A Computer-Based Approach to Solving the Diophantine Equation $7^x - 3^y = 100$

A. Dima

Abstract - Assisted by Octave and by a website where one can find the order of a modulo m (here a and m are relatively prime natural numbers), I found a proof for showing that the diophantine equation $7^x - 3^y = 100$ has only the solution $(x, y) = (3, 5)$ in positive integers.

Keywords : exponential diophantine equation; prime number; order of an integer modulo another integer

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Introduction

Pillai (1931,1936) proved that the equation

$$a^x - b^y = k, \tag{1}$$

where a, b, x, y, k are positive integers, has only finitely many solutions in x, y (a, b, k are fixed) - see [3]. LeVeque (1952) proved that, if $k = 1$, then equation (1) has at most one solution, except when $a = 3$ and $b = 2$, when it has 2 solutions ($x = y = 1$ and $x = 2, y = 3$). I will consider a particular case of equation (1), namely when $a = 7, b = 3, k = 100$. Scott and Styer proved in [2] that if a, b are primes and if $b \equiv 3 \pmod 4$, then equation (1) has at most one solution, except for the cases (2,3,5), (2,3,13) and (13,3,10). Scott and Styer's theorem immediately proves that the solution of the equation considered in the present article is unique. My interest in the particular case of the equation began when I came across [1], where four proofs for it are given, each with a different approach (these various approaches retain the charm of the problem). I wondered if there are any proofs involving a computer program. The first solution presented in the article, due to J.L. Brenner and L.L. Foster, uses congruences modulo 729, 243, 487 and 1459. The second one, which was given by Alexandru Gica, uses the fact that $7^{3^k} \equiv 1 - 3^{k+1} \pmod{3^{k+2}}$ and the properties of the Legendre symbol to compute $\left(\dfrac{916}{1459}\right)$. Ovidiu Avădanei, at the time an undergraduate student, used the norm-euclidean ring $\mathbb{Z}[\frac{1+i\cdot\sqrt{3}}{2}]$ and the fact that $10 + 3^{\frac{y-1}{2}} i\sqrt{3}$ and $10 - 3^{\frac{y-1}{2}} i\sqrt{3}$ are relatively prime in that ring. The fourth proof involves the following theorem: the diophantine equation $u^2 + 3v^2 = t^3$ has the solution $(u, v, t) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with t odd and u, v relatively prime if and only if there exist $a, b \in \mathbb{Z}$, of different parities and a and $3b$ are relatively prime, such that $u = a \cdot (a^2 - 9b^2),\ v = 3b \cdot (a^2 - b^2),\ t = a^2 + 3b^2$.

A Computer-Based Proof

My method also uses congruences, but not modulo 1459, and also makes use of the following lemma:

Lemma 1 [5] **(Lifting The Exponent)** *Suppose x and y are integers, n is a natural number and p is a prime such that $p \nmid x$ and $p \nmid y$. Then the following relations hold:*
a) if p is odd and $p \mid x - y$, then $\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n)$
b) if p and n are odd and $p \mid x + y$, then $\nu_p(x^n + y^n) = \nu_p(x + y) + \nu_p(n)$
c) if $p = 2$ and $4 \mid x - y$, then $\nu_2(x^n - y^n) = \nu_2(x - y) + \nu_2(n)$
d) if $p = 2$, n is even and $2 \mid x - y$, then $\nu_2(x^n - y^n) = \nu_2(x - y) + \nu_2(x + y) + \nu_2(n) - 1$
Here, $\nu_p(a)$ denotes the exponent of p in the prime factorisation of a.

I will denote by $\gamma_a(m)$ the order of a modulo m.

Proposition 2 *Suppose x and y are natural numbers. Then the diophantine equation $7^x - 3^y = 100$ has only one solution, namely $(x, y) = (3, 5)$.*

Proof. I will break the proof in 8 steps.
Step 1. I will prove that x is odd.
Let's suppose that x is even. Then $7^x \equiv 1 \pmod 8$ so $3^y \equiv 5 \pmod 8$, which cannot happen.
So x is odd.
Step 2. I will prove that $x \equiv 3 \pmod{12}$ and $y \equiv 5 \pmod{45}$.
Let's reduce the equation modulo 181, which is prime.
Using [4], $\gamma_{181}(7) = 12$ and $\gamma_{181}(3) = 45$.

y	0	1	2	3	4	5	6	7	8
$3^y + 100 \pmod{181}$	101	103	109	127	0	162	105	115	145
y	9	10	11	12	13	14	15	16	17
$3^y + 100 \pmod{181}$	54	143	48	125	175	144	51	134	21
y	18	19	20	21	22	23	24	25	26
$3^y + 100 \pmod{181}$	44	113	139	36	89	67	1	165	114
y	27	28	29	30	31	32	33	34	35
$3^y + 100 \pmod{181}$	142	45	116	148	63	170	129	6	180
y	36	37	38	39	40	41	42	43	44
$3^y + 100 \pmod{181}$	159	96	88	64	173	138	33	80	40

x	0	1	2	3	4	5	6	7	8	9	10	11
$7^x \pmod{181}$	1	7	49	162	48	155	180	174	132	19	133	26

From the tables, we can see that the only possibilities are: $x \equiv 3 \pmod{12}$ and $y \equiv 5 \pmod{45}$ or $x \equiv 4 \pmod{12}$ and $y \equiv 11 \pmod{45}$ or $x \equiv 0 \pmod{12}$ and $y \equiv 24 \pmod{45}$ or $x \equiv 6 \pmod{12}$ and $y \equiv 35 \pmod{45}$.
But from the previous step we know that x is odd, so the only possibility is that $x \equiv 3 \pmod{12}$ and $y \equiv 5 \pmod{45}$.
This implies that there exists some natural numbers t and u such that $x = 12t + 3$ and $y = 45u + 5$.
Further, notice that $(x, y) = (3, 5)$ is a solution and suppose that there is another solution with $y > 5 \implies x > 3 \implies t, u \geq 1$.
Step 3. I will prove that $27 \mid t$ and $98 \mid u$.
$7^{12t+3} - 3^{45u+5} = 100 = 7^3 - 3^5 \implies$
$7^3 \cdot (7^{12t} - 1) = 3^5 \cdot (3^{45u} - 1) \implies$
$3^5 \mid 7^{12t} - 1$ and $7^3 \mid 3^{45u} - 1$
From [4] we know that $\gamma_{243}(7) = 81 \implies 81 \mid 12t \implies 27 \mid 4t \implies 27 \mid t \implies t = 27 \cdot t'$, where $t' \in \mathbb{N}$.
Also from [4] we know that $\gamma_{343}(3) = 294 \implies 294 \mid 45u \implies 98 \mid 15u \implies 98 \mid u \implies u = 98 \cdot u'$, where $u' \in \mathbb{N}$.
These imply that

$$\frac{7^{324t'} - 1}{3^5} = \frac{3^{4410u'} - 1}{7^3} \tag{2}$$

Step 4. I will prove that $27 \mid u'$.
I reduce equation (2) modulo 487, which is a prime number.
From [4] we know that $\gamma_{487}(7) = 162$ and $\gamma_{487}(3) = 486$.
Because $162 \mid 324t'$, $487 \mid 7^{324t'} - 1$. This implies that:
$487 \mid \frac{7^{324t'}-1}{3^5} \implies 487 \mid \frac{3^{4410u'}-1}{7^3} \implies 487 \mid 3^{4410u'} - 1 \implies 486 \mid 4410u' \implies 27 \mid 245u' \implies 27 \mid u' \implies u' = 27 \cdot u''$, where $u'' \in \mathbb{N}$
Step 5. I will prove that t' is even.
For this, let's reduce equation (2) modulo the prime 1297.
From [4] we know that $\gamma_{1297}(3) = 162$ and $\gamma_{1297}(7) = 648$.
Because $162 \mid 4410 \cdot 27 \cdot u''$, $1297 \mid 3^{4410 \cdot 27 \cdot u''} - 1 \implies 1297 \mid \frac{3^{4410 \cdot 27 \cdot u''}-1}{7^3} \implies 1297 \mid \frac{7^{324t'}-1}{3^5} \implies 1297 \mid 7^{324t'} - 1 \implies 648 \mid 324t' \implies t'$ is even, which implies that there exists $t'' \in \mathbb{N}$ such that $t' = 2t''$.
Step 6. I will prove that $8 \mid u''$.
I use point d) of the lemma for the numbers $7^{648t''} - 1$ and $3^{4410 \cdot 27 \cdot u''} - 1$:
$\nu_2(6) + \nu_2(8) + \nu_2(648t'') - 1 = \nu_2(2) + \nu_2(4) + \nu_2(4410 \cdot 27 \cdot u'') - 1 \implies 3 + \nu_2(t'') = \nu_2(u'') \implies \nu_2(u'') \geq 3 \implies 8 \mid u'' \implies u'' = 8u_1$, where $u_1 \in \mathbb{N}$
Step 7. I will prove that $3 \nmid t''$.
For this, I use part a) of the lemma for the number $7^{648t''} - 1$, with $p = 3$:
$5 = \nu_3(7^{648t''} - 1) = \nu_3(6) + \nu_3(648t'') \implies \nu_3(t'') = 0 \implies 3 \nmid t''$
Step 8. I will prove that $3 \mid t''$, which is a contradiction.
I reduce equation (2) modulo the prime 1905121. I chose this prime because it is congruent to 1 modulo $4410 \cdot 27 \cdot 8$.

We know from [4] that $\gamma_{1905121}(3) = \gamma_{1905121}(7) = 476280$. Because $476280 \mid 4410\cdot 27\cdot 8\cdot u_1$, this implies:
$1905121 \mid 3^{4410\cdot 27\cdot 8\cdot u_1} - 1 \implies 1905121 \mid \frac{3^{4410\cdot 27\cdot 8\cdot u_1}-1}{7^3} \implies 1905121 \mid \frac{7^{648t''}-1}{3^5} \implies 1905121 \mid 7^{648t''} - 1 \implies 476280 \mid 648t'' \implies 3 \mid t''$
To conclude, the only solution to the diophantine equation $7^x - 3^y = 100$ is $(x, y) = (3, 5)$.
□

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References

[1] M. Ţena, Ecuaţia diofantică $7^x = 3^y + 100$, *Gazeta Matematică-B*, Anul CXIX, **10** (2014), available online at the URL: `https://ssmr.ro/gazeta/gmb/2014/10/articol.pdf`

[2] R. Scott, R. Styer, On $p^x - q^y = c$ and related three term exponential Diophantine equations with prime bases, *J. Number Theory*, **105** (2004), 212–234.

[3] T.N. Shorey, R. Tijdeman, *Exponential diophantine equations*, Cambridge University Press, 1986.

[4] K. Matthews, Finding the order of $a \pmod m$, program available online at the URL: `http://www.numbertheory.org/php/order.html`

[5] `https://en.wikipedia.org/wiki/Lifting-the-exponent_lemma`

Andreea Dima
University of Bucharest
14 Academiei Street
Bucharest, Romania
E-mail: `andreeadima21@yahoo.com`

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