On Determining the Equation of a Salkowski Curve Satisfying $\frac{\tau}{\kappa} = \frac{1}{s}$

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Abstract - In this paper, we determine the equation of a Salkowski curve whose ratio of torsion to curvature is given by $\frac{1}{s}$, where s is the arc length of the curve. The Frenet-Serret equations provide the third-order vector differential equation for the unit tangent vector $T(s)$ and the general (series) solution was obtained. In the end, the series solution is entirely determined by the given initial conditions.

Keywords : Salkowski curve; Fundamental Theorem of Curves; Frenet-Serret equations

Mathematics Subject Classification (2020) **:** 53A04

1 Introduction

Geometers have defined and investigated space curves and their properties over the years. Among them, Gaspard Monge wrote and published a paper about space curves in the late 18th century. Monge's work influenced his students to further develop the theory of curves; the most notable of these students, concerning the theory of curves, was M.S. Lancret. In 1802, he conjectured that a space curve was a helix if and only if its ratio of torsion to curvature was constant. It was not until the mid-1840s that A.J.C. Barre de Saint Venant provided proof for Lancret's conjecture. Around the same period, F. Frenet and J. Serret worked independently to obtain what we know today as the Frenet-Serret Theorem. Subsequently, Gaston Darboux's idea of a moving frame was the first step in providing a unified theory of curves. Since the development of local curve theory, research efforts have centered around the classification of curves and the investigation of the geometric properties of curves. The Frenet-Serret Theorem provides the foundation for such work because it removes the need for an explicit equation of the curve. Some current problems include involute-evolute curve pairs, natural mate pairs, and curves lying on surfaces as seen in [3], [4], and [9]. Most notably, B.Y. Chen proved that a curve is a rectifying curve if and only if its ratio of torsion to curvature is a linear function of its arc length [2]. A type of curve relevant to this paper is the Salkowski curve, a curve with constant curvature and non-constant torsion as defined in [10]. Other works focus on studying several kinds of space curves in different ambient spaces. For example, the

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necessary conditions for a curve to be a rectifying curve in four-dimensional Euclidean space, three-dimensional Minkowski space, and four-dimensional Minkowski space have been discussed in [6], [5], and [1], respectively. While most research has a curve as the input and geometric properties as the output, efforts have been made recently to do the opposite. The process of constructing a curve from its torsion and curvature is allowed by the Fundamental Theorem of Curves. One can find an interesting example in the textbook "Elements of Differential Geometry" by Millman and Parker on page 44. It shows the construction of the explicit formula of a circular helix by assuming the ratio of torsion to curvature is constant, and the curvature is positive and constant. In 2015, Seo and Oh used a similar method in [8] to determine the equation of a Salkowski, rectifying curve in three-dimensional Euclidean space. Recently, Yilmaz found the equation of a Salkowski, rectifying curve in Minkowski three-space as seen in [11]. In this paper, we aim to determine the equation of another Salkowski space curve. Here, the curve satisfies the ratio of torsion to curvature of $\frac{1}{s}$, where s is the arc length of the curve. We derive a differential equation in terms of the unit tangent vector and find the series solution to the differential equation. The explicit formula of the curve can be provided after performing the appropriate integrations and applying the given initial conditions.

2 Overview of Local Curve Theory [7]

Let $\alpha = \alpha(t) : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a regular smooth curve on an open interval $I = (a, b)$. The unit tangent vector $\vec{T}(t)$ of $\alpha(t)$ is defined to be

$$\vec{T}(t) = \frac{\alpha'(t)}{||\alpha'(t)||}.$$

If $||\alpha'(t)|| = 1$ for all t, then α is said to be a unit speed curve. The arc length of α is given by

$$s = s\,(t) = \int_{t_0}^{t} ||\alpha'(\sigma_t)||d\sigma_t.$$

Under the arc length reparametrization, the curve $\alpha(s)$ becomes a unit speed curve. It follows directly that the unit tangent vector of a unit speed curve is

$$\vec{T}(s) = \alpha'(s). \tag{1}$$

For the remainder of the paper, $\alpha(t)$ denotes a regular curve that is not necessarily unit speed, while $\alpha(s)$ denotes a unit speed curve. We continue by defining the Frenet-Serret apparatus of a unit speed curve $\alpha(s)$.

Definition 2.1 *Let α be a unit speed curve. The Frenet-Serret apparatus of α is*

$\{\kappa(s), \tau(s), \vec{T}(s), \vec{N}(s), \vec{B}(s)\}$, *where*

$$\begin{aligned}
\vec{T}(s) &= \alpha'(s), \\
\vec{N}(s) &= \frac{\vec{T}'(s)}{||\vec{T}'(s)||}, \\
\vec{B}(s) &= \vec{T}(s) \times \vec{N}(s), \\
\kappa(s) &= ||\vec{T}'(s)||, \text{ and} \\
\tau(s) &= -\langle \vec{B}'(s), \vec{N}(s) \rangle.
\end{aligned}$$

Two principal components form the Frenet-Serret apparatus: the geometric properties of α and an orthonormal basis of $\mathbb{R}^3$ that spans α. The first geometric property is the curvature $\kappa(s)$ of α. The curvature measures how much a curve deviates from being a straight line (i.e. $\kappa(s) = 0$). On the other hand, the torsion $\tau(s)$ measures how much a curve deviates from lying on a plane (i.e. $\tau(s) = 0$). The orthonormal basis consisting of $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ is often called the Frenet frame. The Frenet frame is a moving frame since the orientation of the frame varies along the curve while $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ remain orthogonal to each other. In Figure 1 below, we present a circular helix along with its Frenet frame at different positions along the curve.

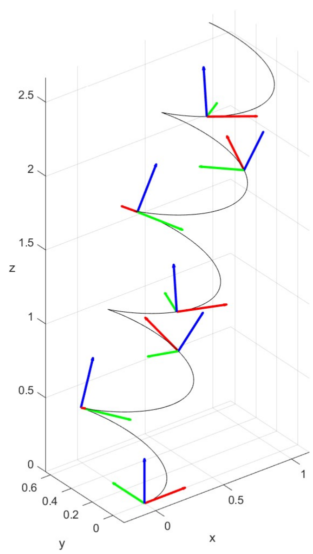

Figure 1: Circular helix with its Frenet frame. The tangent vector is red, the principal normal vector is green, and the binormal vector is blue.

We proceed to show how the Frenet-Serret Theorem connects different elements of the Frenet-Serret apparatus.

Theorem 2.2 *(Frenet-Serret Theorem) Let* $\alpha(s)$ *be a unit speed curve with curvature*

$\kappa(s) \neq 0$ and Frenet-Serret apparatus $\{\kappa(s), \tau(s), \vec{T}(s), \vec{N}(s), \vec{B}(s)\}$, then

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{bmatrix}.$$

Additionally, the equations in Theorem 2.2 are often addressed as the Frenet-Serret equations. We conclude this overview by presenting the Fundamental Theorem of Curves.

Theorem 2.3 *(Fundamental Theorem of Curves) Any regular curve with $\kappa > 0$ is completely determined, up to position, by its curvature and torsion. Let (a, b) be an interval about s_0, $\tilde{\kappa}(s) > 0$ a C^1 function on (a, b), $\tilde{\tau}(s)$ a continuous function on (a, b), x_0 a fixed point of $\mathbb{R}^3$, and $\{\vec{D}, \vec{E}, \vec{F}\}$ a fixed right-handed orthonormal basis of $\mathbb{R}^3$. Then there exists a unique C^3 regular curve $\alpha : (a, b) \longrightarrow \mathbb{R}^3$ such that:*

1. *the parameter is arc length from $\alpha(s_0)$,*
2. *$\alpha(s_0) = x_0$, $\vec{T}(s_0) = \vec{D}$, $\vec{N}(s_0) = \vec{E}$, $\vec{B}(s_0) = \vec{F}$, and*
3. *$\kappa(s) = \tilde{\kappa}(s)$, $\tau(s) = \tilde{\tau}(s)$.*

3 Deriving and Solving a Vector Differential Equation for the Tangent Vector

We are interested in the case where the curvature κ is a positive constant satisfying the following equation of the torsion:

$$\tau(s) = \frac{\kappa}{s}. \tag{2}$$

Given the relation from (2), the Frenet-Serret equations for this case become

$$\vec{T}'(s) = \kappa \vec{N}(s), \tag{3}$$

$$\vec{N}'(s) = -\kappa \vec{T}(s) + \frac{\kappa}{s}\vec{B}(s), \text{ and} \tag{4}$$

$$\vec{B}'(s) = -\frac{\kappa}{s}\vec{N}(s). \tag{5}$$

Now, we derive a differential equation for the tangent vector $\vec{T}(s)$. By taking the derivative of (3) and substituting (4), we get

$$T'' = \kappa N' = -\kappa^2 T(s) + \frac{\kappa^2}{s} B. \tag{6}$$

From (4) and the first equation of (6) for N', we have

$$B = \frac{s}{\kappa}(N' + \kappa T) = \frac{s}{\kappa}(\frac{T''}{\kappa} + \kappa T). \tag{7}$$

By taking another derivative of (6), substituting B from (7) and B' from (5), we obtain the following third-order differential equation for the unit tangent vector $\vec{T}(s)$.

$$\vec{T}'''(s) + \frac{1}{s}\vec{T}''(s) + \kappa^2\left(1 + \frac{1}{s^2}\right)\vec{T}'(s) + \frac{\kappa^2}{s}\vec{T}(s) = \vec{0}. \tag{8}$$

One can observe that solutions to (8) are valid where $s \neq 0$, so we consider a series solution on an interval around a point $s = s_0 \neq 0$.
Let $\vec{a}_0$, $\vec{a}_1$, and $\vec{a}_2$ be any three fixed linearly independent vectors in $\mathbb{R}^3$. With this basis for $\mathbb{R}^3$, the series solution to (8) can be written

$$\vec{T}(s) = f_0(s)\vec{a}_0 + f_1(s)\vec{a}_1 + f_2(s)\vec{a}_2 \tag{9}$$

for three real-valued smooth functions $f_0(s), f_1(s), f_2(s)$.
Differentiating equation (9) and substituting into (8), one obtains the same third-order ordinary differential equation for f_0, f_1, f_2

$$0 = \sum_{n=0}^{2}(f_n''' + \frac{1}{s}f_n'' + \kappa^2(1 + \frac{1}{s^2})f_n' + \frac{\kappa^2}{s}f_n)\vec{a}_n.$$

Since $\vec{a}_0, \vec{a}_1, \vec{a}_2$ are linearly independent, we obtain

$$0 = f_n''' + \frac{1}{s}f_n'' + \kappa^2(1 + \frac{1}{s^2})f_n' + \frac{\kappa^2}{s}f_n$$

for $n = 0, 1, 2$. The differential equations for f_n all have the form

$$0 = f''' + \frac{1}{s}f'' + \kappa^2(1 + \frac{1}{s^2})f' + \frac{\kappa^2}{s}f. \tag{10}$$

Now, we expand $\frac{1}{s}$ and $\frac{1}{s^2}$ as a power series at s_0 as shown below.

$$\frac{1}{s} = \frac{1}{s_0}\sum_{j=0}(-1)^j s_0^{-j}(s - s_0)^j = \frac{1}{s_0}\left[1 - \frac{1}{s_0}(s - s_0) + \frac{1}{s_0^2}(s - s_0)^2 - \frac{1}{s_0^3}(s - s_0)^3 + ...\right], \tag{11}$$

$$\frac{1}{s^2} = \frac{1}{s_0^2}\left[1 - \frac{2}{s_0}(s - s_0) + \frac{3}{s_0^2}(s - s_0)^2 - \frac{4}{s_0^3}(s - s_0)^3 + ...\right]. \tag{12}$$

We use a series solution centered at s_0, setting

$$f(s) = \sum_{n=0}^{\infty} b_n(s - s_0)^n.$$

Next, substituting the expansions (11) and (12) into (10), we observe that the coefficients $b_n, n \geq 3$ are all dependent on b_0, b_1, and b_2.
Here are a couple of more coefficients of b_n.

$$b_3 = -\frac{2}{3!s_0}b_2 - \frac{\kappa^2}{3!}\left(1 + \frac{1}{s_0^2}\right)b_1 - \frac{\kappa^2}{3!s_0}b_0,$$

$$b_4 = \frac{2!\kappa^2}{4!}\left(\frac{2}{\kappa^2 s_0^2} - 1 - \frac{1}{s_0^2}\right) b_2 + \frac{3\kappa^2}{4! s_0^3} b_1 + \frac{2\kappa^2}{4! s_0^2} b_0,$$

$$b_5 = \frac{1}{60}\left(\frac{-6}{s_0^3} + \frac{\kappa^2}{s_0} + \frac{6\kappa^2}{s_0^3}\right) b_2 + \frac{1}{120}\left(\frac{2\kappa^4}{s_0^2} + \frac{\kappa^4}{s_0^4} + \kappa^4 - \frac{11\kappa^2}{s_0^4}\right) b_1$$
$$+ \frac{1}{120}\left(\frac{\kappa^4}{s_0} + \frac{\kappa^4}{s_0^3} - \frac{6\kappa^2}{s_0^3}\right) b_0.$$

We consider the initial conditions for (8) as below,

$$\vec{a}_0 = \vec{T}(s_0), \quad \vec{a}_1 = \vec{T}'(s_0), \quad \vec{a}_2 = \vec{T}''(s_0), \tag{13}$$

where $\vec{a}_0, \vec{a}_1$ and $\vec{a}_2$ are linearly independent vectors in (9). Thus, they yield the following.

- For f_0, we have $b_0 = 1, b_1 = 0, b_2 = 0$.
- For f_1, we have $b_0 = 0, b_1 = 1, b_2 = 0$.
- For f_2, we have $b_0 = 0, b_1 = 0, b_2 = \frac{1}{2}$.

Therefore, the series solution $\vec{T}(s)$ becomes

$$\vec{T}(s) = f_0(s)\vec{a}_0 + f_1(s)\vec{a}_1 + f_2(s)\vec{a}_2, \tag{14}$$

where

$$f_0(s) = 1 - \frac{\kappa^2}{3! s_0}(s-s_0)^3 + \frac{2\kappa^2}{4! s_0^2}(s-s_0)^4 + \frac{\kappa^4}{5! s_0}\left(1 + \frac{1}{s_0^2} - \frac{6}{\kappa^2 s_0^2}\right)(s-s_0)^5 + \cdots,$$
$$f_1(s) = (s-s_0) - \frac{\kappa^2}{3!}\left(1 + \frac{1}{s_0^2}\right)(s-s_0)^3 + \frac{3\kappa^2}{4! s_0^3}(s-s_0)^4 +$$
$$+ \frac{1}{120}\left(\kappa^4 + \frac{2\kappa^4}{s_0^2} + \frac{\kappa^4}{s_0^4} - \frac{11\kappa^2}{s_0^4}\right)(s-s_0)^5 + \cdots,$$
$$f_2(s) = \frac{1}{2}(s-s_0)^2 - \frac{1}{3! s_0}(s-s_0)^3 + \frac{\kappa^2}{4!}\left(\frac{2}{\kappa^2 s_0^2} - 1 - \frac{1}{s_0^2}\right)(s-s_0)^4 +$$
$$+ \frac{1}{5!}\left(\frac{\kappa^2}{s_0} + \frac{6\kappa^2}{s_0^3} - \frac{6}{s_0^3}\right)(s-s_0)^5 + \cdots.$$

4 Obtaining the Final Equation of the Curve

According to (1), the equation of the curve $\alpha(s)$ is given by

$$\alpha(s) = \int_{s_0}^{s} \vec{T}(\sigma_s) d\sigma_s + \alpha(s_0).$$

From (3) and (13), and by the definition of the binormal vector B, we have

$$\vec{T}(s_0) = \vec{a}_0, \vec{N}(s_0) = \frac{\vec{a}_1}{\kappa}, \vec{B}(s_0) = \frac{1}{\kappa}(\vec{a}_0 \times \vec{a}_1). \tag{15}$$

Then, the set of vectors $\{\vec{a}_0, \frac{1}{\kappa}\vec{a}_1, \frac{1}{\kappa}(\vec{a}_0 \times \vec{a}_1)\}$ is orthonormal and the vector $\vec{a}_2$ is a linear combination of these three vectors.
From (6), we have

$$\vec{T}''(s_0) = -\kappa^2 \vec{a}_0 + \frac{\kappa^2}{s_0}\vec{B}(s_0).$$

The third condition in (13) and (15) implies that

$$\vec{a}_2 = -\kappa^2 \vec{a}_0 + \frac{\kappa}{s_0}(\vec{a}_0 \times \vec{a}_1),$$

and by (3), and (9),

$$\vec{T}(s) = \left(f_0(s) - \kappa^2 f_2(s)\right)\vec{a}_0 + f_1(s)\vec{a}_1 + \frac{\kappa}{s_0} f_2(s)(\vec{a}_0 \times \vec{a}_1), \tag{16}$$

$$\vec{N}(s) = \left(\frac{1}{\kappa} f_0'(s) - \kappa f_2'(s)\right)\vec{a}_0 + f_1'(s)\frac{\vec{a}_1}{\kappa} + \frac{f_2'(s)}{s_0}(\vec{a}_0 \times \vec{a}_1), \tag{17}$$

where the functions $f_0(s)$, $f_1(s)$, and $f_2(s)$ are as in (14). With this, we have the following.

Theorem 4.1 *A unit speed curve $\alpha(s)$ with arc length s, torsion $\tau(s) = \frac{\kappa}{s}$, and constant curvature $\kappa > 0$ has the expression*

$$\begin{aligned}\alpha(s) = &\left(\int_{s_0}^{s} f_0(\sigma_s) - \kappa^2 f_2(\sigma_s) d\sigma_s\right)\vec{a_0} + \\ &\left(\int_{s_0}^{s} f_1(\sigma_s) d\sigma_s\right)\vec{a_1} + \left(\frac{\kappa}{s_0}\int_{s_0}^{s} f_2(\sigma_s) d\sigma_s\right)(\vec{a_0} \times \vec{a_1}) + \alpha(s_0),\end{aligned}$$

where the functions $f_0(s), f_1(s)$ and $f_2(s)$ are stated earlier in (14), and $\{\vec{a_0}, \vec{a_1}, \vec{a_0} \times \vec{a_1}\}$ is an orthogonal basis of $\mathbb{R}^3$. Additionally, the Frenet frame vectors for this curve are

$$\begin{gathered}\vec{T}(s) = \left(f_0(s) - \kappa^2 f_2(s)\right)\vec{a_0} + f_1(s)\vec{a_1} + \left(\frac{\kappa}{s_0} f_2(s)\right)(\vec{a_0} \times \vec{a_1}), \\ \vec{N}(s) = \left(\frac{1}{\kappa} f_0'(s) - \kappa f_2'(s)\right)\vec{a_0} + \frac{f_1'(s)}{\kappa}\vec{a_1} + \left(\frac{f_2'(s)}{s_0}\right)(\vec{a_0} \times \vec{a_1}), \\ \textit{and } \vec{B}(s) = \vec{T}(s) \times \vec{N}(s).\end{gathered}$$

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