# Avoiding Small Monochromatic Distances

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**Abstract -** Is it true that for any coloring of the points of $\mathbb{R}$ in two colors there is an $\epsilon > 0$ such that one of the color classes contains pairs of points at every distance smaller than $\epsilon$? We show that the answer to this question is no.

**Keywords :** Euclidean space; Bernstein sets

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## 1 Introduction

The main result in this paper emerged from a puzzle that was presented in the course "math puzzles" by Rom Pinchasi at the Technion in the fall term of 2021:
*Show that for any coloring of the points of* $\mathbb{R}^2$ *in two colors, one of the color classes must contain pairs of points at every possible distance.*

This problem is the particular case of a more general problem by Raiskii, who in 1970 showed in [3] that in any coloring of $\mathbb{R}^n$ to $n+1$ colors, one of the color classes must contain pairs of points at every possible distance. So Raiskii's result gives a solution to the above puzzle even in the case of three colors. Hence the puzzle itself is not so remarkable and we leave it to the reader to enjoy it and find a solution.

After presenting this puzzle in class Rom Pinchasi gave the following variation in the homework puzzles [2]:
*Is it true that for any given coloring of the points of* $\mathbb{R}$ *in two colors, there is an* $\epsilon > 0$ *such that one of the color classes must contain pairs of points at every possible distance smaller than* $\epsilon$.

It was remarked by Raiskii in his original paper [3] that in a 2-coloring of the points of $\mathbb{R}$ one can no longer hope to find a color class that contains pairs of points at every possible distance. Indeed, one can color by red all the points $x$ such that $k \leq x < k+1$ for some even $k \in \mathbb{Z}$ and color by blue all the other points. Then no color class contains pairs of points at distance 1(or any other odd integer for that matter).

The main result in this paper is a negative answer to the homework puzzle of Rom Pinchasi. For a subset $S$ of $\mathbb{R}$ we denote by $S \ominus S$ the Minkowski difference $S \ominus S := \{s_1 - s_2 \mid s_1, s_2 \in S\}$. In Section 2 we prove:

**Theorem 1.1** *There is a partition of* $\mathbb{R}$ *into two sets* $A$ *and* $B$ *such that neither* $A \ominus A$ *nor* $B \ominus B$ *contains an interval.*

Now it is clear that the Minkowski difference of a set $S \ominus S$ is the set of all possible distances between two points in $S$ (and the negatives of those distances). In particular, the partition guaranteed by Theorem 1.1 says that for every $\epsilon > 0$ neither of the sets $A$ nor $B$ contains pairs of points at every distance smaller than $\epsilon$.

We remark that the sets $A$ and $B$ in the partition in Theorem 1.1 are necessarily non-measurable. This is because every measurable set of positive measure must contain a so-called Lebesgue point. This is a point where the set is extremely dense in small neighborhoods of that point. Once we can find an interval in which say $A$ has a density strictly greater than 50%, then necessarily $A \ominus A$ must contain a small interval around 0. Moreover, since clearly for any two sets $X, Y \subset \mathbb{R}$ we have that $X \subset Y$ implies that $X \ominus X \subset Y \ominus Y$, we can deduce from the above that neither $A$ nor $B$ contains any measurable set of positive measure. This last remark may be a bit puzzling as one may wonder if there even exists a set such that neither it nor its complement contains any set of positive measure, but the somewhat surprising answer is that such sets exists and are known as Bernstein sets. Bernstein sets are interesting in their own right (see [1]). We note that the standard (and as far as we know only) way of constructing Bernstein sets is via transfinite induction. We provide an alternative construction, which may be of interest in the study of Bernstein sets.

## 2 Main result

In order to prove Theorem 1.1 we prove a more general result:

**Theorem 2.1** *Given an additive subgroup $P \subset \mathbb{R}$, some arbitrary group $G$ with identity element $e$, and a homomorphism $\phi : P \to G$, there exists a partition $\{X_g\}_{g \in G}$ of $\mathbb{R}$ such that $(X_g \ominus X_g) \cap P \subset \phi^{-1}(e)$ for every $g \in G$.*

First we show that Theorem 2.1 implies Theorem 1.1. And indeed given Theorem 2.1 we can let $G = \mathbb{Z}_2$ the group with two elements, and so we are left with the task of finding an additive subgroup $P \subset \mathbb{R}$ and homomorphism $\phi : P \to G$ such that $\phi^{-1}(1)$ is dense in $\mathbb{R}$.

Let $P_0 = \phi^{-1}(0)$ and $P_1 = \phi^{-1}(1)$. We can now use Theorem 2.1 to find a partition $X_1, X_0$ of $\mathbb{R}$ such that both $(X_1 \ominus X_1) \cap P$ and $(X_0 \ominus X_0) \cap P$ are contained in $P_0$. Consequently,

$$(X_1 \ominus X_1) \cap P_1 = (X_0 \ominus X_0) \cap P_1 = \emptyset.$$

and clearly neither $X_1 \ominus X_1$ nor $X_0 \ominus X_0$ contains an interval since we assumed $\phi^{-1}(1) = P_1$ is dense. Now the task of finding such a group $P$ and homomorphism $\phi$ is quite a simple one. For example one may take $P = \mathbb{Q} + \pi\mathbb{Z} := \{q + k\pi | q \in \mathbb{Q}, k \in \mathbb{Z}\}$ and $\phi(q + k\pi) = k \mod 2$. Checking that $P$ is indeed a group and $\phi$ is a well-defined homomorphism is quite trivial, as is checking that $\phi^{-1}(1)$ is dense since $\{q + \pi | q \in \mathbb{Q}\} \subset \phi^{-1}(1)$, this completes the proof of Theorem 1.1 based on Theorem 2.1.

We now turn to prove Theorem 2.1, but first let us give some motivation for our approach. Intuitively, it seems quite logical that the additive property of homomorphisms will give some connection between the Minkowski difference and homomorphisms. More concretely one may easily check that with the definitions as in Theorem 2.1 we have that for any $g, f \in G$ we have $\phi^{-1}(g) \ominus \phi^{-1}(f) \subset \phi^{-1}(g-f)$ (in fact it is an equality and not just inclusion). The sets $\{\phi^{-1}(g)\}_{g\in G}$ will give us a partition of $P$ such that the Minkowski difference of any two sets in the partition is contained in $\phi^{-1}(e)$. This looks quite similar to the statement of Theorem 2.1 except the result is on the subgroup $P$ and not on $\mathbb{R}$. And as often happens in situations like this the method we use to go from the subgroup to the big group will be to look at the quotient group $\mathbb{R}/P$. Now let us give a formal proof for Theorem 2.1.

**Proof.** We consider the quotient group $\mathbb{R}/P$. Let $A$ be a set of representatives for the cosets of $P$. (We remark that this is a point where we use the Axiom of Choice that seems to be unavoidable in order to prove the theorem.) Every element $x \in \mathbb{R}$ can be uniquely written as $x = a + p$ for some $a \in A$ and $p \in P$. We can now define the desired partition of $\mathbb{R}$. For every $g \in G$ we set $X_g = \{a + p \mid a \in A, \quad \phi(p) = g\}$, informally one may think of those sets as $X_g = A \oplus \phi^{-1}(g)$ where the meaning of $\oplus$ is clear. This is clearly a partition of $\mathbb{R}$. We need to show that it satisfies the desired property, that is, $(X_g \ominus X_g) \cap P \subset \phi^{-1}(e)$ for every $g \in G$.

Let $g \in G$, $d \in (X_g \ominus X_g) \cap P$ and we will show that $d \in \phi^{-1}(e)$, that is $\phi(d) = e$. By the definition of $d$ there are some $x, y \in X_g$ such that $d = x - y$, and since $x - y = d \in P$ both $x$ and $y$ are in the same coset of $P$, i.e. $x + P = y + P$ in the group $\mathbb{R}/P$. Thus we can write $x = a + p$ and $y = a + q$ for some $a \in A$ and $p, q \in P$. Note it is the same $a$ for both of them since, as we remarked, they are in the same coset. Now because $x, y \in X_g$ we have $\phi(p) = \phi(q) = g$. Therefore

$$\phi(d) = \phi(x - y) = \phi(p - q) = \phi(p)\phi(-q) = \phi(p)\phi^{-1}(q) = gg^{-1} = e$$

This shows that $(X_g \ominus X_g) \cap P \subset \phi^{-1}(e)$, as desired.

□

## 3 Open Problems and Concluding Remarks

There are quite a few interesting unsolved problems that arise here. It is not difficult to see that the following generalization of the motivating puzzle presented in the introduction is true:

*Every coloring of the points of $\mathbb{R}^n$ in $n$ colors has at least one color class that contains pairs of points at every possible distance.* And as we remarked in the introduction Raiskii showed in [3] that the above is true even in a coloring of $n + 1$ parts.

In view of Theorem 1.1, it is natural to ask the following question:

**Problem A.** *What is the minimal number $\rho(n)$ such that any coloring of the points of $\mathbb{R}^n$ in $\rho(n)$ colors there is at least one color class that contains pairs of points at every small enough possible distance?*

Theorem 1.1 implies that $\rho(1) = 1$ and the Raiskii result implies that $\rho(n) > n+1$ for all $n > 1$.

Another immediate application of the construction in Theorem 1.1 is that for every $n$ one can partition $\mathbb{R}^n$ into two sets $A$ and $B$ such that neither $A \ominus A$ nor $B \ominus B$ contains an $n$-dimensional ball. Indeed, one only has to partition $\mathbb{R}^n$ into parallel disjoint lines and in each line use the partition of $\mathbb{R}$ described in the proof of Theorem 1.1. Answering Problem $A$ will require, therefore, additional ideas.


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