the circulant Hankel and circulant Toeplitz matrices admitted diagonal structure, we generalize this structure in the hopes of finding a broader class of matrices with named limiting empirical spectral distributions. To this end, notice how the elements of the aforementioned families cascade through the matrix with “slope” ± 1 . It is this notion of slope which we wish to generalize, and will be made more concrete in what follows.

With this change in parameters, the same pattern of equivalent entries being s rows and t columns away persists, and the main observable change is in the number of equivalence classes of matrix entries that appear.

The elements at the indices generated by (s, t) and (t, s) form a group:

$$H = \langle (s, t), (t, s) \rangle \leq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}). \quad (62)$$

Notice that H is a normal subgroup of $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$. We want to understand the number of cosets associated to $((\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}))/H$, as this will give us the number of unique elements within an $N \times N$ matrix with the aforementioned rule.

In the simplest cases we get the matrices we studied in the main portion of the paper. When s and t both equal 1, the resultant ensemble is Hankel, with the number of cosets



increasing as N increases. It is this positive slope that reflects the symmetry of the matrix that is lacking in the circulant Toeplitz. Likewise, if s and t are units with opposite signs, i.e., $s = 1$ and $t = -1$, the resulting matrices are all Toeplitz. Similarly, the number of cosets increases consistently with N . The number of cosets is important to consider because it indicates the amount of variation within the matrix, the more cosets there are, the fewer zero eigenvalues appear.

Now, as we vary s and t , new patterns arise in the family of matrices and consequently the number of cosets. This variance is a function of the positioning of equivalent entries. With these new s and t values, the spacing between the placement of the entries changes, and there are some very interesting patterns to the numbers of cosets and the qualities of symmetry. However, among all these patterns, it appears that the only ones that remain symmetric are circulant Hankel. Besides those, we continue to observe circulant Toeplitz matrices appearing at certain intervals and numbers of cosets.

For N coprime to s and t , we observe that when

$$s \equiv t \pmod{[H : (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})]}, \quad (63)$$

where $[H : (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})]$ is the index of H in $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$, the matrices yielded are circulant Toeplitz. Alternatively, when

$$s \equiv -t \pmod{[H : (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})]} \quad (64)$$

the matrices yielded are circulant Hankel.

However, we found that whenever we consider $N \times N$ matrices from the (s, t) -ensemble with $s, t \neq \pm 1$, then the limiting spectral distribution is uninteresting. This is because the number of cosets for a matrix in this ensemble appears bounded by a constant times $\gcd((s+t)(s-t), N)$. Indeed, the matrix becomes a block matrix with many repetitions of a much smaller matrix, deferring its spectral distribution to that smaller matrix ensemble. When s and t are units (up to sign), we find computationally that the number of cosets is proportional to $\gcd(0, N)$, which is just N . Because the number of cosets is proportional to N , the number of eigenvalues grows as we increase the size of the matrix. However, in the other case, the number of nonzero eigenvalues is fixed, preventing a new distribution from arising.

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