Grim Under a Compensation Variant

A. Davis, E.A. Donovan, and L. Hoots

Abstract - Games on graphs are a well studied subset of combinatorial games. When analyzing a game, heuristics and strategies for winning are often at the forefront of the discussion. One such combinatorial graph game that can be considered is Grim. In Grim there are winning strategies for a variety of well-known families of graphs, many of which favor the first player. Hoping to develop a fairer Grim, we look at Grim played under a slightly different rule set and develop winning strategies for this modified version of the game on various classes of graphs. Throughout, we compare our new results to those previously known and discuss whether our altered Grim is a fairer game than the original.

Keywords : graphs; games on graphs; combinatorial games; alternate turn order

Mathematics Subject Classification (2020) **:** 05C57; 91A46.

1 Introduction

Games have enthralled humankind for almost five millennia. From the Game of Ur to today's professional sports, games have been used to engage one's mind in play whether for entertainment, education, or both. Beyond their longstanding cultural relevance, the study of games have had a substantial impact on mathematics. The study of games of chance gave rise to the field of probability some 300 years ago. Games without a component of chance, such as chess, have only been studied formally by mathematicians for less than a century; giving rise to the field of game theory. One such branch of game theory, combinatorial game theory, will be used in the study of the game under consideration in this paper.

In combinatorial game theory, a two-player game is said to be *combinatorial* if no moves are determined by chance, both players know all possible moves, the game will end after a finite number of moves, and the outcome only depends on which player goes first. We can assume each will play *optimally*, meaning a player will always choose a move that will lead them to win, if possible. Note, if one player does not have a strategy to win, then the other player must have a winning strategy. These definitions and more information can be found in [2].

In this paper we are focused on studying Grim, a game played on graphs. A graph G, denoted by $G = (V, E)$, consists of two sets: V (or $V(G)$ to more readily identify the graph under consideration), a set of vertices; and E, a set of edges, where each edge in this set connects a pair of vertices in V. The *degree* of a vertex is the number of edges incident to this vertex. If a vertex has degree zero, it is *isolated*, if it has degree one it is called a *pendant* vertex.

The ideas behind Grim were first proposed in Fukuyama's work [5, 6] though no formal definition was given until Adams et al. in [1]. Let us now define the game of Grim.

Definition 1.1 *In Grim, a legal move on a graph G consists of deleting a vertex v along with all edges incident to v, as well as any isolated vertices resulting from this vertex deletion.*

When playing under standard turn order, Grim is played by two players on an initial graph G, where Player 1 makes the first move and then the players alternate taking turns. If G initially has any isolated vertices, they are removed before the game begins. The player making the last legal move is the winner on the graph G.

In [1], Adams, et. al determined winning strategies for Grim on a variety of common families of graphs. (See [8] for definitions of these families.) Barretto, Basi, and Miyake [3] extended these results to some additional multipartite graphs. The relevant results are summarized in Section 2 below. This section concludes with two new results that are crucial for later results in this paper.

In Section 3 we play Grim on many well-known families of graphs but use an alternate turn order whose intent is to compensate for the advantage afforded to Player 1 by moving first on certain graph classes. We highlight the changes in comparison to the known results mentioned in Section 2. We close the paper in Section 4, providing direction for future work with the game of Grim.

2 Standard Turn Order Results

For graphs G and H, if H is obtained from G after a legal move in Grim, then we call H a *follower* of G. When playing under standard turn order, players take turns creating such followers as the game alternates back and forth between players making moves. (Symbolically, we can write this turn order as $1, 2, 1, 2, 1, 2, \ldots$)

As in any competitive game, each player is seeking out an optimal strategy so they can win. This leads to the following notation, as defined in [2]. (A more in-depth study of combinatorial games can be found in [2] and [4].)

Definition 2.1 *A game is a $\mathcal{P}$ position if the previous player has a winning strategy. A game is an $\mathcal{N}$ position if the next player has a winning strategy. This notation has three properties:*

1. *Every move from a $\mathcal{P}$ position goes to an $\mathcal{N}$ position.*
2. *There exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position.*
3. *The terminal position for the game is a $\mathcal{P}$ position.*

Results for many common families of graphs are already known for Grim. In [1], Adams et al. determined the winner under the standard turn order and optimal play for paths, cycles, wheels, and complete graphs. These results are compiled into the following theorem:

Theorem 2.2 *Let $n \in \mathbb{N}$.*

1. *If n is odd, and $n \geq 3$, then P_n is an $\mathcal{N}$ position.*
2. *If n is even, and $n \geq 4$, then C_n is a $\mathcal{P}$ position.*
3. *If n is odd, and $n \geq 5$, then W_n is an $\mathcal{N}$ position.*
4. *K_n is an $\mathcal{N}$ position if and only if n is even.*

Additionally, Adams et al. provided some results on complete multipartite graphs. Their results, focused on complete graphs with two, three, or even more partitions, are summarized in Theorem 2.3.

Theorem 2.3 *Let $m, n, n_i \in \mathbb{N}$ for all $i = 1, \ldots, t$.*

1. *$K_{1,n}$ is an $\mathcal{N}$ position for all $|V|$.*
2. *Assume $m, n \geq 2$. $K_{m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*
3. *$K_{1,1,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*
4. *Assume $n \geq 2$. Then $K_{1,2,n}$ is an $\mathcal{N}$ position.*
5. *Assume $m, n \geq 3$. Then $K_{1,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*
6. *Assume $t \geq 4$ and $n_i \geq 3$, for all $i = 2, \ldots, t$. Then $K_{1,n_2,\ldots,n_t}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*
7. *Assume $t \geq 3$ and $n_i \geq 2$, for all $i = 1, \ldots, t$. Then $K_{n_1,n_2,\ldots,n_t}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

Barretto, Basi, and Miyake added to the results on complete multipartite graphs for Grim under standard turn order in [3]. The most relevant of these results, focused on complete graphs containing four partitions, are compiled in the theorem below.

Theorem 2.4 *Let $m, n \in \mathbb{N}$.*

1. *For $n \geq 1$, $K_{1,1,1,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*
2. *For $n \geq 2$, $K_{1,1,2,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*
3. *For $n \geq 3$, $K_{1,1,3,n}$ is an $\mathcal{N}$ position.*
4. *For $m, n \geq 4$, $K_{1,1,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

We now switch our focus to graphs composed of the disjoint union of two graphs. In [1], Adams et al. provided the following result for such a graph.

Theorem 2.5 *$G \cup G$ is a $\mathcal{P}$ position, for all non-empty graphs G.*

We have extended these results to include the disjoint union of two different graphs. These new properties for Grim under standard turn order rules are helpful in proving results about the different turn order we introduce in Section 3.

Theorem 2.6 *Under normal rules of Grim, if a graph G is the disjoint union of two graphs, G_1 and G_2, where G_1 and G_2 are both $\mathcal{P}$ positions, then G is a $\mathcal{P}$ position.*

Proof. By Definition 2.1, every move from a $\mathcal{P}$ position goes to an $\mathcal{N}$ position. Without loss of generality, we say that Player 1's move creates a follower of G_2 that is an $\mathcal{N}$ position. Thus, the follower of G is the disjoint union of two graphs, one that is a $\mathcal{P}$ position and one that is an $\mathcal{N}$ position.

Again, by Definition 2.1, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Player 2 should take this move on G_2, resulting in a follower that is the disjoint union of two graphs, both $\mathcal{P}$ positions.

Since this is how the game started, Player 2 can continue to follow this pattern. By Definition 2.1, the terminal position for a graph is a $\mathcal{P}$ position. Eventually, Player 2 will make a move that completely eliminates one of the disjoint graphs, since they are the only ones that are able to create $\mathcal{P}$ position followers. The follower of this move is a $\mathcal{P}$ position graph. Since it is now Player 1's turn and the remaining graph is a $\mathcal{P}$ position, Player 2 will win. Since Player 2 has a strategy to win G, G is a $\mathcal{P}$ position. □

Corollary 2.7 *Under normal rules of Grim, if a graph G is the union of 2 graphs, G_1 and G_2, where G_1 is a $\mathcal{P}$ position and G_2 is an $\mathcal{N}$ position, then G is an $\mathcal{N}$ position.*

Proof. By Definition 2.1, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Player 1 should take this move on G_2. Thus, the follower is the disjoint union of two graphs and both are $\mathcal{P}$ positions. By Theorem 2.6, this follower is a $\mathcal{P}$ position. By Definition 2.1, every move from a $\mathcal{P}$ position must go to an $\mathcal{N}$ position. Thus, G is not a $\mathcal{P}$ position and must then be an $\mathcal{N}$ position. □

3 Results on the Compensation Turn Order

Has it ever seemed like the first person to play in most games has an unfair advantage? Suppose you are choosing teams, the first person has an edge because they can choose the very best person for the activity. They have more options than every person after them. The same is true in Grim and in other impartial games. In Grim, the first player has more vertices to select for removal than the second player, allowing Player 1 to win more games. In all of the specific families of graphs studied in [1] and [3], the player with a winning strategy was either determined by the parity of the vertex set or the player who went first had a winning strategy. This would imply that the first player has an advantage in Grim. Gaines and Welsh, in [7], studied an alternative turn order that is intended to compensate for this first-player advantage. Their definition of this new turn order is given below.

Definition 3.1 *In the **compensation turn order**, Player 1 takes the first move, then Player 2 gets both the second and third move. The game then returns to normal alternating order on the fourth move, with Player 1's turn. The order is* $1, 2, 2, 1, 2, 1, 2, \ldots$, *symbolically.*

The intention of this new turn order is that this extra move will compensate for the advantage inherent in moving first. Gaines and Welsh found that for several impartial games, this compensation turn order had too pronounced of an effect and appeared to give Player 2 a distinct advantage. For the remainder of this work, we study the effect of this compensation turn order on the game of Grim and determine if it disproportionately favors Player 2 as well. The following result from Gaines and Welsh will be of use.

Theorem 3.2 *Let G be an impartial game, played with the compensation variant. If the first player can on their first turn go to either a terminal position, or to some $\mathcal{N}$ position with no possible moves to other $\mathcal{N}$ positions or terminal positions, then the first player will win. Otherwise, the second player will win.*

This result leads directly to the following observation that we will use extensively.

Observation 3.3 By Theorem 3.2, we can determine the player with the winning strategy within the first two moves of an impartial game:

- If the game has terminated, the last player to have made a legal move has a winning strategy;
- If the game is a $\mathcal{P}$ position after Player 2's first move, Player 1 has a winning strategy;
- If the game is an $\mathcal{N}$ position after Player 2's first move, Player 2 has a winning strategy.

It should be noted that these last two points combined constitute an "if and only if" statement. This observation, combined with the definition of $\mathcal{P}$ position, lead to the following key observation.

Observation 3.4 For playing Grim under compensation turn order, if Player 2 is given a $\mathcal{P}$ position graph on their first turn, then Player 2 has a winning strategy.

We will use these ideas to obtain results on multiple families of graphs. We will begin by focusing on paths, cycles, wheels, and complete graphs which all have corresponding results under standard turn order as noted in Theorem 2.2 from [1].

3.1 Compensation Order on Paths, Cycles, Wheels, and Complete Graphs

Let us begin our exploration of the compensation turn order with paths. As a note, since the notion of a graph being a $\mathcal{P}$ position or an $\mathcal{N}$ position does not make much sense until after Player 2's first move, our results will be phrased to state which player has a winning strategy. Previous results from [1] and [3] denoted graphs as a $\mathcal{P}$ or an $\mathcal{N}$ position, or, in our wording, Player 2 or Player 1 having a winning strategy, respectively.

Theorem 3.5 *For all $n \geq 4$, Player 2 has a winning strategy on P_n.*

Proof. Player 1 creates a follower that is either a path, or the disjoint union of two paths.

Suppose Player 1 creates a single path as a follower. If $n = 4$, then Player 1 creates the follower P_2 or P_3. In either case, Player 2 can win in a single move. If $n = 5$, then Player 1 creates the follower P_3 or P_4. As before, Player 2 can win in one move if given P_3. If given P_4, Player 2 should create the follower P_3 which is an $\mathcal{N}$ position by Theorem 2.2. If $n \geq 6$, Player 1 creates the follower P_{n-1} or P_{n-2}. If the number of vertices in this path is even, Player 2 removes a pendant vertex to create a path of with an odd number of vertices. Similarly, if the number of vertices in the follower is odd, Player 2 removes a vertex adjacent to a pendant vertex (thereby deleting two vertices), again creating a path with an odd number of vertices. In either case, the path Player 2 creates is an $\mathcal{N}$ position by Theorem 2.2. Hence, by Observation 3.3, Player 2 has a winning strategy.

Suppose $n \geq 5$ and Player 1 creates a disjoint union of two paths as a follower (note this is not possible if $n = 4$). Call the graphs G_1 and G_2. There are three possible cases: Either G_1 and G_2 are both $\mathcal{P}$ positions, G_1 and G_2 are both $\mathcal{N}$ positions, or one of G_1 and G_2 is a $\mathcal{P}$ position while the other is an $\mathcal{N}$ position.

Case 1: By Theorem 2.6, the union of two graphs that are both $\mathcal{P}$ positions is a $\mathcal{P}$ position. Thus, Player 2 has a strategy to win by Observation 3.4.

Case 2: By Definition 2.1, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Without loss of generality, Player 2 makes this move on G_1, creating a follower that is the disjoint union of two graphs, where one is a $\mathcal{P}$ position and the other is an $\mathcal{N}$ position. By Corollary 2.7, this graph is an $\mathcal{N}$ position and, by Observation 3.3, Player 2 has a winning strategy.

Case 3: Without loss of generality, assume G_1 is a $\mathcal{P}$ position and G_2 is an $\mathcal{N}$ position. Note that since P_2 and P_3 are $\mathcal{N}$ position graphs, the number of vertices in G_1 must be at least 4 and, by Theorem 2.2, must be even.

If G_2 contains 4 or more vertices, then, reasoning as above, Player 2 removes either 1 or 2 vertices to create a path of odd length, which is an $\mathcal{N}$ position by Theorem 2.2. If $G_2 = P_3$, then Player 2 removes a single vertex to create P_2, which is an $\mathcal{N}$ position. In either case, Player 2 creates the disjoint union of two graphs, where one is a $\mathcal{P}$ position and the other is an $\mathcal{N}$ position. By Corollary 2.7, this graph is an $\mathcal{N}$ position and, by Observation 3.3, Player 2 has a winning strategy.

If $G_2 = P_2$ and $G_1 = P_4$, then Player 2 should create the follower $P_2 \cup P_3$ with their first move, then the follower $P_2 \cup P_2$ with their next. This graph is a $\mathcal{P}$ position by

Theorem 2.5, which means that Player 2 has a winning strategy.

If $G_2 = P_2$ and $G_1 = P_k$ where $k \geq 6$, then Player 2 should create the follower $P_2 \cup (P_2 \cup P_{k-3}) = (P_2 \cup P_2) \cup P_{k-3}$. Note that since k was even, $k-3$ is odd. By Corollary 2.5, $P_2 \cup P_2$ is a $\mathcal{P}$ position and by Theorem 2.2, P_{k-3} is an $\mathcal{N}$ position. By Corollary 2.7, this follower is an $\mathcal{N}$ position and by Observation 3.3, Player 2 has a winning strategy.

We have shown that Player 2 has a winning strategy in every case. Therefore, Player 2 has a winning strategy for all P_n where $n \geq 4$. □

The above yields a surprising result when compared to playing Grim under the standard turn order on paths. Under standard turn order, Player 1 has a winning strategy for graphs of odd order under normal play (see Theorem 2.2). Moreover, in [1], Adams et al. used the Sprague-Grundy function to connect playing Grim on paths to another type of impartial game known as an Octal game, thus showing that there was no discernible pattern for determining the winner on paths of even length under standard turn order. Here, under the compensation variant, we see that not only does one player have a winning strategy for all paths of order greater than three, but that player with a winning strategy is actually Player 2. This is a significant shift from standard turn order results.

Grim played on cycles under the compensation turn order will be analyzed next.

Theorem 3.6 *For all n, Player 2 has a strategy to win on C_n.*

Proof. For $n = 3$ or $n = 4$, Player 1 creates the follower P_2 or P_3, respectively. Either of which can be eliminated by Player 2 on their first move. If $n \geq 5$, Player 1 creates the follower P_{n-1}. Arguing similarly to the proof in Theorem 3.5, Player 2 deletes a pendant vertex or a vertex adjacent to a pendant vertex as needed to create a path with an odd number of vertices. This path will be an $\mathcal{N}$ position by Theorem 2.2. Hence, by Observation 3.3, Player 2 has a winning strategy.

□

Again, the additional turn afforded Player 2 impacts the outcome of the game when playing optimally. Theorem 2.2 states that Player 2 has a winning strategy if the order of the cycle is even. Under the compensation turn order, Player 2 wins on any cycle. Similar to the family of paths, the family of cycles favors Player 2 as the winner.

We will now determine who has a winning strategy on a wheel graph under the compensation variant.

Theorem 3.7 *If n is odd and $n \geq 5$, then Player 2 has a strategy to win on W_n.*

Proof. Suppose $n \geq 5$ and n is odd. Player 1 can either delete the hub or a vertex on the exterior cycle. If Player 1 deletes the hub, then Player 2 will remove a vertex on the exterior cycle. If Player 1 deletes a vertex on the exterior cycle, then Player 2 will delete the vertex that served as the hub. In either case, Player 2 creates the follower P_{n-2}. Since $n \geq 5$ and odd, $n-2 \geq 3$ and is also odd. By Theorem 2.2, P_{n-2} is an $\mathcal{N}$ position. Hence, by Observation 3.3, Player 2 has a winning strategy.

□

In Theorem 2.2, if the number of vertices in a wheel graph is odd, Player 1 has a winning strategy under normal play. When playing using the compensation turn order, this winner is flipped: Player 2 now has a winning strategy when playing Grim on odd wheels. Note that no conclusion has been made for wheels with an even number of vertices. Like Grim on paths under normal turn order, there does not seem to be a discernible pattern to who has an optimal strategy in this game.

Our last result in this section involves complete graphs. Due to the high connectivity of this family, playing Grim on this graph does not allow for additional strategy for Player 2 with the inclusion of the extra turn. Thus, we find that the player with a winning strategy is again tied to the parity of the vertex set, though the winner is reversed from that of the standard turn order rules.

Theorem 3.8 *For $n \geq 4$, Player 2 has a winning strategy on K_n if and only if n is even.*

Proof. By the nature of complete graphs, for $n \geq 4$, regardless of which vertex Player 1 removes, the follower is K_{n-1}. Similarly, Player 2 must create the follower K_{n-2}. Since $n-2$ is even if and only if n is even, Theorem 2.2 implies K_{n-2} is an $\mathcal{N}$ position if and only if n is even. Hence, by Observation 3.3, Player 2 has a winning strategy if and only if n is even. □

3.2 Complete Multipartite Graphs

We will now explore Grim under the compensation variant for various complete multipartite graphs. When playing Grim under the standard turn order, the player who had a winning strategy on many complete multipartite graphs is directly tied to the parity of the vertex set, $|V|$. This dependency on parity also arises under the compensation variant. We begin with complete multipartite graphs with two components.

Theorem 3.9 *For $m \geq 3$ and $n \geq 4$, Player 2 has a winning strategy on $K_{m,n}$ if and only if $|V|$ is odd.*

Proof. Player 1 can create the follower $K_{m-1,n}$ or $K_{m,n-1}$. If $m = 3$ and $n \geq 4$, then Player 1 would not create the follower $K_{m-1,n} = K_{2,n}$ under optimal play since Player 2 could delete the partition of size 2 with their two moves, giving Player 2 a winning strategy regardless of the parity of the vertex set. Since Player 1 would leave the follower $K_{m,n-1}$ instead, Player 2 can create the follower $K_{m-1,n-1}$ or $K_{m,n-2}$. In either case, since $m = 3$ and $n \geq 4$, we know $m \geq m-1 \geq 2$ and $n-1 \geq n-2 \geq 2$. Furthermore, since $m+n-2$, is odd if and only if $|V| = m+n$ is odd, we can apply Theorem 2.3 and conclude the graph is an $\mathcal{N}$ position if and only if $m+n$ is odd.

Suppose $m \geq 4$ and $n \geq 4$. Without loss of generality, suppose Player 1 creates the follower $K_{m,n-1}$. Player 2 can create the followers $K_{m,n-2}$ or $K_{m-1,n-1}$ with their first move.

If Player 2 creates $K_{m,n-2}$, since $n \geq 4$ we we know $n-2 \geq 2$. Since the number of vertices in $K_{m,n-2}$, $m+n-2$, is odd if and only if $|V| = m+n$ is odd, we can apply Theorem 2.3. Hence,$K_{m,n-2}$ is an $\mathcal{N}$ position if and only if $m+n$ is odd.

Similarly, If Player 2 creates $K_{m-1,n-1}$, since $m, n \geq 4$, then $m-1 \geq 3$ and $n-1 \geq 3$. Again, the number of vertices in $K_{m-1,n-1}$, $m+n-2$, is odd if and only if $|V| = m+n$ is odd. Theorem 2.3 implies $K_{m-1,n-1}$ is an $\mathcal{N}$ position if and only if $m+n$ is odd. Hence, by Observation 3.3, Player 2 has a winning strategy if and only if $m+n$ is odd. □

The switch from the standard turn order to the compensation turn order caused a reversal in who has a winning strategy on $K_{m,n}$ based on parity. Player 1 wins when there are an odd number of vertices under the standard order and Player 2 wins when there are an odd number of vertices under the compensation variant. The notable exceptions are $K_{1,n}$, where Player 1 has a winning strategy regardless of turn order, $K_{2,n}$, where Player 2 has a winning strategy under the compensation variant (they can remove the entire partition of size 2 regardless of how Player 1 begins the game) as opposed to the standard turn order where the winner was dependent upon the parity of the vertex set, and Player 2 also winning strategy on $K_{3,3}$ under the compensation variant (by deleting the partition of size two that Player 1 leaves behind regardless of what move they make) which, like the $K_{1,n}$ case for Player 1, doesn't change between the two turn orders.

When we begin looking at complete multipartite graphs with three partitions, the advantage afforded Player 2 by the compensation turn order becomes more apparent. Under the standard turn order, Player 2 has a winning strategy on $K_{1,1,n}$ when the graph has an odd number of vertices. However, under the compensations variant, Player 2 has a winning strategy regardless of the size of the vertex set as they can delete the components of size one. The following theorem shows that Player 2 also has a winning strategy on $K_{1,2,n}$, which was always won by Player 1 under the standard turn order.

Theorem 3.10 *Player 2 has a winning strategy on* $K_{1,2,n}$.

Proof. Observe that, regardless of what move Player 1 makes, Player 2 can win by the end of their second turn on the graph $K_{1,2,2}$. The same is true of $K_{1,2,3}$: if Player 1 makes the follower $K_{2,3}$ or $K_{1,1,3}$, Player 2 can win by the end of their second move. If Player 1 creates $K_{1,2,2}$ the game returns to Player 1 before Player 2 can win: Player 2 will create $K_{1,1,1}$ with their two moves which forces Player 1 to create the follower $K_{1,1}$ (or P_2). This game is then terminated by Player 2. See Figure 1 for this scenario.

If $n \geq 4$, the three possible followers Player 1 can create after their first turn are $K_{2,n}$, $K_{1,1,n}$, or $K_{1,2,n-1}$. For the first two graphs, Player 2 wins on their second turn by removing the partition of size two or the two partitions of size one, respectively.

If Player 1 creates the follower $K_{1,2,n-1}$, then Player 2 should create the follower $K_{1,2,n-2}$ on their first move. Since $n \geq 4$, $n-2 \geq 2$, by Lemma 2.3, this follower is an $\mathcal{N}$ position and by Observation 3.3, Player 2 has a winning strategy. □

In the following three theorems, the player with a winning strategy is completely determined by the parity of the vertex set. The only change from standard play is that instead of Player 1 winning when there is an odd number of vertices and Player 2 winning when there is an even number of vertices, Player 2 wins when $|V|$ is odd and Player 1

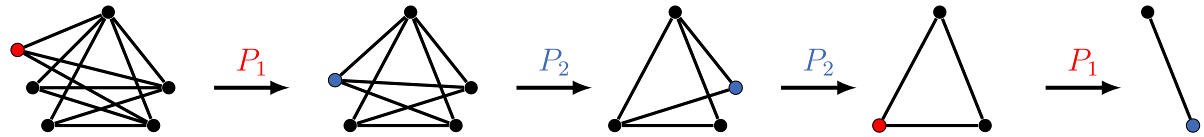

Figure 1: Player 2 has a winning strategy on $K_{1,2,3}$ when Player 1 creates the follower $K_{1,2,2}$.

when $|V|$ is even. This is evidence that, as we saw with K_n in Theorem 3.8, if a graph displays enough structure, the winner is simply the optimal player who happens to take the last move.

Theorem 3.11 *For $m, n \geq 3$, Player 2 has a winning strategy on $K_{1,m,n}$ if and only if $|V|$ is odd.*

Proof. Suppose $m, n \geq 3$ and $|V| = 1+m+n$ is odd. The two possible followers Player 1 can create are $K_{m,n}$ and, without loss of generality, $K_{1,m-1,n}$.

If Player 1 creates $K_{1,m-1,n}$ and $m = 3$, Player 2 can create $K_{1,m-1,n-1} = K_{1,2,n-1}$. If Player 1 creates $K_{1,m-1,n}$ and $m = 4$, Player 2 can create $K_{1,m-2,n-1} = K_{1,2,n}$. In either case, Player 2 wins for all $n \geq 3$ by Theorem 2.3, which means Player 1 would not create the follower $K_{1,m-1,n}$ in these cases since they could win for some values of n by creating a different follower (see below). If $m \geq 5$ and Player 1 creates $K_{1,m-1,n}$, then Player 2 could create $K_{m-1,n}$, $K_{1,m-2,n}$, or $K_{1,m-1,n-1}$. Theorem 2.3 implies all of these graphs are in $\mathcal{N}$ position if and only if $|V|$ is odd. Hence, by Observation 3.3, Player 2 has a winning strategy if and only if $|V|$ is odd.

If Player 1 creates the follower $K_{m,n}$, then, without loss of generality, Player 2 creates the follower $K_{m-1,n}$ on their first turn. Since $1+m+n-2 = (m-1)+n$ is odd if and only if $1+m+n$ is odd (also $m-1, n \geq 2$ since $m, n \geq 3$), so by Theorem 2.3, the graph is an $\mathcal{N}$ position if and only if $|V|$ is odd. Hence, by Observation 3.3, Player 2 has a winning strategy if and only if $|V|$ is odd. $\square$

Theorem 3.12 *Let $G = K_{1,n_2,\ldots,n_t}$, where $n_i \in \mathbb{N}$, $t \geq 4$, and $n_i \geq 5$, for all $i = 2, \ldots, t$. Then Player 2 has a winning strategy on G if and only if $|V|$ is odd.*

Proof. Assume $|V| = 1+n_2+\ldots+n_t$ is odd. Player 1 can create the follower $K_{n_2,n_3,\ldots,n_t}$ or (without loss of generality) $K_{1,n_2-1,n_3,\ldots,n_t}$.

If Player 1 creates $K_{n_2,n_3,\ldots,n_t}$ then Player 2 can create, without loss of generality, the follower $K_{n_2-1,n_3,\ldots,n_t}$. Since $t-1 \geq 3$, $n_2-1, n_3, \ldots, n_t \geq 2$, and $|V|-2$ is odd if and only if $|V|$ is odd, we know by Theorem 2.3 $K_{n_2-1,n_3,\ldots,n_t}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

If Player 1 creates the follower $K_{1,n_2-1,n_3,\ldots,n_t}$ then Player 2 could create $K_{n_2-1,n_3,\ldots,n_t}$, $K_{1,n_2-2,n_3,\ldots,n_t}$, or $K_{1,n_2-1,n_3-1,\ldots,n_t}$. The first case is identical to the case above. In the other two cases, since $t \geq 4$, $n_2-1, n_2-2, n_3-1, n_3, \ldots, n_t \geq 3$, and $|V|-2$ is odd if and only if $|V|$ is odd, we know by Theorem 2.3 $K_{n_2-1,n_3,\ldots,n_t}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd. Hence, by Observation 3.3, Player 2 has a winning strategy if and only if $|V|$ is odd. $\square$

Theorem 3.13 *Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_i \in \mathbb{N}$ for all $i = 1,\ldots,t$. Assume $t \geq 3$ and $n_i \geq 2$, for all $i = 1,\ldots,t$. Then Player 2 has a winning strategy on G if and only if $|V|$ is odd.*

Proof. Assume $|V| = n_1 + n_2 + \ldots + n_t$ is odd. Without loss of generality, assume Player 1 removes a vertex from the partition of size n_1. If $n_1 = 2$ then this leaves the follower $K_{1,n_2,\ldots,n_t}$ where $1 + n_2 + \ldots + n_t$ is even, thus by Theorem 2.3 the follower is in $\mathcal{P}$ Position and, by Observation 3.4 Player 2 has a winning strategy if and only if $|V|$ is odd. If $n_1 \geq 3$ then this leaves the follower $K_{n_1-1,n_2,\ldots,n_t}$ where $1 + n_2 + \ldots + n_t$ is even and $n_1 - 1, n_2, n_3, \ldots, n_t \geq 2$ so, by Theorem 2.3, the follower is a $\mathcal{P}$ position and, by Observation 3.4 Player 2 has a winning strategy if and only if $|V|$ is odd. In either case, Player 2 has a winning strategy if and only if $|V|$ is odd. □

We conclude our analysis of playing Grim under the compensation variant by looking at 4-partite graphs. These reflect results found by Barretto, Basi, and Miyake in [3]. The only change when switching to the compensation variant is that parities of the vertex set that originally gave Player 1 a winning strategy now give Player 2 a winning strategy and vice versa. This switch is especially noteworthy on $K_{1,1,3,n}$, where Player 1 originally won for all $n \geq 3$ in the standard turn order and now Player 2 wins for all $n \geq 3$, once again highlighting the advantage given to Player 2 by the compensation variant.

Theorem 3.14 *Let $G = K_{1,1,1,n}$, then Player 2 has a winning strategy on G if and only if $|V|$ is even.*

Proof. Assume $|V| = 1 + 1 + 1 + n$ is even. First, consider the case where $n = 1$. Then $G = K_4$ and Player 2 has a winning strategy by Theorem 3.8.

Now, let $n \geq 2$ and recall $|V|$ is even. Player 1 can either create the follower $K_{1,1,n}$ or $K_{1,1,1,n-1}$. Now, $K_{1,1,n}$ is a $\mathcal{P}$ position by Theorem 2.3 and $K_{1,1,1,n-1}$ is a $\mathcal{P}$ position by Theorem 2.4 if and only if $n - 1 \geq 1$ and $|V| - 1$ is odd. By Observation 3.4 Player 2 has a winning strategy on $K_{1,1,1,n}$ if and only if $|V|$ is even. □

Theorem 3.15 *For $n \geq 2$, Player 2 has a winning strategy on $K_{1,1,2,n}$ if and only if $|V|$ is even.*

Proof. If $n = 2$ then Player 1 can create the followers: $K_{1,2,2}$, or $K_{1,1,1,2}$. From these, Player 2 can use their first move to create the followers $K_{2,2}$, $K_{1,1,2}$, or $K_{1,1,1,1}$. Player 2 wouldn't create $K_{2,2}$ as this results in a win for Player 1, Player 2 wins in all other cases. So Player 2 has a winning strategy.

Suppose $n \geq 3$ and $|V| = 1 + 1 + 2 + n$ is even. Player 1 can create three different followers: $K_{1,1,2,n-1}$, $K_{1,1,1,n}$, or $K_{1,2,n}$.

For $K_{1,1,2,n-1}$, this graph is a $\mathcal{P}$ position by Theorem 2.4 if and only if $|V| = 1 + 1 + 2 + (n-1)$ is odd. Similarly, $K_{1,1,1,n}$ is also a $\mathcal{P}$ position by Theorem 2.4 if and only if $|V| = 1 + 1 + 1 + n$ is odd. By Observation 3.4, Player 2 has a winning strategy.

For the third possible follower that Player 1 could make, $K_{1,2,n}$, Player 2 can create the followers $K_{1,1,n}$, $K_{2,n}$, or $K_{1,2,n-1}$ from this graph. By Theorem 2.3, $K_{1,1,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even (which is good for Player 2), $K_{2,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd (which is bad for Player 2), and $K_{1,2,n-1}$ is an $\mathcal{N}$ position for all n (which is really good for Player 2). Since creating the follower $K_{1,2,n}$ would give Player 2 a winning strategy for all n, Player 1 would not create this follower when $|V|$ is odd (which the above showed they can win). Hence, Player 2 has a strategy to win if and only if $|V|$ is even. □

Theorem 3.16 *Player 2 has a winning strategy on $K_{1,1,3,n}$ for all $n \geq 3$.*

Proof. If $n \geq 3$ then Player 1 can create three different followers: $K_{1,3,n}$, $K_{1,1,2,n}$, or $K_{1,1,3,n-1}$.

Assume Player 1 creates the follower $K_{1,3,n}$ or $K_{1,1,2,n}$. Then Player 2 can create the follower $K_{1,2,n}$. By Theorem 2.3, $K_{1,2,n}$ is an $\mathcal{N}$ position and by Observation 3.4, Player 2 has a winning strategy. Alternatively, assume that Player 1 creates the follower $K_{1,1,3,n-1}$.

If $n = 3$ then this is $K_{1,1,2,3}$ which is a $\mathcal{P}$ position by Theorem 2.4 and by Observation 3.4, Player 2 has a winning strategy.

If $n = 4$ then this is $K_{1,1,3,3}$. Player 2 can create the follower $K_{1,3,3}$, which is an $\mathcal{N}$ position by Theorem 2.3 and by Observation 3.4, Player 2 has a winning strategy.

If $n \geq 5$ then Player 2 will create the follower $K_{1,1,3,n-2}$. Since $n-2 \geq 3$, this is an $\mathcal{N}$ position by Theorem 2.4 and by Observation 3.4, Player 2 has a winning strategy. □

Theorem 3.17 *Player 1 has a winning strategy on $K_{1,1,m,n}$ for all $m, n \geq 4$ if and only if $|V|$ is even.*

Proof. Assume $|V| = 1+1+m+n$ is even. Player 1 can create the followers; $K_{1,m,n}$ or (without loss of generality) $K_{1,1,m-1,n}$. Since $m, n \geq 3$ and $|V|-1 = 1+m+n$ is odd if and only if $|V| = 1+1+m+n$ is even, Theorem 2.3 implies $K_{1,m,n}$ is a $\mathcal{P}$ position if and only if $|V|$ is even. If $m = 4$, then Player 1 would not create the follower $K_{1,1,m-1,n} = K_{1,1,3,n}$ since Theorem 2.4 implies this is a $\mathcal{P}$ position and Player 1 would lose according to Observation 3.4. If $m \geq 5$, then we would have $m-1, n \geq 4$ and $|V|-1 = 1+1+(m-1)+n$ is odd if and only if $|V| = 1+1+m+n$ is even, Theorem 2.4 implies $K_{1,1,m-1,n}$ is a $\mathcal{P}$ position if and only if $|V|$ is odd. Thus, by Observation 3.4, Player 1 has a winning strategy in both cases and therefore Player 1 has a winning strategy on $K_{1,1,m,n}$ if and only if $|V|$ is odd. □

What we have seen is that, for families of complete multipartite graphs, the switch from the standard turn order to the compensation turn order typically has one of two effects. Either the winner was determined by the parity of the vertex set and the player with the winning strategy on a particular graph switched from one player to the other when shifting from standard turn order to the compensation variant (seen more commonly on graphs with more structure), or Player 1 had a winning strategy under the standard turn order which switched to Player 2 having a winning strategy under the compensation variant. Apart from some individual graphs with small vertex sets (such as $K_{2,2}$ and $K_{3,3}$) and families of graphs with extremely simple structure (such as $K_{1,n}$), where the player with the winning strategy remained the same under both turn orders due to them being able to be terminated in a small number of moves, the the most notable exceptions to this rule we found was the aforementioned $K_{2,n}$ and $K_{1,1,n}$ where the player with the winning strategy was determined by parity of the vertex set under the standard turn order but is always won by Player 2 under the compensation variant; similar to what we saw for paths and cycles in Theorems 3.5 and 3.6.

These results indicate that switching from the standard turn order to the compensation turn order only ever provides a distinct advantage to Player 2. We can think of the parity switches as "neutral" changes where the family of graphs is still split between the two players, even if the player with a winning strategy is reversed from the standard turn order. All other changes we discovered for common families of graphs solely benefit Player 2 when switching to the compensation variant.

4 Conclusion & Future Work

Playing Grim on graphs has raised recent interest in the realm of combinatorial game theory. As such, some notable results have been given in [1] and [3], many of which were listed in Section 2. Like any combinatorial game, we saw in this section that playing multiple games of Grim under the normal turn order was equivalent to playing a single game of Grim on a graph made up of the disjoint union of other graphs. This aligns with what is commonly known in game theory, though our approach is from a graph theoretic perspective.

In Section 3, in order to compensate for Player 1 making the first move in the game, a second move was given to Player 2 after Player 1's first turn. This additional move yielded some interesting results on paths and cycles as seen in Section 3.1. When we moved to more structured graphs, the complete graphs and complete multipartite graphs given in Section 3.2, we found that the structure of a graph largely drove the players' moves and the additional turn given to Player 2 typically had the effect of switching the winner when based on the parity of the graph but in several cases gave Player 2 a winning strategy where Player 1 had a winning strategy under the standard turn order. Overall, the switch from the standard turn order to the compensation variant routinely benefited Player 2 over Player 1, which is consistent with the results of Gaines and Welsh [7].

Additional work can be done on Grim under a compensation turn order on other families of graphs. We noted an apparent trend for the player with a winning strategy was determined by the parity of the vertex set when the graph possessed significant underlying structure. What exactly constitutes "significant underlying structure" is, as of yet, unknown and may be tied to some measure of connectivity. This is an idea that merits further exploration. Moreover, the apparent bias of certain graphs for Player 2 motivates a broader study of graphs, perhaps via a data analysis of random graphs, in order to better define this advantage. There is also the possibility of playing Grim with a different turn order than the two mentioned in this paper or even adding more players to the game. While these extensions may seem novel, recall the changes encountered for paths and cycles: even a small shift in the definition of playing Grim has the potential to lead to interesting results.

References

[1] R. Adams, J. Dixon, J. Elder, J. Peabody, O. Vega, K.Willis, Combinatorial analysis of a subtraction game on graphs, *Int. J. Comb.*, **2016** (2016).

[2] M.H. Albert, R.J. Nowakowski, D. Wolfe, *Lessons in play: an introduction to combinatorial game theory*, CRC Press, 2019.

[3] S. Barretto, J. Basi, M. Miyake, Grim on complete multipartite graphs, *PUMP J. Undergrad. Res.*, **1** (2018), 48–61.

[4] E.R. Berlekamp, J.K. Conway, R.K. Guy, *Winning ways for your mathematical plays, volume 4*, AK Peters/CRC Press, 2004.

[5] M. Fukuyama, A Nim game played on graphs, *Theor. Comput. Sci.*, **304** (2003), 387–399.

[6] M. Fukuyama, A Nim game played on graphs II, *Theor. Comput. Sci.*, **304** (2003), 401–419.

[7] J. Welsh, B. Gaines, Examining a non-standard turn order in impartial games, *MJUM*, **4** (2019).

[8] D.B. West, *Introduction to graph theory*, Prentice Hall, 1996.

Aaron Davis
Murray State University
6C Faculty Hall
Murray, KY 42071
E-mail: `adavis63@murraystate.edu`

Elizabeth A. Donovan
Murray State University
6C Faculty Hall
Murray, KY 42071
E-mail: `edonovan@murraystate.edu`

Lucas Hoots
Morehead State University
Lappin Hall
150 University Blvd.
Morehead, KY 40351
E-mail: `l.hoots@moreheadstate.edu`

Received: February 5, 2022 **Accepted**: November 30, 2024
Communicated by Oscar Vega