The Limiting Spectral Measure for an Ensemble of Generalized Checkerboard Matrices

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Abstract - Random matrix theory successfully models many systems, from the energy levels of heavy nuclei to zeros of L-functions. While most ensembles studied have continuous spectral distribution, Burkhardt et al. introduced the ensemble of k-checkerboard matrices, a variation of Wigner matrices so that entries in a checkerboard pattern are some fixed constant. In this family, $N-k$ of the eigenvalues are of size $O(\sqrt{N})$ and were called bulks while the rest are tightly contrained around certain multiples of N and were called blips.

We extend their work by allowing the fixed entries to take different constant values. We can construct ensembles with blip eigenvalues at any multiples of N we want and with any multiplicity. For example, we can have the blips occur at sequences such as the primes or the Fibonaccis. The presence of multiple blips creates technical challenges to separate them and to look at only one blip at a time. We overcome this by choosing a suitable weight function which allows us to localize at each blip, and then exploiting cancellation to deal with the resulting combinatorics to determine the average moments of the ensemble; we then apply standard methods from probability to prove that almost surely the limiting distributions of the matrices converge to the average behavior as the matrix size tends to infinity. For blips with just one eigenvalue in the limit we have convergence to a Dirac delta spike, while if there are k eigenvalues in a blip we again obtain hollow $k \times k$ GOE behavior.

Keywords : random matrix ensembles; checkerboard matrices; limiting spectral measure; Gaussian orthogonal ensemble

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1 Introduction

1.1 Background

Initially introduced by Wishart [8] for some problems in statistics, random matrix theory has successfully modeled a large number of systems from energy levels of heavy nuclei to zeros of the Riemann zeta function. A simple but important example is the ensemble of real symmetric matrices whose upper triangular entries are independent, identically distributed random variables from some fixed probability distribution with mean 0, variance

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1 and finite higher moments. Wigner's semi-circle law states that as the size of the matrix $N \to \infty$, the properly normalized spectral distribution of a matrix from the ensemble converges almost surely to semi-circle law (or semi-ellipse):

$$\sigma_R(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & \text{if } |x| \le R, \\ 0 & \text{if } |x| > R. \end{cases} \tag{1}$$

See [7] for more details.

Besides the more well-known families such as the Gaussian Orthogonal, Unitary and Symplectic Ensembles, many other special ensembles have been studied; see for example [1], where the additional structures on the entries of the matrices lead to different behaviors of the eigenvalues in the limit.

For most ensembles that people have studied, while it is possible to prove the convergence of the limiting spectral measure, in only a few (such as d-regular graphs [6], block circulant matrices [4] and palindromic Toeplitz matrices [5]) can the limiting distribution be written down in a nice, closed form expression.

This paper is a sequel to [2], where they introduce ensembles of checkerboard matrices which also have a nice, closed-form expression for their limiting distribution. The spectrum splits into two; $N - k$ of the eigenvalues are of size $\sqrt{N}$ (called the **bulk** eigenvalue) and converges to a semicircle, while k of the eigenvalues (called the **blip** eigenvalues) are of order N and converge to the spectral distribution of a $k \times k$ hollow Gaussian orthogonal ensemble.

1.2 Generalized Checkerboard Ensembles

We generalize [2] by allowing the constant w to take different values. While the Checkerboard ensembles in [2] only allow one blip for each ensemble, the generalized Checkerboard ensembles allow arbitrarily many blips for each ensemble. Moreover, we have control over the positions of these blips. That is, given a list of points, the generalized checkerboard ensemble allows the spectrum at those points in a "non-trivial" way. We can always "trivially" construct ensembles with prescribed locations and frequency by taking a diagonal union of block matrices. But then the blocks are independent from each other. The significance of the generalized checkerboard ensemble is that we can control the locations of normalized eigenvalues within an ensemble that doesn't have independent diagonal blocks. It is a "mixed" matrix whose eigenvalues have a nice split limiting distribution.

Definition 1.1 *Fix $k \in \mathbb{N}$ and a k-tuple of real numbers $W = (w_1, \ldots, w_k)$. The $N \times N$ (k, W)-checkerboard ensemble is the ensemble of matrices $A_N = (m_{ij})$ given by*

$$m_{ij} = \begin{cases} a_{ij} & \mathrm{if i \not\equiv j \pmod k}, \\ w_u & \mathrm{if i \equiv j \equiv u \pmod k}, \ \textit{with} \ \mathrm{u \in \{1, 2, \ldots, k\}}, \end{cases} \tag{2}$$

where $a_{ij} = a_{ji}$ are independent and identically distributed random variables with mean 0, variance 1, and finite higher moments.

1.3 Results

What makes the checkerboard ensemble in [2] interesting is that the eigenvalues of a matrix from the ensemble almost surely fall into two separate regimes. With our generalization we can exploit the freedom to choose different constants to force the eigenvalues to fall into more regimes.

Theorem 1.2 *Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of (k, W)-checkerboard matrices. Suppose that W has x non-zero entries and there are s distinct w's, then almost surely as $N \to \infty$, the eigenvalues of A_N fall into $s+1$ regimes: $N - x$ of the eigenvalues are $O(N^{1/2+\epsilon})$ and if w_i' appears k_i times, k_i eigenvalues are of magnitude $Nw_i'/k + O(N^{1/2+\epsilon})$.*

As in [2], we refer to the $N - x$ eigenvalues that are on the order of $\sqrt{N}$ as the eigenvalues in the **bulk**, while for each distinct w_i, the k_i eigenvalues near Nw_i/k are called the eigenvalues in the **blips**. We study the eigenvalue distribution of each regime.

For the remainder of this paper, $\boldsymbol{A_N}$ always refers to an $N \times N$ matrix.

Let ν_{A_N} be the empirical spectral measure of A_N, where we have normalized the eigenvalues by dividing by $\sqrt{N}$:

$$\nu_{A_N}(x) = \frac{1}{N} \sum_{\lambda \text{ an eigenvalue of } A_N} \delta\left(x - \frac{\lambda}{\sqrt{N}}\right). \tag{3}$$

For example, Figure 1 gives this normalized eigenvalue distribution of a collection of 500×500 $(6, W)$-checkerboard matrices with $W = (1, -2, -2, 3, 3, 3)$.

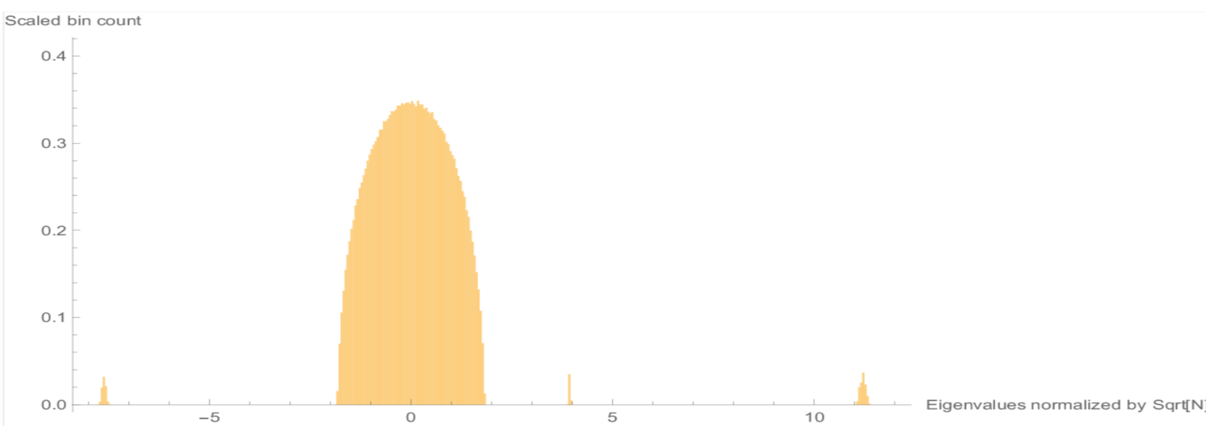

Figure 1: A histogram of the normalized eigenvalue distribution on a probability density scale for 500×500 $(6, W)$-checkerboard real matrices with $W = (1, -2, -2, 3, 3, 3)$ after 500 trials.

Taking $\tilde{\mathcal{A}}_N$ to be the fixed matrix with entries $m_{ij} = w_u$ whenever $i \equiv j \equiv u \pmod{k}$ and zero otherwise, we have that the limiting spectral distribution (LSD) of the (k, W)-checkerboard ensemble is the same as the LSD of the ensemble with $W = \mathbf{0}$, which does not have the k large blip eigenvalues . This overcomes the issue of diverging moments.

Theorem 1.3 *Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of $N \times N$ (k, W)-checkerboard matrices, and let ν_{A_N} denote the empirical spectral measure. Then, ν_{A_N} converges weakly almost surely to the Wigner semicircle measure σ_R with radius*

$$R \;=\; 2\sqrt{1 - 1/k}. \tag{4}$$

The proof is by standard combinatorial arguments similar to the one in [2]. Weyl's Inequality [3] implies that if the spectral radius of P is $O(f)$ then the size of the perturbations are $O(f)$ as well. Hence it suffices to demonstrate that almost surely the spectral radius of a sequence of $(k, 0)$-checkerboard matrices is $O(N^{1/2+\epsilon})$.

Let A_N be a $(k, 0)$-checkerboard matrix. By Remark A.3 in [2] we have that $\mathrm{Var}(\mathrm{Tr}(A_N^{2m})) = O(N^{2m})$ and by the proof of Lemma 2.1 in [2] we get $\mathbb{E}\left[\mathrm{Tr}(A_N^{2m})\right] = O(N^{m+1})$.

Since Lemma B.2 in [2] holds for all $m \in \mathbb{Z}^+$, we have that almost surely $\|A_N\|_{\mathrm{op}}$ is $O(N^{1/2+\epsilon})$. Together with and Weyl's Inequality, we obtain Theorem 1.2

Similar to the previous checkerboard paper [2], each blip may be thought of as deviations about the trivial eigenvalues. Instead of having just one blip as in [2], we now have many different blips. A blip containing $k_i > 1$ eigenvalues has the same distribution as the eigenvalues of the $k_i \times k_i$ hollow Gaussian Orthogonal Ensemble defined below; when $k_i = 1$ the blip has the distribution of a dirac delta function.

Definition 1.4 *The $k \times k$ **hollow Gaussian Orthogonal Ensemble** is given by $k \times k$ matrices $A = (a_{ij}) = A^T$ with*

$$a_{ij} = \begin{cases} \mathcal{N}_{\mathbb{R}}(0,1) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \tag{5}$$

We need to define a weighted blip spectral measure which takes into account only the eigenvalues of one blip. Thus we not only need to get rid of the interference from the bulk, we also need to avoid the interference from the other blips. In order to facilitate the use of eigenvalue trace lemma, similar to [2], we are led to use a polynomial weighting function and we use a sequence of polynomials of degree tending to infinity as the matrix size $N \to \infty$ so that in the limit we mimic a smooth cutoff function. Specifically, let

$$f_i^{2n}(x) \;:=\; \left(\frac{x(2-x)\prod_{w_j \neq w_i}(x - \frac{w_j}{w_i})(2 - x - \frac{w_j}{w_i})}{\prod_{w_j \neq w_i}(1 - \frac{w_j}{w_i})^2} \right)^{2n}. \tag{6}$$

Thus we alter the standard empirical spectral measure in the following way to capture the blip.

Definition 1.5 *Given $k \in \mathbb{N}$ and a k-tuple of real numbers $W = (w_1, \dots, w_k)$, the **empirical blip spectral measure** associated to an $N \times N$ (k, W)-checkerboard matrix A_N around $Nw_i/k \neq 0$ is*

$$\mu_{A_N,i}(x) \;:=\; \frac{1}{k_i} \sum_{\lambda \text{ an eigenvalue of } A} f_i^{2n}\left(\frac{k\lambda}{w_i N}\right) \delta\left(x - \left(\lambda - \frac{w_i N}{k}\right)\right), \tag{7}$$

where k_i is the number of w_i's in $(w_1, \ldots, w_k)$, where $n(N)$ is a function satisfying $\lim_{N\to\infty} n(N) = \infty$ and $n(N) = O(\log\log N)$.

Remark 1.6 The actual choice of weight functions should not change the empirical blip spectral measure in the limit. It will be used in the proof that the weight polynomial $f_i^{2n}(x)$ has a critical point at 1 with $f_i^{2n}(1) = 1$ and has zeroes of order $2n$ at 0 and at all w_j/w_1 with $w_j \neq w_1$. Heuristically, because the fluctuation of eigenvalues in each regime is of order $\sqrt{N}$, we have $f_i^{2n}\left(\frac{k\lambda}{w_iN}\right) \approx 1$ if λ is in the blip around Nw_i/k, and $f_i^{2n}\left(\frac{k\lambda}{w_iN}\right) \approx 0$ if λ is in the bulk or in the blip other than Nw_i/k. More specifically,

$$f_i^{2n}\left(\frac{k\lambda}{w_iN}\right) = \begin{cases} O\left(\frac{\log N}{N^n}\right) & \text{if } \lambda \text{ is } O\left(\sqrt{N}\right) \text{ or } \frac{Nw_j}{k} + O\left(\sqrt{N}\right) \text{ with } w_j \neq w_i, \\ 1 + O\left(\frac{\log N}{N^{2n}}\right) & \text{if } \lambda \text{ is } \frac{Nw_i}{k} + O\left(\sqrt{N}\right). \end{cases} \tag{8}$$

As in [2], we use the method of moments to relate the expected moments of the empirical blip measure around Nw_i/k to those of the $k_i \times k_i$ hollow GOE.

In particular, when there is only one eigenvalue in a blip, we obtain the following.

Theorem 1.7 *Fix $k \in \mathbb{N}$ and a k-tuple of real numbers $W = (w_1, \ldots, w_k)$ where $w_i \neq 0$ and there is exactly one w_i in W. Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of (k, W)-checkerboard matrices. Then the associated empirical blip spectral measure $\mu_{A_N,i}$ around Nw_i/k converges weakly to the Dirac delta distribution centered at $\frac{Nw_i}{k} + \frac{k-1}{w_i}$.*

Thus, when $k_i = 1$, we expect an eigenvalue of magnitude exactly $\frac{Nw_i}{k} + \frac{k-1}{w_i}$ as $N \to \infty$. In general, when $k_i > 1$, the empirical blip spectral measure of one matrix A_N around Nw_i/k no longer converges to the expected value, as the variances of the moments do not necessarily converge to zero as $N \to \infty$. Thus, we follow [2] to modify the moment convergence theorem and average over the eigenvalues of multiple independent matrices.

Definition 1.8 *Fix $k \in \mathbb{N}$, a k-tuple of real numbers $W = (w_1, \ldots, w_k)$, and a function $g : \mathbb{N} \to \mathbb{N}$. The* ***averaged empirical blip spectral measure*** *around Nw_i/k associated to a $g(N)$-tuple of $N \times N$ (k, W)-checkerboard matrices $(A_N^{(1)}, A_N^{(2)}, \ldots, A_N^{(g(N))})$ is*

$$\mu_{i,g,A_N^{(1)},A_N^{(2)},\ldots,A_N^{(g(N))}} := \frac{1}{g(N)} \sum_{j=1}^{g(N)} \mu_{A_N^{(j)},i}. \tag{9}$$

Theorem 1.9 *Fix $k \in \mathbb{N}$, a k-tuple of real numbers $W = (w_1, \ldots, w_k)$. Let $g : \mathbb{N} \to \mathbb{N}$ be such that there exists a $\delta > 0$ for which $g(N) \gg N^\delta$. Let $A^{(j)} = \{A_N^{(j)}\}_{N\in\mathbb{N}}$ be sequences of fixed $N \times N$ matrices, and let $\overline{A} = \{A^{(j)}\}_{j\in\mathbb{N}}$ be a sequence of such sequences. Then, as $N \to \infty$, the averaged empirical blip spectral measures $\mu_{i,g,A_N^{(1)},A_N^{(2)},\ldots,A_N^{(g(N))}}$ around Nw_i/k of the (k, W)-checkerboard ensemble over $\mathbb{R}$ converge weakly almost-surely to the measure with moments equal to the expected moments of the standard empirical spectral measure of the $k_i \times k_i$ hollow Gaussian Orthogonal Ensemble, where k_i is the number of w_i in W.*

Theorem 1.9 can be proven using the exactly same method as in [2]. Thus we omit the proof here. See [2] for the detail of the proof.

In conclusion, we can construct an expanding family to have blips of any desired finite size at any sequence of positions after normalization. Moreover, we extend [2] by showing that the averaged empirical blip spectral measure around $\frac{Nw_i}{k}$ converges to a $k_i \times k_i$ hollow Gaussian with its mean $\frac{k-1}{w_i}$ independent of the choice of all the constants $w_j \neq w_i$. This means that the distribution of different blips don't interfere with each other. When the blip has size $k_i = 1$, we get weak convergence of empirical blip spectral measure around $\frac{Nw_i}{k}$ to a Dirac Delta distribution.

The paper is organized as follows. In §2 we prove our claims concerning the eigenvalues in the blip. We then prove results on the convergence to the limiting spectral measure in §3.

2 The Blip Spectral Measure

In this section, we study the distribution of the eigenvalues at the blips. First, the weight function (6) defined in section 1 enables us to focus on just one blip at a time. Then, we reduce the general cases to the case where all $w_j \neq w_i$ are zero. Finally, we show that the distribution in the special case is hollow $k_1 \times k_1$ gaussian following an argument similar to the one in [2].

Without loss of generality, we focus on the blip around $Nw_1/k \neq 0$ and use the polynomial weight function

$$f_1^{2n}(x) \;=\; \left(\frac{x(2-x)\prod_{w_j\neq w_1}(x-\frac{w_j}{w_1})(2-x-\frac{w_j}{w_1})}{\prod_{w_j\neq w_1}(1-\frac{w_j}{w_1})^2}\right)^{2n}. \tag{10}$$

As discussed in Remark 1.6, our choice of weight functions does not affect our results as long as the functions

1. are essentially 1 close to 1, and
2. vanish to sufficiently high order at 0 and all w_j/w_1.

This two requirement ensures that any contribution from the eigenvalues within the bulk and the other blips are removed.

Definition 2.1 *The **empirical blip spectral measure** associated to an $N \times N$ k-checkerboard matrix A_N around Nw_1/k is*

$$\mu_{A_N,1}(x) \;:=\; \frac{1}{k_1}\sum_{\lambda \text{ an eigenvalue of } A} f_1^{2n}\left(\frac{k\lambda}{w_1 N}\right)\delta\left(x-\left(\lambda-\frac{w_1 N}{k}\right)\right), \tag{11}$$

where k_1 is the number of w_1's in $(w_1,\ldots,w_k)$, and $n(N)$ is a function which satisfies $\lim_{N\to\infty} n(N) = \infty$ and $n(N) = O(\log\log N)$.

Because the fluctuation of the location of the eigenvalues in each regime is of order $\sqrt{N}$, the modified spectral measure of Definition 2.1 weights eigenvalues within this blip by almost exactly 1 and those in the bulk and the other blips by almost exactly zero.

For fixed N, the polynomial f_1^{2n} can be written as $f_1^{2n}(x) = \sum_{\alpha=2n}^{4nl} c_\alpha x^\alpha$, where l is the number of distinct constants in $(w_1, \ldots, w_k)$ and all $c_\alpha \in \mathbb{R}$.

We apply the method of moments to the modified spectral measure (11). By the eigenvalue trace formula and linearity of expectation, the expected m-th moment of the empirical blip spectral measure is

$$\begin{aligned}
\mathbb{E}\left[\mu_{A_N,1}^{(m)}\right] &= \mathbb{E}\left[\frac{1}{k_1}\sum_{\lambda}\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k\lambda}{w_1 N}\right)^{\alpha}\left(\lambda - \frac{w_1 N}{k}\right)^{m}\right] \\
&= \mathbb{E}\left[\frac{1}{k_1}\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^{\alpha}\left(\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{w_1 N}{k}\right)^{m-i}\operatorname{Tr}(A_N^{\alpha+i})\right)\right] \\
&= \frac{1}{k_1}\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^{\alpha}\left(\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{w_1 N}{k}\right)^{m-i}\mathbb{E}\left[\operatorname{Tr}(A_N^{\alpha+i})\right]\right). \qquad (12)
\end{aligned}$$

Recall that

$$\mathbb{E}\left[\operatorname{Tr}(A_N^{\alpha+i})\right] = \sum_{1\le j_1,\ldots,j_{\alpha+i}\le N} \mathbb{E}\left[m_{j_1 j_2} m_{j_2 j_3}\cdots m_{j_{\alpha+i} j_1}\right]. \qquad (13)$$

The calculation of the moment has been transformed into a combinatorial problem of counting different types of products of entries. We follow the vocabulary from [2] to describe the combinatorics problem.

Definition 2.2 *A **block** is a set of adjacent a's surrounded by w's in a cyclic product, where the last entry of a cyclic product is considered to be adjacent to the first. We refer to a block of length ℓ as an ℓ-block or sometimes a block of size ℓ.*

Definition 2.3 *A **configuration** is the set of all cyclic products for which it is specified (a) how many blocks there are, and of what lengths, and (b) in what order these blocks appear (up to cyclic permutation); However, it is not specified how many w's there are between each block.*

Definition 2.4 *A **congruence configuration** is a configuration together with a choice of the congruence class modulo k of every index.*

Definition 2.5 *Given a configuration, a **matching** is an equivalence relation $\sim$ on the a's in the cyclic product which constrains the ways of indexing (see Definition 2.6) the a's as follows: an indexing of a's conforms to a matching $\sim$ if, for any two a's $a_{i_\ell,i_{\ell+1}}$ and $a_{i_t,i_{t+1}}$, we have $\{i_\ell, i_{\ell+1}\} = \{i_t, i_{t+1}\}$ if and only if $a_{i_\ell i_{\ell+1}} \sim a_{i_t,i_{t+1}}$. We further constrain that each a is matched with at least one other by any matching $\sim$.*

Definition 2.6 *Given a configuration, matching, and length of the cyclic product, then an* ***indexing*** *is a choice of*

1. *the (positive) number of w's between each pair of adjacent blocks (in the cyclic sense), and*

2. *the integer indices of each a and w in the cyclic product.*

We see that the congruence classes of the indices of the a's determine which congruence classes of the indices of the w's belong to, and thus which w_j's appear between the blocks.

2.1 Reducing to the case where all $w_j \neq w_1$ are zero

We prove that if there is some $w_j \neq w_1$ in a fixed congruence configuration, then it does not contribute to the expected moment (12) in the limit.

We begin by analyzing the form of the summands in the total contribution of a congruence configuration in Lemma 2.8. The following lemma helps us to derive this form, and its proof is provided in Appendix A.

Lemma 2.7 *Fix $s \in \mathbb{N}$ with $s \geq 2$ and some polynomial $p(x_1, \ldots, x_s) \in \mathbb{R}[x_1, \ldots, x_s]$ of degree q. For $\eta \in \mathbb{N}$ with $\eta \geq \sum_{i=1}^{s} y_i$ and distinct $w_1, \ldots, w_s$, we have*

$$\sum_{\substack{x_1+\cdots+x_s=\eta \\ x_i \geq y_i}} p(x_1, \ldots, x_s) w_1^{x_1} \cdots w_s^{x_s} \;=\; \frac{\sum_{l=1}^{s} w_l^{\eta+2-\sum_{i=1}^{s} y_i} f_{l,\eta}(w_1, \ldots, w_s)}{\left(\prod_{1 \leq i < j \leq s} (w_i - w_j)\right)^{2^q}}, \tag{14}$$

where each $f_{l,\eta}(w_1, \ldots, w_s) \in \mathbb{R}[\eta][w_1, \ldots, w_s]$ is a homogeneous polynomial in $w_1, \ldots, w_s$ of degree $2^q \binom{s}{2} + (\sum_{i=1}^{s} y_i) - 2$. Moreover, the coefficients in the polynomial $f_{l,\eta}(w_1, \ldots, w_s)$ are polynomials in η of degree $\leq q$.

Lemma 2.8 *Fix a congruence configuration and a matching. We have that the contribution to $\mathbb{E}\left[Tr(A_N^{\alpha+i})\right]$ is a sum of terms of the form*

$$p(\alpha+i) w_j^{\alpha+i-\gamma} \left(\frac{N}{k}\right)^{\alpha+i-t} \tag{15}$$

where p is a polynomial of degree $\leq \beta - s + 1$, β is the number of blocks determined by the configuration, s is the number of distinct constants w's determined by the chosen congruence classes, and t is the lost degrees of freedom determined by the matching.

Proof. Suppose that the distinct constants $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$ appear in the configuration, where each w_{j_q} appears x_q times and the x_q w_{j_q}'s are separated by the blocks into y_q parts. There are $\binom{x_q-1}{y_q-1}$ ways to put x_q w_{j_q} into y_q gaps. Note that $\mathbb{E}\left[m_{j_1 j_2} m_{j_2 j_3} \cdots m_{j_{\alpha+i} j_1}\right] = w_{j_1}^{x_1} x_{j_2}^{x_2} \cdots w_{j_s}^{x_s} \mathcal{A}$, where $\mathcal{A}$ is some constant determined by the matching.

Denote the number of a's in this configuration by r, then $\sum_{l=1}^{s} x_l = \alpha + i - r$ and there are

$$\sum_{\substack{x_i \geq y_i \\ x_1+\cdots+x_s=\alpha+i-r}} \prod_{q=1}^{s} \binom{x_q - 1}{y_q - 1} w_{j_q}^{x_q} \tag{16}$$

ways to place the constants $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$. For fixed $y_1, \ldots, y_s$, we can write (16) as

$$\sum_{\substack{x_i \geq y_i \\ x_1+\cdots+x_s=\alpha+i-r}} \widetilde{g}_{y_1,\ldots,y_s}(x_1, \ldots, x_s) w_{j_1}^{x_1} w_{j_2}^{x_2} \cdots w_{j_s}^{x_s} \tag{17}$$

where $\widetilde{g}_{y_1,\ldots,y_s}(x_1, \ldots, x_s) \in \mathbb{R}[x_1, \ldots, x_s]$ is a polynomial in $x_1, \ldots, x_s$ of degree $\sum_{q=1}^{s}(y_q - 1) = (y_1 + \cdots + y_s) - s = \beta - s$. By Lemma 2.7, we can write (17) as a sum of the terms of the form

$$w_j^{\alpha+i-r+2-\sum_{i=1}^{s} y_i} \widetilde{p}(\alpha + i - r) \ = \ \widetilde{p}(\alpha + i - r) w_j^{\alpha+i-r+2-\beta} \tag{18}$$

where $\widetilde{p}(x) \in \mathbb{R}[x]$ is a polynomial of degree $\leq \beta - s$.

Recall that β, r are constants fixed by the configuration. Taking into account cyclic permutation, the contribution is a sum of the terms of the form

$$p(\alpha + i) w_j^{\alpha+i-\gamma} \left(\frac{N}{k}\right)^{\alpha+i-t}$$

where $p(x) \in \mathbb{R}[x]$ is a polynomial of degree $\leq \beta - s + 1$, $\gamma \in \mathbb{Z}$ and $\left(\frac{N}{k}\right)^{\alpha+i-t}$ is from choosing the indices from given equivalence classes modulo k. □

Observe that in (13) there are $(\alpha + i)$ degrees of freedom in choosing $j_1, \ldots, j_{\alpha+i}$. Whenever the lost degrees of freedom $t \geq m + 1$, we have

$$\begin{aligned} &\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^{\alpha} \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} N^{\alpha+i-t}\right) \\ &= N^{m-t} \left(\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1}\right)^{\alpha}\right) \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1}{k}\right)^{m-i}\right) \\ &\ll N^{-1} \left| f_1 \left(\frac{k}{w_1}\right) \right|^{2n}, \end{aligned} \tag{19}$$

then since we have required $n(N) = O(\log \log N)$, we only need to consider the contribution from $\mathbb{E}\left[A_N^{\alpha+i}\right]$ that loses at most m degrees of freedom.

Remark 2.9 Even though each term contributes $O(1/N)$, the contribution adds up to $c^{n(N)}/N$ for some $c \in \mathbb{R}$. Thus, in order to remove the contributions from configurations with more than m blocks in this way, we have to require $n = o(\log N)$, so we correct the assumed growth rate $n(N) \gg N^{\epsilon}$ in [2].

We cite the following lemma from [2], which relates the number of blocks to the lost degree of freedom.

Lemma 2.10 *([2]) Fix the number of blocks β, and consider all classes with β blocks. Then the classes among these with the highest number of degrees of freedom are exactly those which contain only* 1*- or* 2*-blocks,* 1*-blocks are matched with exactly one other* 1*-block, and both a's in any* 2*-block are matched with their adjacent entry and no others.*

Remark 2.11 In [2], they prove Lemma 2.10 by showing that the average number of degrees of freedom lost per block is at least 1, and that the average number of degrees of freedom lost per block is 1 if and only if we have the configurations and matchings specified in Lemma 2.10.

By Lemma 2.10, we can restrict ourselves to the configurations that have no more than m blocks.

The following lemma allows us to cancel the contributions from the congruence configurations that contain some constants $w_j \neq w_1$ and reduce the general case to the special one where all the constant $w_j \neq w_1$ are zero.

Lemma 2.12 *Suppose the polynomial $f(x) := \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[x]$ has a zero of order $n > 0$ at x_0. Then for any polynomial p of degree $d < n$,*

$$\sum_\alpha c_\alpha x_0^\alpha p(\alpha) \; = \; 0. \tag{20}$$

Proof. The polynomial $f_d(x) := \sum_\alpha c_\alpha x^\alpha \alpha^d$ has a zero of order $(n-d)$ at x_0 from the fact that

$$f_d(x) = x f'_{d-1}(x) \tag{21}$$

for all $d \in \mathbb{N}$ with $d < n$. □

We are now ready to show that the contributions from the congruence configurations that contain some constants $w_j \neq w_1$ cancel.

Given any polynomial $p(x) \in \mathbb{R}[x]$ and $\gamma, t \in \mathbb{Z}$, notice the following.

1. If $w_j \neq w_1$, $w_j \neq 0$, and p has degree less than $2n$, then

$$\begin{aligned}
&\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^\alpha \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} p(\alpha+i) w_j^{\alpha+i-\gamma} \left(\frac{N}{k}\right)^{\alpha+i-t}\right) \\
&= \frac{k^t}{w_j^\gamma N^t} \sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{w_j}{w_1}\right)^\alpha \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} p(\alpha+i) \left(\frac{w_j N}{k}\right)^{i}\right) \\
&= \frac{k^t}{w_j^\gamma N^t} \sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} \left(\frac{w_j N}{k}\right)^{i} \sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{w_j}{w_1}\right)^\alpha p(\alpha+i) \\
&= 0,
\end{aligned} \tag{22}$$

where we get $\sum_{\alpha=2n}^{4nl} c_\alpha (\frac{w_j}{w_1})^\alpha p(\alpha+i) = 0$ from Lemma 2.12 using the fact that $f_1^{2n}(x) = \sum_{\alpha=2n}^{4nl} c_\alpha x^\alpha \in \mathbb{R}[x]$ has a zero of order $2n$ at w_j/w_1.

2. If $w_j = w_1$, and p has degree less than m, then

$$\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^\alpha \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} p(\alpha+i) w_1^{\alpha+i-\gamma} \left(\frac{N}{k}\right)^{\alpha+i-t}\right)$$
$$= \frac{k^{t-m}}{w_1^{\gamma-m} N^{t-m}} \sum_{\alpha=2n}^{4nl} c_\alpha \left(\sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} p(\alpha+i)\right)$$
$$= 0, \tag{23}$$

where we get $\sum_{i=0}^{m} \binom{m}{i}(-1)^{m-i} p(\alpha+i) = 0$ from Lemma 2.12 using the fact that $(x-1)^m$ has a zero of order m at 1.

By Lemma 2.8, we know given a configuration with β blocks, the polynomial p in the contribution (15) has degree $\le \beta - s + 1$ where s the number of distinct w's in this configuration determined by the chosen congruence classes. In particular, given $\beta \le m$, the polynomial p has degree $\le m-1+1 = m$, and whenever both w_1 and $w_j \ne w_1$ appear in the configuration, the polynomial p has degree $\le m-2+1 = m-1$.

From (22) and (23), we conclude that the configurations with some $w_j \ne w_1$ do not contribute to the moment. We may therefore assume that all all $w_j \ne w_1$ are zero.

2.2 The special case where all $w_j \ne w_1$ are zero

We have reduced to the special case where k_1 of the w_j's are w_1 and the rest $k - k_1$ are 0.

Following the arguments in §3 of [2], we can show that the contributions to the m-th moment from all configurations with fewer than m blocks cancel, and the contributions from all configurations with matchings that lose more than m degrees of freedom become insignificant as $N \to \infty$. In particular, by Lemma 2.10, we are only left with the configurations with m blocks.

Proposition 2.13 *Fix the number of blocks β, the total contribution of configurations with m_1 1-blocks to $\mathbb{E}\left[Tr(A_N^{\alpha+i})\right]$ is*

$$w_1^{\alpha+i-m_1-2(\beta-m_1)} \left(\frac{(\alpha+i)^\beta}{\beta!} + \tilde{p}(\alpha+i)\right) \binom{\beta}{m_1} (k-1)^{\beta-m_1} \mathbb{E}_{k_1}\left[Tr(B^{m_1})\right] \left(\frac{N}{k}\right)^{\alpha+i-\beta}$$
$$+ O_\beta\left((\alpha+i)^\beta \left(\frac{N}{k}\right)^{\alpha+i-\beta-1}\right) \tag{24}$$

where $\tilde{p}$ is a polynomial of degree $\le \beta - 1$.

The proof follows closely from that of Proposition 3.15 in [2], and is given in Appendix B.

Proposition 2.14 *The expected m-th moment in the limit is*

$$\lim_{N\to\infty} \mathbb{E}\left[\mu_{A_N,1}^{(m)}\right] = \frac{1}{k_1} \sum_{m_1=0}^{m} \binom{m}{m_1} \left(\frac{k-1}{w_1}\right)^{m-m_1} \mathbb{E}_{k_1}\left[Tr(B^{m_1})\right]. \tag{25}$$

The proof follows closely from that of Theorem 3.18 in [2], and is given in Appendix C.

Note that the expected first moment in the limit is

$$\lim_{N\to\infty} \mathbb{E}\left[\mu_{A_N,1}^{(1)}\right] \;=\; \frac{k-1}{w_1}. \tag{26}$$

Following the same calculation as in Theorem 3.18 of [2], we obtain the centered m-th moment

$$\mu_{c,1}^{m} := \lim_{N\to\infty} \mathbb{E}\left[\int (x-\mu_{A_N,1}^{(1)})^m d\mu_{AN,1}\right] \;=\; \frac{1}{k_1}\mathbb{E}_{k_1}\left[\mathrm{Tr}(B^m)\right]. \tag{27}$$

3 Weak convergence for blip of size 1

In this section we use the standard technique to show weak convergence for a blip of size one.

Definition 3.1 *(Weak Convergence). A family of probability distribution μ_n weakly converges to μ if and only if for any bounded, continuous f we have*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\mu_n(dx) = \int_{-\infty}^{\infty} f(x)\mu(dx).$$

Since $\mu_{A_N,1}^{(m)}$ is finite, to prove weak convergence we prove the variance of expected m^{th} moment tends to zero as N goes to infinity. That is,

$$\lim_{N\to\infty} \mathbb{E}[(\mu_{A_N,1}^{(m)})^2] - \mathbb{E}[(\mu_{A_N,1}^{(m)})]^2 = 0.$$

By (12), we have that

$$\begin{aligned}\mathbb{E}[(\mu_{A_N,1}^{(m)})^2] \;&=\frac{1}{k_1^2}\sum_{\alpha=2n}^{4nl}\sum_{\beta=2n}^{4nl} c_\alpha c_\beta \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}(-1)^{i+j}(\frac{w_1N}{k})^{2m-(i+j)-(\alpha+\beta)}\\ &\left(\sum_{\substack{C_{i+\alpha}\\ C_{j+\beta}}}\mathbb{E}[C_{i+\alpha}C_{j+\beta}]\right),\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbb{E}\left[\mu_{A_N,1}^{(m)}\right]^2 \;&=\frac{1}{k_1^2}\sum_{\alpha=2n}^{4n}\sum_{\beta=2n}^{4n} c_\alpha c_\beta \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}(-1)^{i+j}\left(\frac{w_1N}{k}\right)^{2m-(i+j)-(\alpha+\beta)}\\ &\left(\sum_{\substack{C_{i+\alpha},\\ C_{j+\beta}}}\mathbb{E}\left[C_{i+\alpha}\right]\mathbb{E}\left[C_{j+\beta}\right]\right),\end{aligned} \tag{29}$$

where C_t denotes the cycle $m_{i_1i_2}m_{i_2i_3}\dots m_{i_ti_1}$ Notice that the difference cancels unless there exists $a_{t_1t_2}$ such that $a_{t_1t_2}\in C_{i+\alpha}$ and $C_{j+\beta}$. Therefore we only need to count the pair of cycles where $C_{i+\alpha}$ and $C_{j+\beta}$ has at least one common a. We call such pair of cycles the crossover terms.

Lemma 3.2 *The contributions of crossover terms to* $\mathbb{E}[(\mu^{(m)}_{A_N,1})]^2$ *is 0 as* $N\to\infty$.

Proof. $\mathbb{E}[(\mu^{(m)}_{A_N,1})]^2$ is the product of

$$\frac{1}{k_1}\sum_{\alpha=2n}^{4nl}c_\alpha\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}(\frac{w_1N}{k})^{m-i-\alpha}\sum_{C_{i+\alpha}}\mathbb{E}[C_{i+\alpha}]$$

and

$$\frac{1}{k_1}\sum_{\beta=2n}^{4nl}c_\beta\sum_{i=0}^{m}\binom{m}{j}(-1)^{m-j}(\frac{w_1N}{k})^{m-j-\beta}\sum_{C_{j+\beta}}\mathbb{E}[C_{j+\beta}].$$

Suppose we fix a pair of congruence configurations of $C_{i+\alpha}$ and $C_{i+\alpha}$ such that there is a common $a_{t_1t_2}$ in the two cycles, and $C_{i+\alpha}$ has b_1 blocks while $C_{j+\beta}$ has b_2 blocks. If either of b_1,b_2 is less than m, then by 22 and 23 their product makes 0 contribution. So we know the that configuration contributes only when $b_1+b_2\geq 2m$. By Lemma 2.10, each block loses at least 1 degree of freedom. However, due to the common $a_{t_1t_2}$ of $C_{i+\alpha}$ and $C_{j+\beta}$, there is a block that loses at least 2 degrees of freedom, so in total at least $b_1+b_2+1\geq 2m+1$ degree of freedom is lost. Thus by (19), the crossover terms contribute to 0 when $N\to\infty$. □

Now it is sufficient to look at the contribution from crossovers to $\mathbb{E}[(\mu^{(m)}_{A_N,1})^2]$. For general k_1, the contributions of the crossovers doesn't necessarily go to 0 as $N\to\infty$. We want to show that for $k_1=1$, the contribution from the crossovers does go to 0. In order to show this, we first reduce the general W to the simplest case where all $w_j\neq w_1$ are zero.

Lemma 3.3 *The contribution from the congruence configurations that contain* $w_j\neq w_1$ *to* $\mathbb{E}[(\mu^{(m)}_{A_N,1})^2]$ *is* 0.

Proof. Fix a pair of congruence configuration. Say $w_{i_1},w_{j_2},\dots,w_{j_s}$ appears in the cyclic product $C_{i+\alpha}$ and w_{j_q} appears x_q times, separated by the blocks into y_q parts, and $w_{j'_1},\dots,w_{j'_{s'}}$ appears in the cyclic product $C_{j+\beta}$, $w_{j'_q}$ appears ${x'}_{q'}$ times and are separated by the blocks into $y_{q'}$ parts. The sum of y's should be the total number of blocks, so we have $y_1+\dots+y_s+{y'}_1+\dots+{y'}_{s'}=b_1+b_2$. By Lemma 2.10, the total lost degree of freedom is at least b_1+b_2. On the other hand, by (19) we know that the total lost of degree of freedom should be at most $2m$. Therefore we have $b_1+b_2\leq 2m$.

By Lemma 2.8, with the congruence configuration fixed, the total number of ways to place w_j and $w_{j'}$ is

$$(\alpha+i)\sum_{\substack{x_i\geq y_i\\ x_1+\dots+x_s=\alpha+i-r_1}}\prod_{q=1}^{s}\binom{x_q-1}{y_q-1}w_{j_q}^{x_q}(\beta+j)\sum_{\substack{x'_{i'}\geq y'_{i'}\\ x'_1+\dots+x'_{s'}=\beta+i-r_2}}\prod_{q'=1}^{s'}\binom{x'_{q'}-1}{y'_{q'}-1}w_{j'_q}^{x'_q}\qquad(30)$$

where r_1, r_2 are the number of a in each cycle. Since $y_i, y'_{i'}$ are fixed, the above expression can be written as

$$(\alpha+i) \sum_{\substack{x_i \geq y_i \\ x_1+\cdots+x_s=\alpha+i-r_1}} p_{y_1,\ldots,y_s}(x_1,\ldots,x_s) w_{j_1}^{x_1} \cdots w_{j_s}^{x_s} (\beta+j) \cdot \sum_{\substack{x'_{i'} \geq y'_{i'} \\ x'_1+\cdots+x'_{s'}=\beta+i-r_2}} p_{y'_1,\ldots,y'_{s'}}(x'_1,\ldots,x'_{s'}) w_{j'_1}^{x'_1} \cdots w_{j'_{s'}}^{x'_{s'}} \tag{31}$$

where $p_{y_1,\ldots,y_s}$ and $p_{y'_1,\ldots,y'_{s'}}$ are polynomials with variables $x_i, x'_{i'}$, and the sum of their degree is $y_1+\cdots+y_s+y'_1+\cdots+y'_{s'}-s-s' = b_1+b_2-s-s'$. Then (31) is a sum of terms of the form $p_1(\alpha+j)w_j^{\alpha+j-\gamma_1} p_2(\beta+i)w_{j'}^{\beta+i-\gamma}$, where sum of degrees of p_1 and p_2 is $b_1+b_2+2-s-s'$, s or s' should be at least 2. Since $s, s' \geq 1, b_1+b_2 \leq 2m$, the sum of degree of p_1 and p_2 would be at most $2m-1$. Therefore at least one of p_1, p_2 will have degree $\leq m-1$. Without loss of generality say p_1 has degree $\leq m-1$. Then by (22) and (23)

$$\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1 N}\right)^\alpha \left(\sum_{i=0}^{m} \binom{m}{i} \left(-\frac{w_1 N}{k}\right)^{m-i} p_1(\alpha+i) w_j^{\alpha+i-\gamma} \left(\frac{N}{k}\right)^{\alpha+i-t}\right) \;=\; 0. \tag{32}$$

Therefore, the contribution of the terms $p_1(\alpha+j)w_j^{\alpha+j-\gamma_1} p_2(\beta+i)w_{j'}^{\beta+i-\gamma}$ to will be 0. Thus the contribution from congruence configurations containing $w_j \neq w_1$ to $\mathbb{E}[(\mu_{A_N,1}^{(m)})^2]$ is 0 as $N \to \infty$. □

Now we can restrict ourselves to the simplest case where $w_j \neq w_1$ are all 0. We want to prove that when $k_1 = 1$, the contribution from the crossovers to $\mathbb{E}[(\mu_{A_N,1}^{(m)})^2]$ is 0. Assume $w_1 \neq 0$ and $w_2 = \cdots = w_k = 0$.

Theorem 3.4 *When $k_1 = 1$ and $w_j = 0$ for all $w_j \neq w_1$, we have*

$$\lim_{N\to\infty} Var\left[\mu_{A_N,1}^{(m)}\right] \;=\; 0. \tag{33}$$

Proof. We are left to prove that the contributions from crossovers to $\mathbb{E}[(\mu_{A_N,1}^{(m)})^2]$ is 0.

Fix the pair of congruence configuration at $C_{i+\alpha}$ and $C_{j+\beta}$. Suppose there are b_1 blocks in $C_{i+\alpha}$ and b_2 blocks in $C_{j+\beta}$. If $b_1 < m$ or $b_2 < m$, then

$$\sum_{i=0}^{m}\sum_{j=0}^{m} \binom{m}{i}\binom{m}{j}(-1)^{2m-i-j} i^{p'} j^{q'} \;=\; \sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i} i^{p'} \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-i} j^{q'} \;=\; 0$$

for all integers $0 \leq p' \leq b_1$ and $0 \leq q' \leq b_2$, so that the contributions from this configuration cancel out. So we only need to look at configurations with $b_1 \geq m$ and $b_2 \geq m$.

Now notice that in this W, $w_j = 0$ for all $j \not\equiv 1$ (k). Thus, if there is some 1-block in $C_{i+\alpha}$ or $C_{j+\beta}$, then both $\mathbb{E}\left[C_{i+\alpha}C_{j+\beta}\right]$ and $\mathbb{E}\left[C_{i+\alpha}\right]\mathbb{E}\left[C_{j+\beta}\right]$ are 0. Therefore, we can restrict ourselves to the configurations where all the blocks are 2-blocks.

By Lemma 2.10 and Equation (19), We only need to consider congruence configurations where $b_1 + b_2 \leq 2m$. Combining with $b_1 \geq m, b_2 \geq m$, We require $b_1 = b_2 = m$, and both a's in 2-blocks matched with their adjacent entry. But then crossover matchings between $C_{i+\alpha}$ and $C_{j+\beta}$ become impossible. Therefore, we conclude that

$$\lim_{N\to\infty} \mathrm{Var}\left[\mu_{A_N,1}^{(m)}\right] \;=\; 0. \tag{34}$$

□

A Proof of Lemma 2.7

Lemma A.1 *Fix $s \in \mathbb{N}$ with $s \geq 2$. For $\eta \in \mathbb{N}$ with $\eta \geq s$ and distinct $w_1, \ldots, w_s$, we have*

$$\sum_{\substack{x_1+\cdots+x_s=\eta \\ x_1,\ldots,x_s\geq 1}} w_1^{x_1}\ldots w_s^{x_s} \;=\; \frac{\sum_{l=1}^{s} w_l^{\eta+2-s} f_l(w_1,\ldots,w_s)}{\prod_{1\leq i<j\leq s}(w_i - w_j)}, \tag{35}$$

where each $f_l(w_1, \ldots, w_s) \in \mathbb{R}[w_1, \ldots, w_s]$ is a homogeneous polynomial of degree $\binom{s}{2} + s - 2$.

Proof. Induct on s. When $s = 2$, by geometric progression, for all $\eta \in \mathbb{N}$ with $\eta \geq 2$, we have

$$\sum_{\substack{x_1+x_2=\eta \\ x_1,x_2\geq 1}} w_1^{x_1} w_2^{x_2} \;=\; \frac{w_1^{\eta} w_2 - w_2^{\eta} w_1}{w_1 - w_2}. \tag{36}$$

Suppose $s \in \mathbb{N}$ with $s \geq 2$ and equation (35) holds for all $\eta \in \mathbb{N}$ with $\eta \geq s$. Then for $\eta \geq s + 1$,

$$\begin{aligned}
\sum_{\substack{x_1+\cdots+x_{s+1}=\eta \\ x_1,\ldots,x_{s+1}\geq 1}} w_1^{x_1}\ldots w_{s+1}^{x_{s+1}} &= \sum_{x_{s+1}=1}^{\eta-s} \sum_{\substack{x_1+\cdots+x_s=\eta-x_{s+1} \\ x_1,\ldots,x_s\geq 1}} w_1^{x_1}\ldots w_s^{x_s} w_{s+1}^{x_{s+1}} \\
&= \sum_{x_{s+1}=1}^{\eta-s} \frac{\sum_{l=1}^{s} w_l^{\eta-x_{s+1}+2-s} f_l(w_1,\ldots,w_s)}{\prod_{1\leq i<j\leq s}(w_i - w_j)} w_{s+1}^{x_{s+1}} \\
&= \sum_{l=1}^{s} \frac{f_l(w_1,\ldots,w_s)}{\prod_{1\leq i<j\leq s}(w_i - w_j)} \frac{w_l^{\eta+1-s} w_{s+1} - w_l w_{s+1}^{\eta+1-s}}{1 - \frac{w_{s+1}}{w_l}} \\
&= \frac{\sum_{l=1}^{s} w_l^{\eta+1-s}(w_l w_{s+1} \prod_{\substack{1\leq i\leq s \\ i\neq l}}(w_i - w_{s+1}) f_l(w_1,\ldots,w_s))}{\prod_{1\leq i<j\leq s+1}(w_i - w_j)} \\
&\quad - \frac{w_{s+1}^{\eta+1-s}(\sum_{l=1}^{s} w_l^2 \prod_{\substack{1\leq i\leq s \\ i\neq l}}(w_i - w_{s+1}) f_l(w_1,\ldots,w_s))}{\prod_{1\leq i<j\leq s+1}(w_i - w_j)},
\end{aligned} \tag{37}$$

where each $w_l w_j \prod_{\substack{1\le i\le s\\ i\neq l}}(w_i - w_{s+1}) f_l(w_1,\dots,w_s)$ is a homogeneous polynomial in $w_1,\dots,w_{s+1}$ of degree $2+(s-1)+(\binom{s}{2}+s-2) = \binom{s+1}{2}+s+1-2$. $\square$

Lemma A.2 *Fix $s \in \mathbb{N}$ with $s \ge 2$, $q \in \mathbb{N}\cup\{0\}$, and $\alpha_1,\dots,\alpha_q \in \mathbb{N}_{\le s}$ (may not be distinct). For $\eta \in \mathbb{N}$ with $\eta \ge s$ and distinct $w_1,\dots,w_s$, we have*

$$\sum_{\substack{x_1+\dots+x_s=\eta\\ x_1,\dots,x_s\ge 1}} x_{\alpha_1}\dots x_{\alpha_q} w_1^{x_1}\dots w_s^{x_s} \;=\; \frac{\sum_{l=1}^s w_l^{\eta+2-s} f_{l,\eta}(w_1,\dots,w_s)}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}, \tag{38}$$

where each $f_{l,\eta}(w_1,\dots,w_s) \in \mathbb{R}[\eta][w_1,\dots,w_s]$ is a homogeneous polynomial in $w_1,\dots,w_s$ of degree $2^q\binom{s}{2}+s-2$. Furthermore, the coefficients in the polynomial $f_{l,\eta}(w_1,\dots,w_s)$ are polynomials in η of degree $\le q$.

Proof. Induct on q. The case $q=0$ was proved in Lemma A.1. Suppose $q\in\mathbb{N}$ and we have

$$\sum_{\substack{x_1+\dots+x_s=\eta\\ x_1,\dots,x_s\ge 1}} x_{\alpha_1}\dots x_{\alpha_{q-1}} w_1^{x_1}\dots w_s^{x_s} \;=\; \frac{\sum_{l=1}^s w_l^{\eta+2-s} f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}} \tag{39}$$

for all $\eta\in\mathbb{N}$ with $\eta\ge s$, where each $f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)$ is a homogeneous polynomial in $w_1,\dots,w_s$ of degree $2^{q-1}\binom{s}{2}+s-2$ and the coefficients are polynomials in η of degree $\le q-1$. Then

$$\begin{aligned}
&\sum_{\substack{x_1+\dots+x_s=\eta\\ x_1,\dots,x_s\ge 1}} x_{\alpha_1}\dots x_{\alpha_q} w_1^{x_1}\dots w_s^{x_s} \;=\; w_{\alpha_q}\frac{\partial}{\partial w_{\alpha_q}} \sum_{\substack{x_1+\dots+x_s=\eta\\ x_1,\dots,x_s\ge 1}} x_{\alpha_1}\dots x_{\alpha_{q-1}} w_1^{x_1}\dots w_s^{x_s}\\
&= \frac{\sum_{l=1}^s w_l^{\eta+2-s} w_{\alpha_q} \frac{\partial f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)}{\partial w_{\alpha_q}} (\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}\\
&\quad + \frac{w_{\alpha_q}^{\eta+2-s}(\eta+2-s) f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)(\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}\\
&\quad - \frac{\sum_{l=1}^s w_l^{\eta+2-s} f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s) w_{\alpha_q} \frac{\partial}{\partial w_{\alpha_q}} (\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}.
\end{aligned} \tag{40}$$

Note that by induction hypothesis, we have

1. $w_{\alpha_q}\frac{\partial f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)}{\partial w_{\alpha_q}}(\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}$ is a homogeneous polynomial in $w_1,\dots,w_s$ of degree $1+(2^{q-1}\binom{s}{2}+s-2-1)+2^{q-1}\binom{s}{2} = 2^q\binom{s}{2}+s-2$ and the coefficients are polynomials in η of degree $\le q-1$;

2. $(\eta+2-s) f_{l,\eta,\alpha_1,\dots,\alpha_{q-1}}(w_1,\dots,w_s)(\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}$ is a homogeneous polynomial in $w_1,\dots,w_s$ of degree $(2^{q-1}\binom{s}{2}+s-2)+2^{q-1}\binom{s}{2} = 2^q\binom{s}{2}+s-2$ and the coefficients are polynomials in η of degree $\le q$;

3. $f_{l,\eta,\alpha_1,\ldots,\alpha_{q-1}}(w_1,\ldots,w_s)w_{\alpha_q}\frac{\partial}{\partial w_{\alpha_q}}(\prod_{1\le i<j\le s}(w_i-w_j))^{2^{q-1}}$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $(2^{q-1}\binom{s}{2}+s-2)+1+(2^{q-1}\binom{s}{2}-1)=2^q\binom{s}{2}+s-2$ and the coefficients are polynomials in η of degree $\le q-1$.

Therefore, after collecting the terms, we get (38). □

Lemma A.3 *Fix $s\in\mathbb{N}$ with $s\ge 2$ and some polynomial $p(x_1,\ldots,x_s)\in\mathbb{R}[x_1,\ldots,x_s]$ of degree q. For $\eta\in\mathbb{N}$ with $\eta\ge s$ and distinct $w_1,\ldots,w_s$, we have*

$$\sum_{\substack{x_1+\cdots+x_s=\eta\\ x_1,\ldots,x_s\ge 1}} p(x_1,\ldots,x_s)w_1^{x_1}\cdots w_s^{x_s} = \frac{\sum_{l=1}^s w_l^{\eta+2-s}f_{l,\eta}(w_1,\ldots,w_s)}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}, \tag{41}$$

where each $f_{l,\eta}(w_1,\ldots,w_s)\in\mathbb{R}[\eta][w_1,\ldots,w_s]$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $2^q\binom{s}{2}+s-2$. Furthermore, the coefficients in the polynomial $f_{l,\eta}(w_1,\ldots,w_s)$ are polynomials in η of degree $\le q$.

Proof. By Lemma A.2, fix any $d\in\mathbb{N}\cup\{0\}$ with $d\le q$, and $\alpha_1,\ldots,\alpha_d\in\mathbb{N}_{\le s}$, we have

$$\sum_{\substack{x_1+\cdots+x_s=\eta\\ x_1,\ldots,x_s\ge 1}} x_{\alpha_1}\ldots x_{\alpha_d}w_1^{x_1}\ldots w_s^{x_s} = \frac{\sum_{l=1}^s w_l^{\eta+2-s}\tilde{f}_{l,\eta}(w_1,\ldots,w_s)(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q-2^d}}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}, \tag{42}$$

for some degree $2^d\binom{s}{2}+s-2$ homogeneous polynomials $\tilde{f}_{l,\eta}(w_1,\ldots,w_s)\in\mathbb{R}[\eta][w_1,\ldots,w_s]$ whose coefficients are polynomials in η of degree $\le d$. Then $\tilde{f}_{l,\eta}(w_1,\ldots,w_s)(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q-2^d}$ are homogeneous polynomials in $w_1,\ldots,w_s$ of degree $2^d\binom{s}{2}+s-2+(2^q-2^d)\binom{s}{2}=2^q\binom{s}{2}+s-2$, and the coefficients are polynomials in η of degree $\le d\le q$, and the result follows. □

Lemma A.4 *Fix $s\in\mathbb{N}$ with $s\ge 2$, $q\in\mathbb{N}\cup\{0\}$, $y_1,\ldots,y_s\in\mathbb{N}$, and $\alpha_1,\ldots,\alpha_q\in\mathbb{N}_{\le s}$ (may not be distinct). For $\eta\in\mathbb{N}$ with $\eta\ge\sum_{i=1}^s y_i$ and distinct $w_1,\ldots,w_s$, we have*

$$\sum_{\substack{x_1+\cdots+x_s=\eta\\ x_i\ge y_i}} x_{\alpha_1}\ldots x_{\alpha_q}w_1^{x_1}\ldots w_s^{x_s} = \frac{\sum_{l=1}^s w_l^{\eta+2-\sum_{i=1}^s y_i}f_{l,\eta}(w_1,\ldots,w_s)}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}}, \tag{43}$$

where each $f_{l,\eta}(w_1,\ldots,w_s)\in\mathbb{R}[\eta][w_1,\ldots,w_s]$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $2^q\binom{s}{2}+(\sum_{i=1}^s y_i)-2$. Moreover, the coefficients in the polynomial $f_{l,\eta}(w_1,\ldots,w_s)$ are polynomials in η of degree $\le q$.

Proof. By Lemma A.3, we have

$$\sum_{\substack{x_1+\cdots+x_s=\eta\\ x_i\ge y_i}} x_{\alpha_1}\ldots x_{\alpha_q}w_1^{x_1}\ldots w_s^{x_s} = \prod_{i=1}^s w_i^{y_i-1}\frac{\sum_{l=1}^s w_l^{\eta+2-\sum_{i=1}^s y_i}\tilde{f}_{l,\eta}(w_1,\ldots,w_s)}{(\prod_{1\le i<j\le s}(w_i-w_j))^{2^q}},$$

where each $\tilde{f}_{l,\eta}(w_1,\ldots,w_s) \in \mathbb{R}[\eta][w_1,\ldots,w_s]$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $2^q\binom{s}{2}+s-2$, and the coefficients of $f_{l,\eta}(w_1,\ldots,w_s)$ are polynomials in η of degree $\leq q$. Then each $f_{l,\eta}(w_1,\ldots,w_s) = \prod_{i=1}^{s} w_i^{y_i-1}\tilde{f}_{l,\eta}(w_1,\ldots,w_s)$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $2^q\binom{s}{2} + (\sum_{i=1}^{s} y_i) - 2$, and the coefficients are polynomials in η of degree $\leq q$. $\square$

Lemma A.5 *Fix $s \in \mathbb{N}$ with $s \geq 2$ and some polynomial $p(x_1,\ldots,x_s) \in \mathbb{R}[x_1,\ldots,x_s]$ of degree q. For $\eta \in \mathbb{N}$ with $\eta \geq \sum_{i=1}^{s} y_i$ and distinct $w_1,\ldots,w_s$, we have*

$$\sum_{\substack{x_1+\cdots+x_s=\eta \\ x_i\geq y_i}} p(x_1,\ldots,x_s)w_1^{x_1}\ldots w_s^{x_s} = \frac{\sum_{l=1}^{s} w_l^{\eta+2-\sum_{i=1}^{s} y_i} f_{l,\eta}(w_1,\ldots,w_s)}{(\prod_{1\leq i<j\leq s}(w_i-w_j))^{2^q}}, \qquad (44)$$

where each $f_{l,\eta}(w_1,\ldots,w_s) \in \mathbb{R}[\eta][w_1,\ldots,w_s]$ is a homogeneous polynomial in $w_1,\ldots,w_s$ of degree $2^q\binom{s}{2}+(\sum_{i=1}^{s} y_i)-2$. Moreover, the coefficients in the polynomial $f_{l,\eta}(w_1,\ldots,w_s)$ are polynomials in η of degree $\leq q$.

Proof. By Lemma A.4, fix any $d \in \mathbb{N} \cup \{0\}$ with $d \leq q$, and $\alpha_1,\ldots,\alpha_d \in \mathbb{N}_{\leq s}$, we have

$$\sum_{\substack{x_1+\cdots+x_s=\eta \\ x_i\geq y_i}} x_{\alpha_1}\ldots x_{\alpha_d} w_1^{x_1}\ldots w_s^{x_s} \qquad (45)$$

$$= \frac{\sum_{l=1}^{s} w_l^{\eta+2-s}\tilde{f}_{l,\eta}(w_1,\ldots,w_s)(\prod_{1\leq i<j\leq s}(w_i-w_j))^{2^q-2^d}}{(\prod_{1\leq i<j\leq s}(w_i-w_j))^{2^q}} \qquad (46)$$

for some degree $2^d\binom{s}{2} + (\sum_{i=1}^{s} y_i) - 2$ homogeneous polynomials $\tilde{f}_{l,\eta}(w_1,\ldots,w_s) \in \mathbb{R}[\eta][w_1,\ldots,w_s]$ whose coefficients are polynomials in η of degree $\leq d$. Then we have $\tilde{f}_{l,\eta}(w_1,\ldots,w_s)(\prod_{1\leq i<j\leq s}(w_i-w_j))^{2^q-2^d}$ are homogeneous polynomials in $w_1,\ldots,w_s$ of degree $2^d\binom{s}{2} + (\sum_{i=1}^{s} y_i) - 2 + (2^q-2^d)\binom{s}{2} = 2^q\binom{s}{2} + (\sum_{i=1}^{s} y_i) - 2$, and the coefficents are polynomials in η of degree $\leq d \leq q$, and the result follows. $\square$

B Proof of Proposition 2.13

By Lemma 2.10, the configurations with the highest number of degrees of freedom contain only 1- and 2-blocks. The number of ways to arrange the constants w_1's and the blocks (take all blocks to be identical) is

$$\frac{(\alpha+i)^\beta}{\beta!} + \tilde{p}(\alpha+i),$$

where $\tilde{p}$ is a polynomial of degree $\leq \beta - 1$, and the number of ways to choose the 1-blocks among all the blocks is $\binom{\beta}{m_1}$.

Now we assign the equivalence classes modulo k of the inner indices of the 2-blocks. The number of ways to assign inner indices of 2-blocks is $(k-1)^{\beta-m_1}$. The number of ways to assign indices of the 1-blocks is the same as the number of cyclic product $b_{i_1i_2}b_{i_2i_3}\cdots b_{i_{m_1}i_1}$, where i_j's are chosen from k_1 residues modulo k with the b's matched in pairs under the restriction that $i_j \neq i_{j+1}$ for all j. Thus it is the expected trace of m_1^{th} power of $k_1 \times k_1$ GOE, which is $\mathbb{E}_{k_1}\left[\operatorname{Tr}\left(B^{m_1}\right)\right]$.

Finally, for each index, once we have specified its congruence class modulo k, the number of ways to choose it from $\{1, 2, \ldots, N\}$ is $\left(\frac{N}{k}\right)^{\alpha+i-\beta} + O(\frac{N}{k})^{\alpha+i-\beta-1}$.

C Proof of Proposition 2.14

By Proposition 2.13 and (12), we get the contribution from the configurations with β blocks to the expected m^{th} moment of the blip is

$$\frac{1}{k_1}\sum_{\alpha=2n}^{4nl} c_\alpha \left(\frac{k}{w_1N}\right)^\alpha \sum_{i=0}^{m}\binom{m}{i}\left(-\frac{w_1N}{k}\right)^{m-i}$$
$$\left(\sum_{m_1=0}^{\beta} w_1^{\alpha+i-m_1-2(\beta-m_1)}\binom{\beta}{m_1}(k-1)^{\beta-m_1}\mathbb{E}_{k_1}\left[\operatorname{Tr}(B^{m_1})\right]\right)$$
$$\left(\frac{(\alpha+i)^\beta}{\beta!}+\tilde{p}(\alpha+i)\right)\left(\frac{N}{k}\right)^{\alpha+i-\beta} + O_\beta\left((\alpha+i)^\beta\left(\frac{N}{k}\right)^{\alpha+i-\beta-1}\right). \tag{47}$$

Recall that by (19) and Lemma 2.10, the contribution becomes insignificant as $N \to \infty$ if $\beta > m$. On the other hand, given any polynomial $p(x) \in \mathbb{R}[x]$ of degree less than m and $t \in \mathbb{Z}$, we have $\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{w_1N}{k}\right)^{m-i} p(\alpha+i)\left(\frac{N}{k}\right)^{\alpha+i-t} = \left(\frac{N}{k}\right)^{m+\alpha-t}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}p(\alpha+i) = 0$ from Lemma 2.12 using the fact that $(x-1)^m$ has a zero of order m at 1, so the contribution cancels out if $\beta < m$. Therefore, only the configurations with m blocks will contribute.

We set $\beta = m$ in (47), and use the identity

$$\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}i^j \;=\; \begin{cases} 0 & \text{if } j = 0, 1, \ldots, m-1, \\ m! & \text{if } j = m, \end{cases}$$

and the fact that $\sum_{\alpha=2n}^{4nl} c_\alpha = f_1^{2n}(1) = 1$ to get the expected m-th moment

$$\lim_{N\to\infty}\mathbb{E}\left[\mu_{A_N,1}^{(m)}\right] \;=\; \frac{1}{k_1}\sum_{m_1=0}^{m}\binom{m}{m_1}\left(\frac{k-1}{w_1}\right)^{m-m_1}\mathbb{E}_{k_1}\left[\operatorname{Tr}(B^{m_1})\right]. \tag{48}$$

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