An Examination of Linear Combinations of Skew-Adjoint Type Algebraic Curvature Tensors

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Abstract In this paper a new potential invariant of algebraic curvature tensors, the signature, will be examined. Furthermore, linear combinations of skew-adjoint type algebraic curvature tensors will be thoroughly examined so as to provide some insight into the possible forms of algebraic curvature tensors.

Keywords : algebraic curvature tensors; signature conjecture

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1 Introduction

Algebraic curvature tensors come in a variety of different forms, and there are a variety of different properties that are of interest. In fact, it can often be difficult to distinguish between two algebraic curvature tensors, hence the development of invariants that can distinguish between different algebraic curvature tensors is useful. We will be examining a new potential invariant, the signature. Furthermore, the study of linear combinations of skew-adjoint type algebraic curvature tensors turns out to be quite interesting as it provides insight to the possible forms of such algebraic curvature tensors and differs from the self-adjoint case in several subtle ways.

We begin with some basic definitions along with some notes on notation. [4] provides a much more detailed examination of generic *algebraic curvature tensors* and their properties, and is a good starting point for those wishing to learn more about this subject. We will then proceed to examine the signature as a potential invariant. Next we will look at the kernel of linear combinations of skew-adjoint type algebraic curvature tensors and discuss decomposability. Finally we will discuss $\eta(R)$, which is a measurement of how efficiently one can express an algebraic curvature tensor.

Definition 1.1 *On any vector space V, an* inner product *is a map $\phi : V \times V \to \mathbb{R}$ with the following properties:*

1) multilinear entries;

2) $\phi(u, v) = \phi(v, u)$ for all $u, v \in V$;

3) $\phi(v, v) \geq 0$ for all $v \in V$, with equality if and only if $v = 0$ (positive definite);

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Definition 1.2 *On any vector space V,* algebraic curvature tensor *is a map $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:*

1) multilinear in all four entries;

2) $R(x, y, z, w) = -R(y, x, z, w)$;

3) $R(x, y, z, w) = R(z, w, x, y)$;

4) $R(x, y, z, w) + R(x, w, y, z) + R(x, z, w, y) = 0$;

We will now introduce some basic properties of algebraic curvature tensors.

Definition 1.3 *The* kernel *of an algebraic curvature tensor R, denoted $Ker(R)$, is the set of all $x \in V$ such that $R(x, y, z, w) = 0$ for any $y, z, w \in V$.*

We can use Properties 2 and 3 of algebraic curvature tensors to show that for any $x \in Ker(R)$ and $y, z, w \in V$, $R(x, y, z, w) = R(y, x, z, w) = R(y, z, x, w) = R(y, z, w, x) = 0$ [2].

Given a manifold with connection M and a point p on M, one can extract the tangent space V at p, an algebraic curvature tensor defined on V, and, provided the manifold had a metric, an inner product ϕ. The tuple (V, ϕ, R) is called a model space, and the pair (V, R) is called a weak model space. Understanding of the model space at a point helps understand the manifold at the point, hence a proper understanding of algebraic curvature tensors defined on a vector space with an inner product is important to a proper understanding of manifolds. One important property a model space, or a weak model space, can have is decomposability.

Definition 1.4 *A* model space (V, ϕ, R) *is* decomposable *if there exist sub-vector spaces V_1 and V_2 such that $V_1 \bigoplus V_2 = V$, and $(V_1, \phi_1, R_1) \bigoplus (V_2, \phi_2, R_2) = (V, \phi, R)$, where each ϕ_i and R_i are respectively ϕ and R restricted to V_i . Similarly, a* weak model space (V, R) *is* decomposable *if there exist sub-vector spaces V_1 and V_2 such that $V_1 \bigoplus V_2 = V$, and $(V_1, R_1) \bigoplus (V_2, R_2) = (V, R)$.*

From now on we fix some vector space V of dimension n and an inner product ϕ acting on V. The set of all possible algebraic curvature tensors acting on V is denoted by $\mathbb{A}(V)$. There is a special suspace of algebraic curvature tensors, called canonical algebraic curvature tensors, which are known to be a spanning set of $\mathbb{A}(V)$. Canonical algebraic curvature tensors are associated with linear transformations, and there are two basic types. Firstly we have those defined with self-adjoint linear transformations.

Definition 1.5 *Let $T : V \to V$ be a self-adjoint linear transformation. The canonical algebraic curvature tensor $R_T^S : V \times V \times V \times V \to \mathbb{R}$ is defined as follows:*

$$R_T^S(x, y, z, w) = \phi(Tx, w)\phi(Ty, z) - \phi(Tx, z)\phi(Ty, w)$$

Such a canonical algebraic curvature tensor is of the self-adjoint type *and is referred to as such.*

The second flavor of canonical algebraic curvature tensors are defined with skew-adjoint linear transformations.

Definition 1.6 *Let $T : V \to V$ be an skew-adjoint linear transformation. The canonical algebraic curvature tensor $R_T : V \times V \times V \times V \to \mathbb{R}$ is defined as follows:*

$$R_T(x, y, z, w) = \phi(Tx, w)\phi(Ty, z) - \phi(Tx, z)\phi(Ty, w) - 2\phi(Tx, y)\phi(Tz, w)$$

Such a canonical algebraic curvature tensor is of the skew-adjoint type *and is referred to as such.*

By picking a basis for V we can identify V with $\mathbb{R}^n$ and T with a $n \times n$ matrix A. It is convenient to use matrices for calculations, so from now on we will fix a basis $\{e_1, \ldots, e_n\}$ of V and for a self-adjoint $n \times n$ matrix A the expression R^S_A denotes the self-adjoint type canonical algebraic curvature tensor R^S_T where the self-adjoint linear transformation T is determined by the matrix A and this choice of basis. Similarly R_A denotes the skew-adjoint type canonical algebraic curvature tensor R_T where T is the skew-adjoint linear transformation determined by the skew-adjoint matrix A. R^S_A and R_A are meaningless if the matrix A is not self-adjoint or skew-adjoint respectively.

One key property of canonical algebraic curvature tensors is: given any self-adjoint matrix A, $\alpha \in (R)$, and $x, y, z, w \in V$, $R_{\alpha A}(x, y, z, w) = \alpha^2 R_A(x, y, z, w)$[4]. One can easily use the definition to expand the left hand side and demonstrate this:

$$\begin{aligned}
R_{\alpha A}(x, y, z, w) &= \phi(\alpha Ax, w)\phi(\alpha Ay, z) - \phi(\alpha Ax, z)\phi(\alpha Ay, w) - 2\phi(\alpha Ax, y)\phi(\alpha Az, w) \\
&= \alpha^2\phi(Ax, w)\phi(Ay, z) - \alpha^2\phi(Ax, z)\phi(Ay, w) - 2\alpha^2\phi(Ax, y)\phi(Az, w) \\
&= \alpha^2 R_A(x, y, z, w).
\end{aligned}$$

It should be noted that both types (Definition 1.5 and Definition 1.6) form spanning sets of $\mathbb{A}(V)$[4]. Hence the notion of the least number of each type of canonical curvature tensor required to express any curvature tensor R is of interest.

Definition 1.7 *For any $R \in \mathbb{A}(V)$, $\nu(R) := min\{k | \sum_{i=1}^k \alpha_i R^S_{A_i} = R\}$, where each α_i is an element of V, and $R^S_{A_i}$ is a self-adjoint type canonical algebraic curvature tensor, as in Definition 1.5.*

Definition 1.8 *For any $R \in \mathbb{A}(V)$, $\eta(R) := min\{k | \sum_{i=1}^k \alpha_i R_{A_i} = R\}$, where each α_i is an element of V, and R_{A_i} is a skew-adjoint type canonical algebraic curvature tensor as in Definition 1.6.*

We can also consider ν and η as functions of n, the dimension of the vector space, instead of individual algebraic curvature tensors.

Definition 1.9 *For any vector space V of dimension n,*

$$\eta(n) := \max_{R \in \mathbb{A}(V)} \{\eta(R)\}$$

Similarly

$$\nu(n) := \max_{R \in \mathbb{A}(V)} \{\eta(R)\}$$

This paper will primarily focus on this skew-adjoint type of canonical algebraic curvature tensor, as the self-adjoint type is fairly well understood. This paper will also serve to highlight some of the similarities and differences between the two types. For instance, in [4] Gilkey shows that $\lfloor \frac{n}{2} \rfloor \leq \nu(n) \leq \frac{n(n+1)}{2}$, whereas in [5] Lopez shows that $\eta(n) \leq \frac{n^2(n^2-1)}{12} - \binom{n}{2}$. Clearly there is a big discrepancy between the two types here; $\frac{n(n+1)}{2}$ is much less than $\frac{n^2(n^2-1)}{12} - \binom{n}{2}$ for large n.

1.1 Notation

Throughout this paper R will always be an algebraic curvature tensor, R_A will always be a canonical algebraic curvature tensor of the skew-adjoint type, Definition 1.6 (so A is assumed to be skew-adjoint), and R_A^S will always be a canonical algebraic curvature tensor of the self-adjoint type, definition 1.5 (so A is assumed to be self-adjoint). Occasionally R_{ijkl} will be used to denote $R(e_i, e_j, e_k, e_l)$ where each e_s is a basis vector of V.

There is one more useful piece of notation to introduce which simplifies an important definition. We will sometimes replace individual entries of a matrix with 2×2 blocks when all of the other entries are 0. For instance:

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix} \tag{1}$$

where α_i is the 2×2 block: $\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$. In the event that $\dim(V)$ is odd there is an extra row and column of zeros, see Definition 1.10. When we write down such a matrix we really mean that $A = \alpha_1 \bigoplus \alpha_2 \bigoplus \dots \bigoplus \alpha_{\frac{n}{2}}$. This brings us to the definition.

Definition 1.10 *A square skew-adjoint matrix A is called* block diagonalizable *if there exists a basis in which the only non-zero entries of A are the $i, i+1$, and $i+1, i$ entries, where i must be odd.*

Note that if A is block diagonal, then

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix} \quad \text{or if } A \text{ has odd dimensions} \quad A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

for some α_is as described in Equation (1).

2 Signature Conjecture

In this section we will investigate the extent to which the expression of any algebraic curvature tensor R as a linear combination of canonical algebraic curvature tensors of the skew-adjoint type is unique. Our first remark has to be that any linear combination of skew-adjoint type curvature tensors is definitely not unique. The polarization identity [4]:

$$2R_A + 2R_B = R_{A-B} + R_{A+B} = R$$

provides an easy example of two linear combinations of different skew-adjoint type curvature tensors that both equal the same algebraic curvature tensor. One can come up with many other such examples.

However; there might still be something we can say on this subject. First note that any linear combination of skew-adjoint type curvature tensors: $\sum_{i=1}^{m} \alpha_i R_{A_i}$ can be written as the sum or difference of m skew-adjoint type curvature tensors: $\sum_{i=1}^{m} \pm R_{B_i}$ where $B_i = \sqrt{|\alpha_i|}A_i$, since $|\alpha_i| R_{A_i} = R_{\sqrt{|\alpha_i|}A_i}$. We can now more easily define the *signature* of a linear combination of skew-adjoint type curvature tensors:

Definition 2.1 *The* signature *of a linear combination of skew-adjoint type curvature tensors: $\sum_{i=1}^{m} \alpha_i R_{A_i} = \sum_{i=1}^{m} \pm R_{B_i}$ is the ordered pair (p, q) where p is the number of positive signs in the sum, q is the number of negative signs, and $B_i = \sqrt{|\alpha_i|}A_i$.*

In [7], Ragosta, proposes the following conjecture about the signature of linear combinations of self-adjoint type canonical algebraic curvature tensors:

Conjecture 2.2 If $R = \sum_{i=1}^{m} \pm R^S_{A_i}$, $\nu(R) = m$, and the rank of each $R^S_{A_i}$ is greater than 3, then any other linear combination of m canonical algebraic curvature tensors of the self-adjoint type that equals R must preserve the signature. Put more plainly, if $\sum_{i=1}^{m} \pm R^S_{A_i} = \sum_{i=1}^{m} \pm R^S_{B_i}$ is a minimal expression, and the rank of each A_i and B_i is greater than 3, then both sums have the same number of positive and negative terms.

We can adapt this to the skew-adjoint type by replacing each self-adjoint matrix a skew-adjoint one, and getting rid of the rank greater than 3 restriction:

Conjecture 2.3 If $R = \sum_{i=1}^{m} \pm R_{A_i}$ and $\eta(R) = m$, then any other linear combination of m skew-adjoint type curvature tensors that equals R must preserve the signature.

Note that minimality is a very important assumption; the signature of $\sum_{i=1}^{m} \pm R_{A_i}$ can only equal the signature of $\sum_{i=1}^{h} \pm R_{B_i}$ if $m = h$. The rest of the section is devoted to proving the signature conjecture for the skew-adjoint type when $\eta(R) = 2$. We start with a couple lemmas.

Lemma 2.4 *Let A and B be the following non zero block diagonalized skew-adjoint ma-*

trices:

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix} \quad B = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{\frac{n}{2}} \end{bmatrix}$$

where α_i is the 2×2 block: $\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$, and β_i is the 2×2 block: $\begin{bmatrix} 0 & b_i \\ -b_i & 0 \end{bmatrix}$. If $a_i b_j = b_i a_j$ for all i and j, and there exists at least one i such that both a_i and b_i are non zero, then A and B are multiples of each other. Additionally, $A = \frac{a_i}{b_i} B$ for any i where both a_i and b_i are non zero.

Proof. Pick any i such that a_i and b_i are non zero. Since $a_i b_j = b_i a_j$ for all j, we can divide by $a_i b_i$ so $\frac{b_j}{b_i} = \frac{a_j}{a_i}$. Now

$$\frac{a_i}{b_i}\beta_j = \begin{bmatrix} 0 & b_j \frac{a_i}{b_i} \\ -b_j \frac{a_i}{b_i} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_i \frac{b_j}{b_i} \\ -a_i \frac{b_j}{b_i} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_i \frac{a_j}{a_i} \\ -a_i \frac{a_j}{a_i} & 0 \end{bmatrix} = \alpha_j$$

Hence A and B are multiples, and $A = \frac{a_i}{b_i} B$. □

From [3] we have the following lemma:

Lemma 2.5 *if $\sum_{i=1}^{m} R_{A_i} = R_B$, where all A_i and B are skew adjoint, then all the A_is are simultaneously block diagonalizable.*

We now propose the following theorem which will be instrumental to proving the signature conjecture for $\eta(R) = 2$, and has even more far reaching implications.

Theorem 2.6 *If $\sum_{i=1}^{m} R_{A_i} = R_B$, where each A_i and B are skew adjoint matrices, then there exists real numbers c_i such that $A_i = c_i B$, or in other words all the matrices involved are multiples of one another.*

Proof. We know that all the A_is and B are simultaneously block diagonalizable, so let's start by picking a basis, $\{e_1, \dots, e_n\}$ such that all the A_is and B are block diagonalized. We can now write down each matrix:

$$A_i = \begin{bmatrix} \alpha_{i_1} & 0 & \dots & 0 \\ 0 & \alpha_{i_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{i_{\frac{n}{2}}} \end{bmatrix} \quad B = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{\frac{n}{2}} \end{bmatrix}$$

where α_{i_j} is the 2×2 block: $\begin{bmatrix} 0 & a_{i_j} \\ -a_{i_j} & 0 \end{bmatrix}$, and β_j is the 2×2 block: $\begin{bmatrix} 0 & b_j \\ -b_j & 0 \end{bmatrix}$. Note that if n is odd there are only $\frac{n-1}{2}$ blocks and an extra row and column of zeros in each matrix. Since all the matrices are block diagonalizable, we can re-order the basis so that the n^{th} row and column are identically zero in all the matrices.

Clearly any number of the A_is could be zero without changing the overall sum, but in that case $A_i = c_i B$ where $c_i = 0$, so from now on we will assume that each $A_i \neq 0$. The only issue with this assumption would be if every $A_i = 0$, but in that case $B = 0$, so $A_i = B$ for every i and we may proceed with our assumption. Now we will consider $\sum_{i=1}^{m} R_{A_i}$. We are assuming that the sum equals R_B for some skew-adjoint matrix B that we have block diagonalised with our choice of basis. Since algebraic curvature tensors can be uniquely determined by their output on a basis, we will look at $R_B(e_k, e_l, e_l, e_k)$. We may assume that $k < l$ since $R_B(e_k, e_l, e_l, e_k) = -R_B(e_k, e_l, e_l, e_k)$ and $R_B(e_k, e_k, e_k, e_k) = 0$. $R_B(e_k, e_l, e_l, e_k) = \sum_{i=1}^{m} R_{A_i(e_k, e_l, e_l, e_k)}$, and $A_i(e_k, e_l, e_l, e_k) = 0$ for all k and l except when l is even and $k = l-1$, in which case $A_i(e_k, e_l, e_l, e_k) = 3a_{i_{\frac{l}{2}}}^2$. In which case we conclude that $b_{\frac{l}{2}}^2 = \sum_{i=1}^{m} a_{i_{\frac{l}{2}}}^2$.

Going through the other cases, we find that the only other permutation of basis vectors e_k, e_l, e_r, e_s such that $R_B(e_k, e_l, e_r, e_s)$ is non zero is when l and s are distinct and even, $k = l-1$, and $r = s-1$. In this case $R_B(e_{l-1}, e_l, e_{s-1}, e_s) = -2\sum_{i=1}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}}$. But $R_B(e_{l-1}, e_l, e_{s-1}, e_s) = -2b_{\frac{l}{2}} b_{\frac{s}{2}}$, so $b_{\frac{l}{2}} b_{\frac{s}{2}} = \sum_{i=1}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}}$.

We have already shown that $b_{\frac{l}{2}}^2 = \sum_{i=1}^{m} a_{i_{\frac{l}{2}}}^2$, so we can now combine these two equations:

$$b_{\frac{l}{2}}^2 b_{\frac{s}{2}}^2 = \left(\sum_{i=1}^{m} a_{i_{\frac{l}{2}}}^2\right)\left(\sum_{i=1}^{m} a_{i_{\frac{s}{2}}}^2\right) = \left(\sum_{i=1}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}}\right)\left(\sum_{i=1}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}}\right) = (b_{\frac{l}{2}} b_{\frac{s}{2}})^2$$

If we expand, we can cancel all the terms of the form $a_{i_{\frac{l}{2}}}^2 a_{i_{\frac{s}{2}}}^2$, and move everything to the left side:

$$\sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} a_{i_{\frac{l}{2}}}^2 a_{j_{\frac{s}{2}}}^2 - \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}} a_{j_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} = 0$$

Note that $a_{i_{\frac{l}{2}}}^2 a_{j_{\frac{s}{2}}}^2 + a_{j_{\frac{l}{2}}}^2 a_{i_{\frac{s}{2}}}^2 - a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}} a_{j_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} - a_{j_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}} = (a_{i_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} - a_{j_{\frac{l}{2}}} a_{i_{\frac{s}{2}}})^2$. Hence:

$$\sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} a_{i_{\frac{l}{2}}}^2 a_{j_{\frac{s}{2}}}^2 - \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} a_{i_{\frac{l}{2}}} a_{i_{\frac{s}{2}}} a_{j_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} (a_{i_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} - a_{j_{\frac{l}{2}}} a_{i_{\frac{s}{2}}})^2 = 0$$

Note that the factor of $\frac{1}{2}$ after the first equality since we have counted each of the terms twice. This means that $a_{i_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} = a_{j_{\frac{l}{2}}} a_{i_{\frac{s}{2}}}$ for all i and $j \neq i$. Now let $i = m$, pick any $j < m$, and choose any l such that $a_{m_{\frac{l}{2}}} \neq 0$. Such an l must exist since we have assumed that each $A_i \neq 0$. Since $A_j \neq 0$, we can also find an s such that $a_{j_{\frac{s}{2}}} \neq 0$. We know that $a_{m_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} = a_{j_{\frac{l}{2}}} a_{m_{\frac{s}{2}}}$, and we now know that $a_{m_{\frac{l}{2}}} a_{j_{\frac{s}{2}}} \neq 0$. Therefore $a_{j_{\frac{l}{2}}} \neq 0$. We can now apply Lemma 2.1, which tells us that $A_j = \frac{a_{j_{\frac{l}{2}}}}{a_{m_{\frac{l}{2}}}} A_m$. So we can now simplify the sum:

$$\sum_{j=1}^{m} R_{A_i} = \sum_{j=1}^{m} \left(\frac{a_{j_{\frac{l}{2}}}}{a_{m_{\frac{l}{2}}}}\right)^2 R_{A_m} = R_{A_m} \sum_{j=1}^{m} \left(\frac{a_{j_{\frac{l}{2}}}}{a_{m_{\frac{l}{2}}}}\right)^2 = B$$

Therefore R_{A_m} is a multiple of B, and since each R_{A_i} is a multiple of R_{A_m}, each R_{A_i} must also be a multiple of B.

□

This theorem is extremely useful and lets us prove some interesting results.

Corollary 2.7 *If $\sum_{i=1}^{m} R_{A_i} = R$, and $\eta(R) = m$ and $m > 1$, then there do not exist $n \times n$ skew-adjoint matrices B and C_j for $1 \leq j < m$ such that $\sum_{i=1}^{m} R_{A_i} = R_B - \sum_{j=1}^{m-1} R_{C_j}$.*

Proof. We will consider the m skew-adjoint $n \times n$ matrices A_i, the algebraic curvature tensor $R = \sum_{i=1}^{m} R_{A_i}$ with $\eta(R) = m$, and any other m skew-adjoint $n \times n$ matrices B and C_j. We will assume, for a contradiction, that $\sum_{i=1}^{m} R_{A_i} = R_B - \sum_{j=1}^{m-1} R_{C_j}$. Therefore $\sum_{i=1}^{m} R_{A_i} + \sum_{j=1}^{m-1} R_{C_j} = R_B$, so we can apply Theorem 2.2 which tells us that all the matrices involved are multiples of B. Therefore $\sum_{i=1}^{m} R_{A_i}$ is also a multiple of R_B. Since $R = \sum_{i=1}^{m} R_{A_i}$ must be a multiple of R_B, $\eta(R) = 1$. This is a contradiction, as $\eta(R) = m > 1$. □

We can use Corollary to conclude that the Signature Conjecture 2.0.2 is true when $\eta(R) = 2$.

Corollary 2.8 *The Signature Conjecture* 2.0.2 *is true when $\eta(R) = 2$, i.e. if A, B, are skew-adjoint $n \times n$ matrices, and $R_A + R_B = R$ is a minimal expression of R, then there do not exist skew-adjoint $n \times n$ matrices C and D such that $R_A + R_B = R_C - R_D$.*

Proof. This is just a specific case of the previous corollary. Simply let $m = 2$, $A = A_1$, $B = A_2$, $C = B$, and $D = C_1$ and the result follows. □

3 The Kernel of $R_A \pm R_B$

We start this section by introducing some of the work that has been done on the kernel of canonical algebraic curvature tensors. In [4], Gilkey proves that $Ker(R_A) = Ker(A)$, and $Ker(R_A^S) = Ker(A)$ if the rank of $A \neq 1$. Furthermore, in [8], Williams proves that $\dim(Ker(R_A^S \pm R_B^S)) = 0, 1$, or n if the eigenvalues of A are positive.

In this section we hope to achieve a similar result for $Ker(R_A \pm R_B)$. Hence the following theorems.

Theorem 3.1 *If A_k, for $1 \leq k \leq m$ are skew-adjoint matrices and the algebraic curvature tensor R equals $\sum_{k=1}^{m} R_{A_k}$, then the kernel of R is $\bigcap_k Ker(A_k)$*

Proof. Let $a_{k_{ij}}$ represent the ijth entries of the matrix A_k. It is clear that $Ker(R) \supset \bigcap_k Ker(R_{A_k})$. Hence if $Ker(R) = \{0\}$, then $\bigcap_k Ker(R_{A_k}) = \{0\}$ as well.

The other case, where $Ker(R) \neq \{0\}$, is more interesting. We will start by picking a basis, $\{e_1, e_2, \ldots, e_l\}$ of $Ker(R)$, then extend this basis to a basis of the whole space $\{e_1, \ldots, e_l, e_{l+1}, \ldots, e_n\}$, and write down the matrices A and B in this new basis. We will now consider some basis vector $e_i \in Ker(R)$, and any other basis vector e_j.

$$0 = R(e_i, e_j, e_j, e_i) = \sum_{k=1}^{m} R_{A_k}(e_i, e_j, e_j, e_i) = 3 \sum_{k=1}^{m} a_{k_{ij}}^2 \tag{2}$$

So $a_{k_{ij}}^2 = 0$ for all k. Since e_j was an arbitrary basis vector this means that $a_{k_{ij}} = 0$ for every j whenever $e_i \in Ker(R)$. Hence $e_i \in Ker(R_{A_k})$ whenever $e_i \in Ker(R)$. Therefore $Ker(R) = \bigcap_k Ker(R_{A_k})$. □

Unfortunately this proof does not work in an expression of the form $R_A - R_B$, as the introduction of the minus sign means we would get $a_{ij} = \pm b_{ij}$ instead of $a_{ij} = b_{ij} = 0$ from Equation (2). Hence we will have to work a lot harder to prove the following theorem.

Theorem 3.2 *Suppose R is an algebraic curvature tensor, and A and B are skew-adjoint matrices such that $R_A - R_B = R$. Then either the kernel of R is $Ker(R_A) \cap Ker(R_B)$, or $B = \pm A$ so $R = 0$ and has kernel equal to V.*

Proof. As before, let a_{ij} and b_{ij} represent the ijth entries of the matrices A and B, and notice that if $Ker(R) = \{0\}$, then $Ker(R_A) \cap Ker(R_B) = \{0\}$ as well. In the case that $Ker(R) \neq \{0\}$, pick a basis $\{e_1, e_2, \ldots, e_k\}$ of $Ker(R)$, extend this basis to a basis of the whole space $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$, and write down the matrices A and B in this new basis as before. Now let e_i be a specific basis vector of $Ker(R)$, and let e_j, e_k, e_l be any other distinct basis vectors.

$$0 = R(e_i, e_j, e_j, e_i) = R_A(e_i, e_j, e_j, e_i) - R_B(e_i, e_j, e_j, e_i) = 3a_{ij}^2 - 3b_{ij}^2$$

Hence $a_{ij} = \pm b_{ij}$. Since j was arbitrary, we also know that $a_{ik} = \pm b_{ik}$ and $a_{il} = \pm b_{il}$

$$0 = R(e_i, e_j, e_k, e_i) = R_A(e_i, e_j, e_k, e_i) - R_B(e_i, e_j, e_k, e_i) = 3a_{ij}a_{ik} - 3b_{ij}b_{ik}$$

Since $a_{ij} = \pm b_{ij}$ and $a_{ik} = \pm b_{ik}$, this tells us that either $a_{ij} = b_{ij}$ and $a_{ik} = b_{ik}$ or $a_{ij} = -b_{ij}$ and $a_{ik} = -b_{ik}$. If it happens that $a_{ij} = -b_{ij}$, without loss of generality we can just replace the matrix B with $-B$, since $R_B = R_{-B}$. Hence we will assume that $a_{ij} = b_{ij}$, $a_{ik} = b_{ik}$, and $a_{il} = b_{il}$ from now on.

Now there are 2 cases to consider. Either $a_{ij} = b_{ij} = 0$ for all j, then we must have that $e_i \in Ker(R_A)$ and $e_i \in Ker(R_B)$, or there exists a j such that $a_{ij} \neq 0$, in which case we will let j be such that $a_{ij} \neq 0$ and continue the proof.

$$0 = R(e_j, e_i, e_k, e_j) = R_A(e_j, e_i, e_k, e_j) - R_B(e_j, e_i, e_k, e_j) = 3a_{ji}a_{jk} - 3b_{ji}b_{jk}$$

So $a_{ji}a_{jk} = b_{ji}b_{jk}$. We have chosen j such that $a_{ij} \neq 0$, and we know that $a_{ij} = b_{ij}$ so we can divide: $a_{jk} = b_{jk}$.

$$0 = R(e_i, e_j, e_k, e_l) = a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} - b_{il}b_{jk} + b_{ik}b_{jl} + 2b_{ij}b_{kl}$$

$$0 = R(e_i, e_l, e_j, e_k) = -a_{ik}a_{jl} + a_{ij}a_{kl} - 2a_{il}a_{jk} + b_{ik}b_{jl} - b_{ij}b_{kl} + 2b_{il}b_{jk}$$

Subtracting yields: $0 = 3a_{il}a_{jk} - 3a_{ij}a_{kl} - 3b_{il}b_{jk} + 3b_{ij}b_{kl}$

But we know that $a_{ij} = b_{ij}$, $a_{il} = b_{il}$, and $a_{jk} = b_{jk}$. Hence $0 = a_{ij}(a_{kl} - b_{kl})$, but $a_{ij} \neq 0$, so $a_{kl} = b_{kl}$. Since k and l were arbitrary, and each of $a_{ij} = b_{ij}$, $a_{ik} = b_{ik}$, and $a_{il} = b_{il}$, we can conclude that $A = B$. But we might have switched the sign of B, so $A = \pm B$.

Thus we have shown that either $e_i \in Ker(R_A) \cap Ker(R_B)$, or $A = \pm B$ so $R = R_A - R_B = 0$ □

When $m = 2$ in Theorem 3.1, $Ker(R_A + R_B) = Ker(A) \cap Ker(B)$. If we combine this result with Theorem 3.2, we see that if $Ker(A) \cap Ker(B) = \{0\}$, then $\dim(Ker(R_A \pm R_B)) = 0, n$. This contrasts with the self-adjoint case, where [8] proves that $\dim(Ker(R^S_A \pm R^S_B)) = 0, 1, n$ if the eigenvalues of A are positive.

4 Decomposability of $R_A \pm R_B$

Here we want to study the decomposability of model spaces of the form $(V, \phi, R_A \pm R_B)$; however, the inclusion of an inner product adds an extra layer of complexity which detracts somewhat from the focus on canonical algebraic curvature tensors. Hence we will study the decomposability of weak model spaces of the form $(V, R_A \pm R_B)$.

It is easy to show that if an algebraic curvature tensor R has a non-trivial kernel, then the weak model space (V, R) is decomposable. In fact, it can at least be decomposed into $Ker(R)$ and $V \cap Ker(R) \cup \{0\}$, as well as any subspace of $Ker(R)$, and depending on R it could be decomposed into other subspaces.

Moving away from general algebraic curvature tensors, it is known that (V, R_A) can only be decomposed if A has non-trivial kernel [4]. Clearly if R_A and R_B share a common kernel, then $R_A + R_B$ will have non-zero kernel by Theorem 3.1 and thus is decomposable. Thus we want to investigate the case where R_A and R_B do not necessarily share a common kernel.

Theorem 4.1 *If an algebraic curvature tensor R equals $\sum_{k=1}^{m} R_{A_k}$ for some skew-adjoint linear maps A_k, and the weak model space (V, R) can be decomposed into (V_1, R) and (V_2, R), then all the linear maps A_k preserve the vector spaces V_1 and V_2. i.e. $A_k(V_1) \subset V_1$ and $A_k(V_2) \subset V_2$.*

Proof. Let $\{e_1, e_2, \ldots, e_p\}$ be a basis of V_1 and $\{f_1, f_2, \ldots, f_q\}$ be a basis of V_2. $\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_q\}$ is a basis of V. We will express each A_k in this basis. Now let us consider $R(e_i, f_l, f_l, e_i)$. Since R is decomposable, $R(e_i, f_l, f_l, e_i) = 0$; however, it also equals $\sum_{k=1}^{m} A_k(e_i, f_l, f_l, e_i) = 3\sum_{k=1}^{m} a^2_{k_{il}}$. Hence we conclude that $a_{k_{il}} = 0$ whenever $i \leq p$ and $l > p$. This means that each A_k must preserve V_1 and V_2, since now $A_k e_i = \sum_{l=1}^{n} a_{k_{li}} e_l$ and $A_k f_i = \sum_{l=n}^{m} a_{k_{li}} f_l$. □

Theorem 4.2 *If an algebraic curvature tensor, R, equals $R_A - R_B$ for some skew-adjoint linear maps A and B, and the weak model space (V, R) can be decomposed into (V_1, R) and (V_2, R) where V_1 has dimension n and V_2 has dimension m, then there exists a basis of V such that the entries of A and B satisfy the following properties:*

1) $a_{ik} = \delta b_{ik}$ *for all* $i \leq n$ *and* $k > n$*, where* δ *is either* 1 *or* -1
2) given any $i \leq n$*, either* $a_{ik} = b_{ik} = 0$ *for all* $k > n$*, or* $a_{ij} = \delta b_{ij}$ *for all* $j \leq n$
3) $a_{ij}a_{kl} = b_{ij}b_{kl}$ *for all* $i, j \leq n$ *and* $k, l > n$

Note that it is still possible for A and B to preserve V_1 and V_2, but this is not necessarily the case as it was for $R = R_A + R_B$.

Proof. As before, let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V_1 and $\{f_1, f_2, \ldots, f_m\}$ be a basis of V_2. So $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_m\}$ is a basis of V. Express A and B as matrices in this basis, and consider $R(e_i, f_k, f_k, e_i)$:

$$0 = R(e_i, f_k, f_k, e_i) = R_A(e_i, f_k, f_k, e_i) - R_B(e_i, f_k, f_k, e_i) = 3a_{ik}^2 - 3b_{ik}^2$$

Hence $a_{ik} = \pm b_{ik}$ for all $i \leq n$ and $k > n$. Now let's consider $R(e_i, f_k, f_l, e_i)$:

$$R(e_i, f_k, f_l, e_i) = 3a_{il}a_{ik} - 3b_{il}b_{ik} = 0$$

Since $a_{il} = \pm b_{il}$ and $a_{ik} = \pm b_{ik}$, $a_{il}a_{ik} = b_{il}b_{ik}$ if and only if $a_{ih} = \delta b_{ih}$ for all $i \leq n$ and $h > n$ where δ is either 1 or -1. This proves 1). With this in mind we can now consider $R(e_i, e_j, f_k, e_i)$:

$$R(e_i, e_j, f_k, e_i) = 3a_{ik}a_{ij} - 3b_{ik}b_{ij} = 0$$

Since $a_{ik} = \delta b_{ik}$, either $a_{ik} = 0$ for all $k > n$, or there exists one k with $a_{ik} \neq 0$ so we can divide to get $a_{ij} = \delta b_{ij}$ for all $j \leq n$. This proves 2).

To prove 3) we need to consider $R(e_i, e_j, f_k, f_l)$:

$$R(e_i, e_j, f_k, f_l) = a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} + b_{il}b_{jk} - b_{ik}b_{jl} - 2b_{ij}b_{jl} = 0$$

We know that $a_{il}a_{jk} - b_{il}b_{jk} = 0$ and $-a_{ik}a_{jl} + b_{ik}b_{jl} = 0$, so $a_{ij}a_{kl} = b_{ij}b_{kl}$ for all $i, j \leq n$ and $k, l > n$. □

In the case that we are considering an expression of the form: $R_A \pm R_B$, theorem 4.2 gives us all the information about $R_A - R_B$, and theorem 4.1 tells us that in the $R_A + R_B$ case both A and B must preserve the decomposable subspaces. However, we can actually say a little more about A and B in this case by considering $R(e_i, e_j, f_k, f_l)$:

$$\begin{aligned} R(e_i, e_j, f_k, f_l) &= R_A(e_i, e_j, f_k, f_l) + R_B(e_i, e_j, f_k, f_l) \\ &= a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} + b_{il}b_{jk} - b_{ik}b_{jl} - 2b_{ij}b_{jl} \end{aligned}$$

But we just showed that $a_{il} = a_{ik} = a_{jk} = a_{jl} = b_{il} = b_{ik} = b_{jk} = b_{jl} = 0$. Therefore we have that $a_{ij}a_{kl} = -b_{ij}b_{kl}$ for all $i, j \leq p$ and $k, l > p$, where $p = dim(V_1)$. This gives us an easy way of constructing examples of two skew-adjoint matrices A and B such that neither R_A nor R_B are decomposable, but $R_A + R_B$ is decomposable.

Example 4.3 We will be considering the following matricies A and B, which are defined on $\mathbb{R}^4$ with the standard basis:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that $b_{ij} = -a_{ij}$ if $i, j > 2$ and $b_{ij} = a_{ij}$ otherwise. Since A and B have a trivial kernel, [4] tells us that R_A and R_B are indecomposable. However, $R_A + R_B$ decomposes into two dimensional subspaces; one spanned by e_1 and e_2 and the other by e_3 and e_4.

5 The Invariant $\eta_m(n)$

We have already introduced $\eta(n)$ in the introduction. In this section we will investigate a slightly different concept, $\eta_m(n)$.

Definition 5.1 *Let R be an algebraic curvature tensor. Define the rank m η-invariant to be: $\eta_m(R) := \min\{k | \sum_{i=1}^{k} \alpha_i R_{A_i} = R$, and $rank(A_i) \geq m\}$.*

It globalizes in a similar fashion to the normal invariant.

Definition 5.2 *Let R be an algebraic curvature tensor. Define the globalized rank m η-invariant to be: $\eta_m(n) := \max_{R \in \mathbb{A}} \{\eta_m(R)\}$.*

The following lemma is vital in establishing some interesting results about $\eta_m(n)$.

Lemma 5.3 *If D is a skew-adjoint matrix of rank $2m$, then there exist skew-adjoint matrices A, B, and C each of rank $2m + 2 \leq n$ such that $R_D = R_A + R_B - R_C$*

Proof. Let D be any skew-adjoint matrix of rank $2m$. Since D is skew-adjoint, there exists a basis that block diagonalizes D, so we can write D as:

$$\begin{bmatrix} \delta_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_m & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

Where δ_i is the 2×2 block: $\begin{bmatrix} 0 & d_i \\ -d_i & 0 \end{bmatrix}$. Now if we consider the following matrices, where i is the 2×2 block: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$:

$$A = \begin{bmatrix} \frac{5}{2}\delta_1 & 0 & \dots & 0 & \dots & 0 & \\ 0 & \frac{5}{2}\delta_2 & \dots & 0 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & \frac{5}{2}\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 3i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \end{bmatrix} \quad B = \begin{bmatrix} 5\delta_1 & 0 & \dots & 0 & \dots & 0 & \\ 0 & 5\delta_2 & \dots & 0 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 5\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 4i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{11}{2}\delta_1 & 0 & \dots & 0 & \dots & 0 & \\ 0 & \frac{11}{2}\delta_2 & \dots & 0 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & \frac{11}{2}\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 5i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \end{bmatrix}$$

Clearly, A, B, and C are all of rank $2m+2$. We now claim that $R_A + R_B - R_C = R_D$. To prove this claim we just need to check that $R_{Aijkl} + R_{Bijkl} - R_{Cijkl} = R_{Dijkl}$ for all i, j, k, l. Since all the matrices involved are block diagonal, there are not that many possible permutations of i, j, k, l which yield non zero entries, and one can check them all. We only include three sample calculations, one for each of the cases:

$$R_{A1221} + R_{B1221} - R_{C1221} = \frac{5}{2}^2 d_1^2 + 25d_1^2 - \frac{11}{2}^2 d_1^2 = d_1^2[\frac{25}{4} + \frac{100}{4} - \frac{121}{4}] = d_1^2 = R_{D1221}$$

$$R_{A1234} + R_{B1234} - R_{C1234} = \frac{5}{2}^2 d_1 d_2 + 25d_1 d_2 - \frac{11}{2}^2 d_1 d_2 = d_1 d_2 = R_{D1234}$$

$$R_{A12(2m)(2m+1)} + R_{B12(2m)(2m+1)} - R_{C12(2m)(2m+1)} = d_1[\frac{15}{2} + \frac{40}{2} - \frac{55}{2}] = 0 = R_{D12(2m)(2m+1)}$$

□

Note that this is just one example from an infinite number of possible A, B, and C.

Theorem 5.4 *For each n we have: $\eta_{2m+2}(n) \leq 3^m \eta(n)$.*

Proof. The result follows from the above result: any canonical algebraic curvature tensor of the skew-adjoint type can be written in 3^m rank $2m+2$ skew-adjoint type tensors, so $\eta_{2m+2}(n) \leq 3^m \eta(n)$. □

This is interesting, because in [7], Ragosta proves that, for the self-adjoint type: $\mu_m(n) \leq 2^m \mu(n)$. This means that one can move up single dimensions in the self-adjoint case, something that is impossible in the skew-adjoint case because ever skew-adjoint matrix has an even rank, but using these estimates the skew-adjoint type seems to be more efficient, needing only $3^m \eta(n)$ rank $2m+2$ matrices, whereas in the self-adjoint case one might need up to $2^{2m}\mu(n) = 4^m \mu(n)$. The word only is a bit gratuitous here; 3^m and 4^m are much larger than $\eta(n)$ and $\mu(n)$ for large m, so it is highly unlikely that $\eta_{2m+2}(n) = 3^m \eta(n)$ for large n.

Corollary 5.5 *The canonical algebraic curvature tensors with the skew-adjoint type in which the defining matrices all have rank greater that or equal to $2m$ form a spanning set of all algebraic curvature tensors.*

We can use this fact to improve upon the previous estimate for large n. Since every spanning set of a vector space must contain a basis, and we know that the skew-adjoint type curvature tensors of rank $\geq 2m$ form a spanning set of $\mathbb{A}(n)$, we conclude that $\eta_{2m}(n) \leq dim(\mathbb{A}(n)) = \frac{n^2(n^2-2)}{12}$.

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