Using the Particle Model to Find Structure in Eilenberg-MacLane Spaces

R. Chaiser, K. Pelatt, and C. Thatcher

Abstract - We introduce two models, the bar construction and the particle model, for a collection of spaces known as Eilenberg-MacLane spaces. These spaces can be used as building blocks for other spaces and are of great interest throughout topology. We use these models to compute the homology groups for one Eilenberg-MacLane space in an intuitive and accessible way, and then show how the models can be used to determine the cell structure for additional cases using combinatorial arguments. An introduction to cellular homology and examples are included along the way.

Keywords : Eilenberg-MacLane spaces; cellular homology; combinatorial cell structure; particle model; bar construction

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1 Introduction

A common question in topology is "what is the shape" of a particular topological space? Further one could ask, how do we tell if two topological spaces have the same shape? What do we even mean by "same" when describing shapes? One way to start answering these questions is to say that two spaces have the same shape if one can be continuously deformed into the other. This is one example of how two spaces can be homeomorphic. In general, two topological spaces are homeomorphic if there is a continuous map from one to the other that has a continuous inverse. The map itself is called a homeomorphism. Many of the continuous maps we know from calculus and other classes are in fact homeomorphisms. For example, the map $f(x) = e^x$ is a homeomorphism between $\mathbb{R}$ and $(0, \infty)$ and stereographic projection provides a homeomorphism between S^2 without the north pole and $\mathbb{R}^2$.

While we can determine that two spaces are homeomorphic by finding a specific homeomorphism, the map is dependent on how we define the spaces themselves. Small changes to our spaces that do not change the underlying shape (such as increasing the amplitude of a sine wave or decreasing the radius of a sphere slightly) would require us to write down a completely different map. Given this we might ask if there are other ways to determine if two spaces are homeomorphic or, alternatively, is there a way to determine if two spaces are *not* homeomorphic?

To answer this, we need to think more about what it means for two spaces to have the same shape: What characteristics are inherent? A characteristic that does not change under homeomorphism is called a topological invariant. Some examples include number of components, dimension, Euler characteristic, and holes. In theory, topological invariants should be more computable than individual maps, and collections of them should be sufficient to distinguish certain types of spaces. For example, surfaces can be classified by two invariants: Euler characteristic and orientation. However, computing them can be difficult and finding a complete collection only happens in special cases. Regardless, many are useful for distinguishing spaces, and we will introduce some particularly useful invariants called homology groups. First we start with the idea of holes and the fundamental group.

One characteristic of the shape of a space is the presence of holes. For example, a torus has a hole through the middle of it, whereas a sphere does not seem to have any holes at all. One would expect that holes could be used to distinguish these two spaces, but how do we define "hole" precisely? To start, we could consider loops in the space, or more specifically, maps of S^1 into the space. A hole can be identified by a loop around it that is not collapsible (able to be continuously shrunk to a point). For example, in Figure 1, loop a on the torus is not collapsible and identifies the hole through the middle. On the other hand, all of the loops on the sphere are collapsible and seem to show a lack of holes. While loops can be used to identify holes, they also identify other structure in the space. For example, loop b in the torus identifies some empty space inside. The fundamental group (also known as the first homotopy group) looks at all maps from S^1 into the space and encodes these inherent structures.

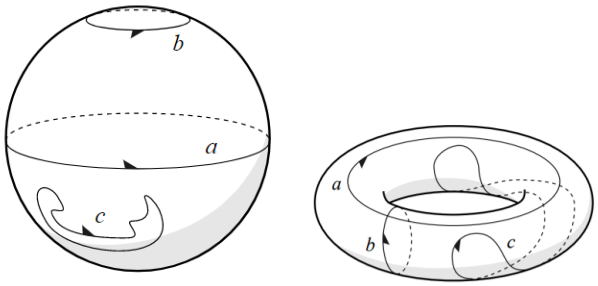


Figure 1: Representative Loops on a Sphere and a Torus [8, 9]

The idea of non-collapsible loops can be generalized to higher dimensions by considering maps of S^n into the space. Specifically, the nth homotopy group of a space is nontrivial if there is some map of S^n into the space that cannot be collapsed. For example, the identity map $S^2 \to S^2$ cannot be collapsed and thus S^2 has nontrivial 2nd homotopy group. While the precise definitions of the homotopy groups are beyond the scope of this paper, the reader can find them in many algebraic topology books such as [4]. These groups will be particularly important for defining the spaces we are considering in this paper, Elienberg-MacLane spaces. However, homotopy groups are hard to compute in general,

so after we define our spaces in Section 2, we will define a related but more computable collection of invariants called homology groups. These groups measure the presence of voids, and it is these groups that we determine for certain Eilenberg-MacLane spaces.

This paper is structured as follows. We first introduce two models, the bar construction and the particle model, for a collection of spaces called Eilenberg-MacLane spaces. These spaces are of interest for many reasons including being used as building blocks for other spaces and in various computations in topology. We then provide an introduction to a collection of topological invariants called cellular homology groups and provide some common but related examples. We then move to computing cellular homology groups with arbitrary coefficients. Finally, we compute the homology groups for one Eilenberg-MacLane space in an intuitive and accessible way, and then use the particle model to assist in determining the cell structure of additional cases of Eilenberg-MacLane spaces using combinatorial arguments.

2 Eilenberg-MacLane Spaces and the Particle Model

The homotopy groups of a space measure the presence of non-collapsible spheres within it. Since higher dimensional spheres can often be mapped into a space creating higher homotopy groups, the homotopy groups of a space are generally quite difficult to compute, even for spheres, so spaces with relatively few homotopy groups are of interest. One particular collection of spaces of interest are spaces with a single nontrivial homotopy group. These spaces are known as Eilenberg-MacLane spaces, and some of them are well known, for example, a circle, a wedge of spheres, a torus, and $\mathbb{R}P^\infty$. We note that while they have simple homotopy groups, they are often infinite dimensional. In this section we describe two related models for them - the bar construction and the particle model - that will be useful for visualization and computing the homology groups.

2.1 Modeling Eilenberg-MacLane Spaces With the Bar Construction, BG

An Eilenberg-MacLane space with its nontrivial homotopy group in degree n and isomorphic to G is written $K(G, n)$. As an example, the circle S^1 is an Eilenberg-MacLane space: $K(\mathbb{Z}, 1)$. Though the following definition allows for an arbitrary abelian group G, we will focus our attention only on the groups $G = \mathbb{Z}/p\mathbb{Z}$, where p is prime. We will denote $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ going forward.

Definition 2.1 *[1] Let $(G, \circ)$ be an abelian group with group operation $\circ$. We define*

$$BG = \bigcup_{n \in \mathbb{Z}_{\geq 0}} (\Delta^n \times G^n)/\sim,$$

where $\Delta^n = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \,|\, 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1\}$ is the standard simplex, and the equivalence relation $\sim$ is such that

- *If $g_i = e$, the identity in G, or if $t_i = t_{i+1}$, then delete t_i and combine g_i and g_{i+1} via G's operation: $g_i \circ g_{i+1}$.*
- *If $t_1 = 0$ then t_1 and g_1 are deleted.*
- *If $t_n = 1$ then t_n and g_n are deleted.*

We write a point in BG as $(t_1, \ldots, t_n) \times (g_1, \ldots, g_n)$ and refer to the t_i corresponding to a group element g_i as the group element's "time coordinate." If G is discrete, the space BG, known as the classifying space for G, is a model for $K(G, 1)$. Additionally, BG provides models for $K(G, n)$, for $n > 1$, through iteration, as described in Section 2.4.

Adem and Milgram [1] define the following natural addition on BG, which makes BG an abelian group.

Definition 2.2 *We define an addition $* : BG \times BG \to BG$ such that for two points $(t_1, \ldots, t_p) \times (g_{t_1}, \ldots, g_{t_p})$ and $(s_1, \ldots, s_q) \times (g_{s_1}, \ldots, g_{s_q})$ in BG,*

$$\big((t_1, \ldots, t_p) \times (g_{t_1}, \ldots, g_{t_p})\big) * \big((s_1, \ldots, s_q) \times (g_{s_1}, \ldots, g_{s_q})\big) \\ = (\lambda_1, \ldots, \lambda_{p+q}) \times (g_{\lambda_1}, \ldots, g_{\lambda_{p+q}}),$$

where the λ_i are the t_i and s_i arranged in increasing order.

Example 2.3 Let $G = \mathbb{Z}_3$. We have

$$\left(\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{3}\right) \times (1, 1, 1)\right) * \left(\left(\frac{1}{7}, \frac{1}{2}\right) \times (2, 2)\right) = \left(\frac{1}{10}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}\right) \times (1, 2, 1, 1, 2)\,.$$

If we add two points with a matching time component, we need to be a bit more careful. Using Definition 2.2 and the equivalences in Definition 2.1, we have

$$\left(\left(\frac{1}{2}, \frac{2}{3}\right) \times (1, 2)\right) * \left(\left(\frac{1}{3}, \frac{1}{2}\right) \times (2, 1)\right) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right) \times (2, 1, 1, 2) \\ \sim \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right) \times (2, 2, 2)\,.$$

Proposition 2.4 *The set $(BG, *)$ is an abelian group.*

Proof. The additive identity in BG is the unique point $() \times ()$. Since $() \times ()$ has no time elements, we have

$$((t_1, \ldots, t_n) \times (g_1, \ldots, g_n)) * (() \times ()) = (() \times ()) * ((t_1, \ldots, t_n) \times (g_1, \ldots, g_n)) \\ = (t_1, \ldots, t_n) \times (g_1, \ldots, g_n).$$

Additive inverses in BG are obtained from the additive inverses of elements in G. From Definition 2.1, we have

$$\begin{aligned}((t_1,\dots,t_n)\times(g_1,\dots,g_n))*((t_1,\dots,t_n)\times(g_1^{-1},\dots,g_n^{-1}))&\\ &=(t_1,t_1,\dots,t_n,t_n)\times(g_1,g_1^{-1},\dots,g_n,g_n^{-1})\\ &=(t_1,\dots,t_n)\times(g_1\circ g_1^{-1},\dots,g_n\circ g_n^{-1})\\ &=(t_1,\dots,t_n)\times(e,\dots,e)\\ &=()\times().\end{aligned}$$

Thus, $((t_1,\dots,t_n)\times(g_1,\dots,g_n))^{-1}=(t_1,\dots,t_n)\times(g_1^{-1},\dots,g_n^{-1})$.

Since the sum of points in BG is completely determined by arranging the time elements in increasing order, we immediately get both associativity and commutativity. This is clear if all of the time elements are distinct, but perhaps less clear if there are matching time elements. If ever there are matching time elements, associativity and commutativity are both inherited from G under the equivalence relation in Definition 2.1. □

2.2 Cells

We now define collections of points called cells which will be needed when we calculate the homology groups below. A *cell* in BG is an element from $(G\setminus\{e\})^n$ associated to a copy of the open standard simplex Δ^n. We can denote a cell by its group elements using what we call *bar notation*: $[g_1|g_2|\cdots|g_n]$ (see [3]). Categorizing cells by dimension will be necessary when computing the homology of a CW complex so we now provide definitions for dimension.

Definition 2.5 *[3] If G is a graded group, let $d(g_i)$ be the dimension of g_i in G. If G is not a graded group, such as $G=\mathbb{Z}_p$, the dimension of every element in G is 0. For a cell $[g_1|\cdots|g_m]\in BG$, we define the* **tensor dimension** *of $[g_1|\cdots|g_m]$ by*

$$d_t[g_1|\cdots|g_m]=d(g_1)+\cdots+d(g_m).$$

We define the **simplicial dimension** *of $[g_1|\cdots|g_m]$ by*

$$d_s[g_1|\cdots|g_m]=m.$$

Finally, we define the **total dimension** *of $[g_1|\cdots|g_m]$ to be*

$$d_B[g_1|\cdots|g_m]=d_t[g_1|\cdots|g_m]+d_s[g_1|\cdots|g_m].$$

*An n-**cell** in BG is a cell with total dimension n.*

For G not a graded group, an n-cell in BG consists of all points with a particular ordering of n group elements and distinct time coordinates. Since each cell is determined by the ordered list of group elements, we will refer to the cell by just the the collection of n ordered group elements with bars between them. Note, the tensor dimension will not be needed until we iterate the construction. We now provide some examples of cells in BG for $G=\mathbb{Z}_2$ and $G=\mathbb{Z}_3$.

Example 2.6 Let $G = \mathbb{Z}_2 = \langle 1 \rangle$. Since there is only one nonzero element in $\mathbb{Z}_2$, each n-cell in $B\mathbb{Z}_2$ is the collection of points of the form $(t_1, \ldots, t_n) \times (1, \ldots, 1)$, where $0 < t_1 < \cdots < t_n < 1$. The following table illustrates how we use bar notation to represent cells.

n	n-cells
0	[]
1	[1]
2	[1|1]
3	[1|1|1]
4	[1|1|1|1]

Example 2.7 Let $G = \mathbb{Z}_3$. Since there are two nonzero elements in $\mathbb{Z}_3$, we have more variety in the form of our n-cells, which we see in the following table.

n	n-cells
0	[]
1	[1], [2]
2	[1|1], [1|2], [2|1], [2|2]
3	[1|1|1], [1|1|2], [1|2|1], [2|1|1], [1|2|2], [2|1|2], [2|2|1], [2|2|2]

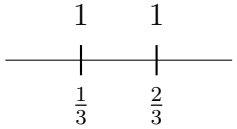
These examples suggest a way to count the number of n-cells in $B\mathbb{Z}_p$. Since each n-cell in $B\mathbb{Z}_p$ is defined by a unique element in $(\mathbb{Z}_p \setminus \{0\})^n$, the cardinality of $(\mathbb{Z}_p \setminus \{0\})^n$ is also the number of n-cells in $B\mathbb{Z}_p$. Since the cardinality of $(\mathbb{Z}_p \setminus \{0\})^n$ is $(p-1)^n$, we have the following observation.

Remark 2.8 The number of n-cells in $B\mathbb{Z}_p$ is $(p-1)^n$.

2.3 The Particle Model

We now introduce the *particle model* for $B\mathbb{Z}_p$, which will aid in visualizing the bar notation, determining the number of cells in the iterated bar construction in Section 2.4, and computing the homology of $B\mathbb{Z}_p$ in Section 5.1. In this section we focus on the particle model for $B\mathbb{Z}_2$. In the next section we generalize it further to $B\mathbb{Z}_p$.

The particle model for points in $B\mathbb{Z}_p$ represents $(t_1, \ldots, t_n) \times (g_1, \ldots, g_n) \in B\mathbb{Z}_p$ by n marks on the unit interval at each time coordinate. The ith mark is labeled above by the corresponding group element g_i and below by the corresponding time coordinate $t_i \in [0, 1]$. For example, in the particle model, the point $((\frac{1}{3}, \frac{2}{3}) \times (1, 1))$ looks like



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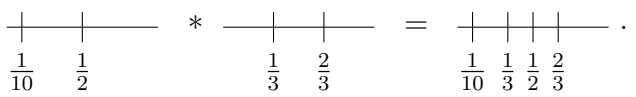
For points in $B\mathbb{Z}_2$, we leave off the group elements since there is only one nontrivial group element in $\mathbb{Z}_2$. The addition defined on $B\mathbb{Z}_2$ can now be visualized: the addition on $B\mathbb{Z}_2$ consists of placing all of the points on a single unit interval in their correct locations. For example,

$$\left(\frac{1}{10}, \frac{1}{2}\right) \times (1,1) * \left(\frac{1}{3}, \frac{2}{3}\right) \times (1,1) = \left(\frac{1}{10}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right) \times (1,1,1,1)\,,$$

would be represented as



An n-cell in $B\mathbb{Z}_p$, represented by $[g_1|\cdots|g_n]$ in the bar notation, can by represented in the particle model by placing the n ordered group elements on the unit interval. Notice that, just as with the bar construction, the particle model for cells disregards the time coordinates.

The table below shows the n-cells for $B\mathbb{Z}_2$ in both the bar notation and the particle model.

n	Bar notation	Particle model
0	[]	
1	[1]	
2	[1|1]	
3	[1|1|1]	
4	[1|1|1|1]	

2.4 Iterating the Bar Construction

Recall that in Definition 2.1, the only restriction on G was that it be abelian. By Theorem 2.4, BG is an additive abelian group, and thus we may iterate this construction so that

$$B(BG) = B^2G = \bigcup_{n \in \mathbb{Z}_{\geq 0}} (\Delta^n \times (BG)^n)/\sim,$$

where $\sim$ is the same equivalence relation from Definition 2.1. In general, an element in B^2G is of the form $(s_1, \ldots, s_m) \times (G_1, \ldots, G_m)$, where each G_m is of the form $(t_1, \ldots, t_{m_k}) \times (g_1, \ldots, g_{m_k})$. The bar notation and particle model extend naturally to B^2G. A cell in B^2G is an element from $(BG \setminus \{e\})^n$ associated to a copy of the open standard simplex Δ^n. The dimension of a cell in B^2G is then its simplicial dimension plus its tensor dimension, but we note that the tensor dimension is the sum of the simplicial dimensions of each G_i, as specified in Definition 2.5.

Example 2.9 The cell $[G_1|G_2] = [[1|1]|[1|1|1]]$ in $B^2\mathbb{Z}_2$ has tensor dimension $d_t[G_1|G_2] = d(G_1) + d(G_m) = 2 + 3 = 5$ and simplicial dimension $d_s[G_1|G_2] = 2$. Its total dimension is then 7 and the cell is a 7-cell.

The particle model for B^2G then depicts a point as a collection of points from BG vertically within the unit square. The time coordinates associated to each G_i are on the horizontal axis and the time coordinates associated to the g_j within a specific G_i are on the vertical axis. When considering the n-cells, we drop the specific time coordinates, and depict the cells from the particle model for BG vertically in the unit square. Since the particle model for points requires a lot of labeling, we predominantly use the model for cells.

Example 2.10 Here we see the particle model for the 7-cell $[[1|1]|[1|1|1]]$ in $B\mathbb{Z}_2$.

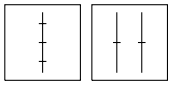
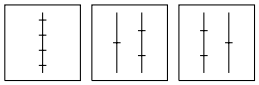
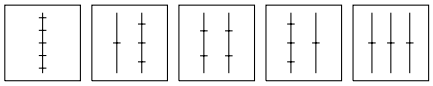
We also note that you can continue to iterate this construction so that $B(B^{n-1}G) = B^nG$ for any integer $n > 1$. The bar notation and particle model extend as expected. For example, the particle model for cells of B^3G consists of cells from the particle model of B^2G arranged in the unit cube.

Remark 2.11 There are no 1-cells in B^2G.

In [3], Eilenberg and MacLane show that B^nG has no cells of dimension d for $0 < d < n$. We provide a short proof by contradiction for the case B^2G. Suppose that we have a 1-cell. Such a cell would necessarily be of the form $[g]$, where the total dimension of g in BG is 0, as this would give us simplicial dimension 1 and tensor dimension 0, and so our cell would have total dimension 1. However, the only element in BG that has total dimension 0 is $[]$, which is the identity in BG. By Definition 2.1, our cell is equivalent to a 0-cell in B^2G. Thus we have our contradiction.

For $B\mathbb{Z}_2$ we can determine the number of cells in each dimension using the particle model, since the dimension of a cell can be found by adding the number of vertical lines to the total number of points on the vertical lines. We can also use the particle model to determine the number of cells in each dimension for $B\mathbb{Z}_p$.

Example 2.12 In the following table, we give the particle model representation for all n-cells in $B^2\mathbb{Z}_2$, for $0 \leq n \leq 6$.

n	n-cells	Number of n-cells
0		1
1	$\emptyset$	0
2		1
3		1
4		2
5		3
6		5

The number of n-cells in $B^2\mathbb{Z}_2$ appears to be a Fibonacci number. This is in fact the case.

Theorem 2.13 *The number of n-cells in $B^2\mathbb{Z}_2$ is F_{n-1}, where F_n is the nth Fibonacci number, with $F_0 = 0$ and $F_1 = 1$.*

Proof. Let f_n be the number of n-cells in $B^2\mathbb{Z}_2$. Given a particular n-cell, look at the rightmost element in the particle model representation of the cell. If the rightmost element is a line with two or more points, removing a point produces an $(n-1)$-cell in $B^2\mathbb{Z}_2$, of which there are f_{n-1}. If the rightmost element is a line with only one point, then removing that point also requires us to remove the line since an empty line represents the identity in $B\mathbb{Z}_2$. Thus, if the rightmost element is a line with a single point, removing that point produces an $(n-2)$-cell in $B^2\mathbb{Z}_2$, of which there are f_{n-2}. Summing over all possibilities, we find that $f_n = f_{n-1} + f_{n-2}$. From Example 2.12, we see that our initial terms are $f_1 = 0$ and $f_2 = 1$. With an index shift, we have our desired result: the number of n-cells in $B^2\mathbb{Z}_2$ is $f_n = F_{n-1}$. □

A common combinatorial interpretation of the Fibonacci numbers is that F_{n+1} is the number of tilings of a $1 \times n$ board with squares and dominoes. In our proof of the previous theorem, we may think of a line with a single dash as a "domino" and any additional dashes after the first as "squares". There is a natural extension of our proof for the previous theorem to counting the number of n-cells in $B^2\mathbb{Z}_p$ for any prime p.

Theorem 2.14 *Let p be prime. Let S_n be the number of n-cells in $B^2\mathbb{Z}_p$. Then S_n is given by the recursion $(p-1)S_{n-1} + (p-1)S_{n-2}$ with initial conditions $S_1 = 0$ and $S_2 = p - 1$.*

Proof. Let S_n denote the number of n-cells in $B^2\mathbb{Z}_p$. Given a particular n-cell, look at the rightmost element in the particle model representation of that cell. If the rightmost

element in the cell is a line with 2 or more symbols, removing one produces an $(n-1)$-cell in $B^2\mathbb{Z}_p$, of which there are S_{n-1}. However, there are $p-1$ different symbols that we could have removed, one for each of the $p-1$ nonzero elements in $\mathbb{Z}_p$. Thus, there are $(p-1)S_{n-1}$ n-cells in $B^2\mathbb{Z}_p$ whose rightmost element is a line with two or more symbols. If instead the rightmost element in the n-cell is a line with one symbol, then there are $p-1$ possibilities for the one symbol and removing it produces an $(n-2)$-cell in $B^2\mathbb{Z}_p$, of which there are S_{n-2}. Thus, there are $(p-1)S_{n-2}$ n-cells in $B^2\mathbb{Z}_p$ whose rightmost element is a line with a single symbol. Summing over all possibilities, we find that the number of n-cells in $B^2\mathbb{Z}_p$ is $S_n = (p-1)S_{n-1} + (p-1)S_{n-2}$. By Theorem 2.11, $S_1 = 0$. By Definitions 2.1 and 2.5, a 2-cell in $B^2\mathbb{Z}_2$ is of the form $[g]$ where g is a 1-cell in $B\mathbb{Z}_p$. By Theorem 2.8, there are $p-1$ 1-cells in $B\mathbb{Z}_p$. Thus, $S_2 = p-1$. □

3 Cellular Homology

Now that we have seen some of the basic structure of $B\mathbb{Z}_p$, our goal is to compute a its homology groups. To do this, we provide some intuition for homology followed by a brief introduction to cellular homology and how it may be computed for simple, but related, examples.

Homotopy groups measure the presence of holes, or non-contractible spheres within a space, and while useful, they can often be hard to compute. We turn now to a related but different topological invariant that measures the presence of voids, or emptiness, in a space. To build a bit of intuition for how we think of detecting n-dimensional voids in a space, let us consider how to detect 2-dimensional voids. To do this we can think of placing all possible loops in our space and checking whether or not each loop bounds a 2-dimensional disk in the space. If a loop bounds a 2-dimensional disk, then we have not detected a void, but if the loop does not bound a 2-dimensional disk, then we have detected a void. The process for detecting $(n+1)$-dimensional voids is similar, except now we think of embedding copies of S^n in our space and checking if they bound $(n+1)$-dimensional disks.

In Figure 1 in Section 1, we have the sphere and the torus with representative loops placed on each. On the sphere, we can see that each of the three loops is the boundary of some 2-dimensional disk embedded in the sphere and so those loops do not detect any voids. On the torus, loops a and b do not bound 2-disks and thus loops a and b bound 2-dimensional voids in the torus. We can see that they bound two distinct voids because neither a nor b can be continuously deformed into the other. This suggests that the torus has at least two 2-dimensional voids, and we hypothesize that the sphere has none.

In general, we cannot analyze all possible loops embedded in a given space. Instead, we add structure to our space so that systematically detecting voids becomes, in theory, computationally feasible. One way to do this is to build the space by gluing together n-balls via "attaching maps". A space constructed in this way is called a *CW complex* (defined in Appendix A). This additional structure allows us to systematically analyze how S^n can be embedded into the space and define an algebraic topological invariant called *homology* that measures the number of $(n+1)$-dimensional voids in the space. The

homology we calculate using the CW complex structure is called *cellular homology*. The definitions given in Sections A and 3.1 are standard and the structure given here most closely follows that given by Massey in [6]. In Section 3.2, we use cellular homology to show that S^2, T, and $\mathbb{R}P^2$ are not homeomorphic by showing that their homology groups are not the same.

The n-cells in BG defined in the previous section provide a CW structure for Eilenberg-MacLane spaces, and this is the structure we will use to compute their cellular homology in Section 5.2. We first go through the definitions needed for cellular homology and provide examples of computing cellular homology groups. While the examples are standard, the process used to compute homology will provide a good foundation for when we shift to Eilenberg-MacLane spaces.

3.1 The Homology of CW Complexes

Definition 3.1 *Let X be a CW complex. The n***th chain group** *of X, denoted $C_n(X)$, is the free abelian group generated by the n-cells of X. Recall that the free abelian group generated by the n-cells of X is the group consisting of all linear combinations of the n-cells of X with coefficients in $\mathbb{Z}$. An element of $C_n(X)$ is called an n***-chain**.

The characteristic map of a cell determines the orientation and boundary of that cell. Throughout, we will not be concerned with defining the characteristic maps for our spaces and we will work with CW complexes where the characteristic maps, or the necessary information they give, is clear.

For an n-cell σ, its *boundary* $\partial_n\sigma$ is the sum of its $(n-1)$-dimensional faces (where the sign indicates the orientation of the face). Given the boundaries of the n-cells of a CW complex X, we may extend the boundary map linearly to all of $C_n(X)$ to form a group homomorphism.

Definition 3.2 *For a CW complex X with $C_n(X) = \langle \sigma_0, \sigma_1, \ldots, \sigma_k \rangle$, we define the n***th boundary homomorphism** *$\partial_n : C_n(X) \to C_{n-1}(X)$ as the linear extension*

$$\partial_n \sigma = \partial_n \left(\sum_{i=0}^{k} \alpha_i \sigma_i \right) = \sum_{i=0}^{k} \alpha_i \partial_n(\sigma_i),$$

where $\sigma \in C_n(X)$.

We note that the boundary of a vertex is taken to be 0. The following lemma is well known and a proof is given in Munkres [7].

Lemma 3.3 *For a CW complex X, $\partial_{n-1} \circ \partial_n$ is the zero homomorphism.*

Lemma 3.3 implies that $\operatorname{im} \partial_{n+1}$ is a normal subgroup of $\ker \partial_n$, and thus we can make the following definition.

Definition 3.4 *The n***th homology group** *of a CW complex X is*

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

We note that $\ker \partial_n$ detects n-cycles, or embedded n-spheres, in our space. Homology then identifies the cycles that do not bound an $(n+1)$-disk, and hence detects the voids we were looking for. The rank of $H_0(X)$ is the number of path connected components of X and the rank of $H_n(X)$ is the number of unique $(n+1)$-dimensional voids in our space. Notice that if X is a CW complex with no n-cells, then $H_n(X) = \{0\}$.

Theorem 3.5 *[7] If two CW complexes X and Y are homeomorphic, then they have the same sequence of homology groups.*

The above theorem shows that the sequence of homology groups is a topological invariant. Furthermore, the homology of a topological space X is independent of the CW structure used to calculate the groups. This follows from an isomorphism between the cellular homology groups and the homology groups arising from a different construction, known as singular homology, that does not rely on the CW structure chosen (see [4]).

3.2 Examples of Computing Cellular Homology

In this section we compute the homology of the torus and the projective plane as examples of cellular homology. In both cases we use a CW structure that arises from thinking about each space as a quotient of the unit square. We additionally determine the homology of the 2-sphere in the appendix using a CW structure that arises in a different way. We note that these examples are directly related to Eilenberg-MacLane spaces - the torus is $K(\mathbb{Z} \times \mathbb{Z}, 1)$ and an infinite version of the projective plane is $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$.

Example 3.6 (Torus) We will use T to denote the torus constructed as a quotient space of $I \times I$ and with the CW complex structure shown in Figure 2.

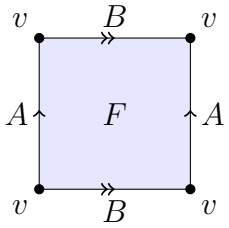


Figure 2: The torus with a CW complex structure.

Our chain groups for T are $C_0(T) = \langle v \rangle$, $C_1(T) = \langle A, B \rangle$, $C_2(T) = \langle F \rangle$, and $C_n(T) = \{0\}$ for all $n > 2$. Computing the boundary maps, we have $\partial_0 v = 0$, $\partial_1 A = \partial_1 B = v - v = 0$, and $\partial_2 F = B - A - B + A = B - B + A - A = 0$. Thus, the homology groups for the

torus are

$$\begin{aligned}
H_0(T) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v \rangle}{\{0\}} = \langle v \rangle \cong \mathbb{Z}, \\
H_1(T) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle A, B \rangle}{\{0\}} = \langle A, B \rangle \cong \mathbb{Z} \times \mathbb{Z}, \\
H_2(T) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\langle F \rangle}{\{0\}} = \langle F \rangle \cong \mathbb{Z}, \\
H_n(T) &= \{0\} \quad \text{for } n > 2.
\end{aligned}$$

Example 3.7 (Projective Plane) The real projective plane, $\mathbb{R}P^2$, is the quotient space obtained from identifying edges of the unit square as in Figure 3.

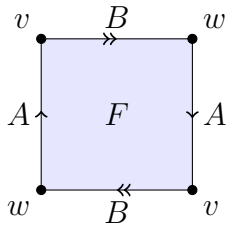


Figure 3: The real projective plane with a CW complex structure.

The chain groups are thus $C_0(\mathbb{R}P^2) = \langle v, w \rangle$, $C_1(\mathbb{R}P^2) = \langle A, B \rangle$, $C_2(\mathbb{R}P^2) = \langle F \rangle$, and $C_n(\mathbb{R}P^2) = \{0\}$ for $n > 2$. By definition, $\ker \partial_0 = \langle v, w \rangle$. We have $\partial_1 A = v - w = -\partial_1 B$, so $\operatorname{im} \partial_1 = \langle v - w \rangle$. Since the rank of $C_1(\mathbb{R}P^2)$ is two and the rank of $\operatorname{im} \partial_1$ is one, we know that the kernel of ∂_1 must have rank one. To determine what elements are in $\ker \partial_1$, suppose that $xA + yB \in C_1(\mathbb{R}P^2)$ is an element of $\ker \partial_1$. Then

$$0 = \partial_1(xA + yB) = x\partial_1 A + y\partial_1 B = x(v - w) + y(w - v) = (x - y)(w - v).$$

Thus, $x = y$ and we conclude $\ker \partial_1 = \{x(A + B) \mid x \in \mathbb{Z}\} = \langle A + B \rangle$. Finally,

$$\partial_2 F = B + A + B + A = 2(A + B),$$

and so $\operatorname{im} \partial_2 = \langle 2(A + B) \rangle$. Since the rank of $C_2(\mathbb{R}P^2)$ and $\operatorname{im} \partial_2$ are the same, we must conclude that $\ker \partial_2 = \{0\}$. Thus, the homology groups for the projective plane are

$$\begin{aligned}
H_0(\mathbb{R}P^2) &= \frac{\langle v, w \rangle}{\langle v - w \rangle} \cong \langle w \rangle \cong \mathbb{Z}, \\
H_1(\mathbb{R}P^2) &= \frac{\langle A + B \rangle}{\langle 2(A + B) \rangle} \cong \mathbb{Z}_2, \\
H_n(\mathbb{R}P^2) &= \{0\} \quad \text{for } n \geq 2.
\end{aligned}$$

We may interpret $H_1(X) \cong \mathbb{Z}_2$ as meaning there is a loop bounding a 2-dimensional void in the real projective plane that vanishes if we traverse it twice.

We can now compare the homology groups for S^2, T, and $\mathbb{R}P^2$. Note that the calculation of the homology groups of S^2 is contained in the appendix.

$$\begin{array}{rccccccc} S^2\colon & \mathbb{Z}, & \{0\}, & \mathbb{Z}, & \{0\}, & \{0\}, & \dots \\ T\colon & \mathbb{Z}, & \mathbb{Z}\times\mathbb{Z}, & \mathbb{Z}, & \{0\}, & \{0\}, & \dots \\ \mathbb{R}P^2\colon & \mathbb{Z}, & \mathbb{Z}_2, & \{0\}, & \{0\}, & \{0\}, & \dots \end{array}$$

Since the sequences of homology groups are different for all three spaces, we see that no two of them are homeomorphic. Further we see the presence of the two 2-dimensional voids in first homology group of T and the lack of any 2-dimensional voids in the first homology group of S^2.

We note that homology groups can only determine when two spaces are *not* homeomorphic; they are not a complete set of topological invariants and therefore cannot be used to determine if two spaces are homeomorphic.

4 Homology With General Coefficients

In the previous section, we made a specific choice to define chain groups with coefficients in $\mathbb{Z}$. There is a natural generalization of homology theory in which the chain groups have coefficients from any abelian group G. The algebra involved in computing homology can be greatly simplified with certain coefficient groups, as we demonstrate here, and the results can often be translated back to integer coefficients using universal coefficient theorems.

Definition 4.1 *[7] Let X be a CW complex and G an abelian group. The n***th chain group of X with coefficients in** *G, denoted $C_n(X;G)$, is the set of all linear combinations of n-cells in X with coefficients from G.*

Notice that if X has k n-cells, then $C_n(X;G) \cong G^k$.

Definition 4.2 *[7] We define the* **boundary map** *$\partial_n : C_n(X;G) \to C_{n-1}(X;G)$ by the G-linear extension of the boundary of an n-cell; that is, for an n-cell σ and for $g \in G$*

$$\partial_n(g\sigma) = g(\partial_n\sigma).$$

*The n***th homology group of X with coefficients in** *G is thus defined as*

$$H_n(X;G) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

Example 4.3 (Torus) We repeat Example 3.6 with $\mathbb{Z}_2$-coefficients. The chain groups for T are now

$$\begin{aligned} C_0(T;\mathbb{Z}_2) &= \langle v\rangle \cong \mathbb{Z}_2, \\ C_1(T;\mathbb{Z}_2) &= \langle A, B\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_2, \\ C_2(T;\mathbb{Z}_2) &= \langle F\rangle \cong \mathbb{Z}_2, \end{aligned}$$

and $C_n(T;\mathbb{Z}_2) = \{0\}$ for all $n > 2$. Computing the boundary maps, we have, as before, $\partial_0 v = 0$, $\partial_1 A = \partial_1 B = 0$, and $\partial_2 F = 0$. Thus, the homology groups for the torus with $\mathbb{Z}_2$-coefficients are

$$\begin{aligned}
H_0(T;\mathbb{Z}_2) &= \frac{\langle v\rangle}{\{0\}} = \langle v\rangle \cong \mathbb{Z}_2,\\
H_1(T;\mathbb{Z}_2) &= \frac{\langle A, B\rangle}{\{0\}} = \langle A, B\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,\\
H_2(T;\mathbb{Z}_2) &= \frac{\langle F\rangle}{\{0\}} = \langle F\rangle \cong \mathbb{Z}_2,\\
H_n(T;\mathbb{Z}_2) &= \{0\} \quad \text{for } n > 2.
\end{aligned}$$

Example 4.4 (Projective plane) We repeat Example 3.7 with $\mathbb{Z}_2$-coefficients. The chain groups are now

$$\begin{aligned}
C_0(\mathbb{R}P^2;\mathbb{Z}_2) &= \langle v, w\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,\\
C_1(\mathbb{R}P^2;\mathbb{Z}_2) &= \langle A, B\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,\\
C_2(\mathbb{R}P^2;\mathbb{Z}_2) &= \langle F\rangle \cong \mathbb{Z}_2,
\end{aligned}$$

and all others are $\{0\}$. As before, $\ker\partial_0 = \langle v, w\rangle$. We now have $\partial_1 A = v + w = \partial_1 B$, and so $\operatorname{im}\partial_1 = \langle v + w\rangle$. Again, $\ker\partial_1$ must have rank one. In this case, if $xA + yB \in C_1(\mathbb{R}P^2)$ is also an element of $\ker\partial_1$, then

$$0 = \partial_1(xA + yB) = x\partial_1 A + y\partial_1 B = x(v + w) + y(v + w) = (x + y)(w + v),$$

which implies $x \equiv y \pmod 2$. Thus, $\ker\partial_1 = \{x(A + B) \mid x \in \mathbb{Z}_2\} = \langle A + B\rangle$. Finally, we see a stark contrast when using $\mathbb{Z}_2$-coefficients when computing $\operatorname{im}\partial_2$,

$$\partial_2 F = B + A + B + A = 2(A + B) \equiv 0.$$

Thus $\operatorname{im}\partial_2 = \{0\}$ and $\ker\partial_2 = \langle F\rangle \cong \mathbb{Z}_2$. Therefore, the homology groups for $\mathbb{R}P^2$ with $\mathbb{Z}_2$-coefficients are

$$\begin{aligned}
H_0(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\langle v, w\rangle}{\langle v + w\rangle} \cong \mathbb{Z}_2,\\
H_1(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\langle A + B\rangle}{\{0\}} \cong \mathbb{Z}_2,\\
H_2(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\langle F\rangle}{\{0\}} \cong \mathbb{Z}_2\\
H_n(\mathbb{R}P^2;\mathbb{Z}_2) &= \{0\} \text{ for } n > 2.
\end{aligned}$$

Notice that in the computation with $\mathbb{Z}_2$-coefficients, the algebra was easier than with $\mathbb{Z}$-coefficients. In particular, we did not have to worry about the signs when finding

$\ker \partial_1$. While this simplification was not substantial, this is not the case in situations with significantly more complicated algebra (such as those in following sections). Notice that for real projective space, $H_2(\mathbb{R}P^2;\mathbb{Z}) = \{0\}$ shows the lack of a three-dimensional void. With $\mathbb{Z}_2$-coefficients, $H_2(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$ does not show the absence of this void. The homology with $\mathbb{Z}_2$-coefficients does hint at the twisting of the unit square when identifying opposite edges, however. In addition, homology with $\mathbb{Z}_2$-coefficients can still be used to distinguish all three spaces as before. Further, homology groups with $\mathbb{Z}_p$ coefficients have more structure and nicer properties, some of which we will see in the Eilenberg-MacLane examples.

5 The Homology of Eilenberg-MacLane Spaces

5.1 Boundary Maps for the nth Chain Group of BG

The nth chain group of BG, $C_n(BG;G)$, is generated by the n-cells of BG with coefficients in G. As was the case when defining the dimension of a cell in BG, there are two boundary maps that we can define for cells in BG and the map we use for computing homology is the sum of those two maps.

Definition 5.1 *[3] Let $[g_1|\cdots|g_m] \in BG$ be an n-cell. The* **simplicial boundary** *of $[g_1|\cdots|g_m]$ is the map $\partial_s : C_n(BG;G) \to C_{n-1}(BG;G)$ defined by*

$$\begin{aligned}\partial_s[g_1|\cdots|g_m] = {} & [g_2|\cdots|g_m] \\ & + \sum_{i=1}^{m-1}(-1)^{d_B[g_1|\cdots|g_i]}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m] \\ & + (-1)^n[g_1|\cdots|g_{m-1}],\end{aligned}$$

where by $[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m]$ we mean combining g_i and g_{i+1} via the group operation $\circ$ for G. The **residual boundary** *of $[g_1|\cdots|g_m]$ is the map $\partial_r : C_n(BG;G) \to C_{n-1}(BG;G)$ defined by*

$$\partial_r[g_1|\cdots|g_m] = \sum_{i=1}^{m}(-1)^{d_B[g_1|\cdots|g_{i-1}]+1}[g_1|\cdots|\partial g_i|\cdots|g_m],$$

where ∂g_i is the boundary of g_i in G. We define the **total boundary** *of $[g_1|\cdots|g_m]$ to be the map $\partial_n : C_n(BG;G) \to C_{n-1}(BG;G)$ defined by*

$$\partial_n = \partial_s + \partial_r.$$

Note that since elements in $\mathbb{Z}_p$ have no boundary, cells in $B\mathbb{Z}_p$ have no residual boundary, and thus the total boundary of a cell in $B\mathbb{Z}_p$ is its simplicial boundary. The simplicial boundary of an n-cell in BG has a nice geometric interpretation. If an n-cell σ has simplicial dimension k, then the simplicial boundary of σ is the boundary of Δ^k, where each side is associated with a particular element of G^{n-1}.

The particle model specifically helps to visualize the boundary maps. For example, applying the boundary map to an n-cell in $B\mathbb{Z}_2$ corresponds to the left-most symbol and the right-most symbol sliding off of the unit interval. Specifically, we depict $\partial_2[1|1] = [1] + [1] = 2[1] \equiv 0$ as

$\partial_2($ $) =$ $+$ $= 2($ $) = 0.$

With these definitions, we define the nth homology group of BG by $H_n(BG; G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, as before.

5.2 The Homology of $B\mathbb{Z}_p$

The homology groups of $K(\mathbb{Z}_p, n)$ are well-known from Cartan [2]. The homology groups of the collection of $K(\mathbb{Z}_p, n)$ for all $n > 1$ and a fixed p form what is known as a Hopf ring. While this is beyond the scope of this paper, the homology groups are nice and worth studying. The original proofs are very algebraic. A newer description of this structure can be found in Wilson's survey on Hopf rings [10]. Here we will use bar notation to give a different calculation for the homology groups of $K(\mathbb{Z}_2, 1)$, and then we will use the particle model to visualize this structure.

Theorem 5.2 *[2] For a prime p, $H_n(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p$, for each $n \geq 0$.*

Proof. We provide the proof for $p = 2$. The proof for odd primes is outside the scope of this paper. As noted in Section 5.1, the residual boundary map of elements in $B\mathbb{Z}_2$ is trivial, so $\partial_n = \partial_s$ for each n. By Remark 2.8, $C_n(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for each n. Let $[g_1|g_2|\cdots|g_n]$, where $g_i = 1$ for each i, be the nontrivial n-cell in $C_n(B\mathbb{Z}_2; \mathbb{Z}_2)$. Computing its boundary, we find

$$\begin{aligned}\partial_n[g_1|g_2|\cdots|g_n] &= \partial_s[g_1|g_2|\cdots|g_n] \\ &= [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}].\end{aligned}$$

For each i, $g_i \circ g_{i+1} = 1 \circ 1 \equiv 0 \pmod 2$. So $[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_n]$ is an $(n-2)$-cell in $B\mathbb{Z}_2$ by Definitions 2.1 and 2.5, and thus is trivial in the boundary map as it is not in the codomain of ∂_n. Since each $g_i = 1$, we have

$$\begin{aligned}[g_2|\cdots|g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}] &= [g_2|\cdots|g_n] + (-1)^n[g_2|\cdots|g_n] \\ &\equiv 0 \pmod 2.\end{aligned}$$

Thus ∂_n is the zero map, and we have

$$H_n(B\mathbb{Z}_2; \mathbb{Z}_2) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{C_n(B\mathbb{Z}_2; \mathbb{Z}_2)}{\{0\}} \cong \mathbb{Z}_2$$

for each $n \geq 0$. □

The previous theorem implies there is a unique generator of $H_n(B\mathbb{Z}_2;\mathbb{Z}_2)$, which we will denote b_n. We may also choose a generator for $H_n(B\mathbb{Z}_p;\mathbb{Z}_p)$ by taking the element associated with $1 \in \mathbb{Z}_p$, which we will also denote by b_n. Given the choice of generators, we can define an addition on the collection of homology groups for $B\mathbb{Z}_p$ that is induced by the addition on $B\mathbb{Z}_p$. One may think of adding two generators as "shuffling" their group elements together.

Definition 5.3 *[3] The addition on the collection of homology groups of $B\mathbb{Z}_p$*

$$* : H_i(B\mathbb{Z}_p;\mathbb{Z}_p) \times H_j(B\mathbb{Z}_p;\mathbb{Z}_p) \to H_{i+j}(B\mathbb{Z}_p;\mathbb{Z}_p)$$

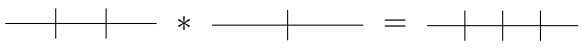
is defined by

$$b_i * b_j = b_j * b_i \equiv \binom{i+j}{i} b_{i+j} \pmod{p}.$$

Using multinomial coefficients and noting that the addition is commutative, we can show that the addition is also associative. Recall the multinomial coefficient $\binom{n}{j,k}$ is defined as $\binom{n}{j,k} = \frac{n!}{j!k!}$. We can then see that

$$\begin{aligned}
(b_i * b_j) * b_k &= \binom{i+j}{i} b_{i+j} * b_k \\
&= \binom{i+j}{i}\binom{i+j+k}{k} b_{i+j+k} \\
&= \binom{i+j+k}{k,i} b_{i+j+k} \\
&= \binom{i+j+k}{i}\binom{j+k}{k} b_{i+j+k} \\
&= \binom{j+k}{k} b_{j+k} * b_i \\
&= b_i * (b_j * b_k).
\end{aligned}$$

For example, $b_2 * b_1 = \binom{3}{1} b_3 \equiv b_3$, is the result of the addition of the generator in degree 2 and the generator in degree 1. In the particle model, this would look like:



From Definition 5.3, we have that $b_i * b_j = 0$ only when $\binom{i+j}{i} \equiv 0 \pmod{p}$. The following theorem provides a method for determining whether $\binom{i+j}{i} \equiv 0 \pmod{p}$.

Theorem 5.4 (Kummer's Theorem [5]) *For a prime number p, the largest power of p that divides $\binom{n}{k}$ is the number of carries when adding k and $n-k$ in base-p.*

Thus, $\binom{i+j}{i} \equiv 0 \pmod{p}$ if there is at least one carry required to add the base-p representations of i and j. We will use this fact to prove the following theorem.

Theorem 5.5 *The generator b_n of $H_n(B\mathbb{Z}_p;\mathbb{Z}_p)$ can be expressed as a sum of the generators b_{p^i} of $H_{p^i}(B\mathbb{Z}_p;\mathbb{Z}_p)$, and this is determined by the base-p representation of n.*

While this theorem is stated in [10] for $p = 2$, a proof is not provided. We provide a proof below for general prime p.

Proof. Let $n = \sum_{i=0}^{k} c_i p^i$, where $c_i \in \{0, 1, 2, \ldots, p-1\}$, be the unique base-$p$ representation of n. We would like to represent b_n as

$$\sum_{i=0}^{k} c_i b_{p^i}, \tag{1}$$

where we use $\sum$ to mean the addition from Definition 5.3 and $c_i b_{p^i}$ to mean the c_i-fold sum of b_{p^i} with itself. By Definition 5.3,

$$c_i b_{p^i} = \sum_{j=1}^{c_i} b_{p^i} = \binom{c_i p^i}{p^i, p^i, \ldots, p^i} b_{p^i} = \prod_{j=0}^{c_i} \binom{(c_i - j)p^i}{p^i} b_{p^i}. \tag{2}$$

For (1) to be a representation of b_n, (2) must be nonzero (mod p) for each i such that $c_i \neq 0$. By Kummer's Theorem, this will be the case if no carries are required when adding the base-p representations of p^i and $(c_i - j)p^i - p^i = (c_i - j - 1)p^i$.

Consider the case when $(c_i - j - 1)$ is maximal; that is, when $c_i = p - 1$ and $j = 0$. Then we are concerned with the base-p addition of p^i and $(p-2)p^i$. Certainly this addition produces no carries, since $1 + (p - 2) = p - 1$ is a valid digit in a base-p representation. It follows that $\binom{(c_i - j)p^i}{p^i}$ is not divisible by p for each j, $0 \leq j \leq c_i$, when $c_i = p - 1$. Similarly, $\binom{(c_i - j)p^i}{p^i}$ is not divisible by p for each c_i such that $0 < c_i < p - 1$. Thus, (2) is nonzero (mod p) for each i such that $c_i \neq 0$. Therefore (1) is a representation for b_n. □

6 Conclusion

The bar construction and the particle model both provide useful ways to perform computations on $B\mathbb{Z}_p$. Specifically, the particle model provides a nice visual way to represent cells in $B\mathbb{Z}_2$, and $B\mathbb{Z}_p$ more generally. It further gives a simple way to denote homology classes. Counting the number of cells becomes much more clear, as does determining dimensions and the addition operation. This advantage continues as we iterate to $B^2\mathbb{Z}_2$ and even $B^3\mathbb{Z}_p$. Even though visualizing the entire particle model is impossible for four or more iterations, the idea of placing elements from $B^n\mathbb{Z}_p$ along a unit interval is useful for calculations in $B^{n+1}\mathbb{Z}_p$, and provides clarity for the multiplication operation on the Hopf ring. This Hopf ring structure will be described in future work.

Appendix A CW Complexes

Although these definitions can generally be found in an introductory topology book such as Massey [6], we provide them here for any readers who may not be familiar with them.

Definition A.1 *An* **(open)** n**-cell**, *denoted* e^n, *is an open space that is homeomorphic to the open disk* $\mathring{B}^n$. *A* **CW complex** *is a Hausdorff space* X *together with an ascending sequence of closed subspaces*

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$$

satisfying:

- X^0 *is a collection of 0-cells with the discrete topology.*
- *For* $n > 0$, $X^n \backslash X^{n-1}$ *is a disjoint collection of* n*-cells* e^n_λ *with* **characteristic maps** $f_\lambda : \mathrm{Cl}(B^n) \to X^n$ *such that* f *maps* $\mathring{B}^n$ *homeomorphically onto* e^n *and* $f(\partial \mathring{B}^n) \subset X^{n-1}$. *We think of the characteristic maps as instructions for how to attach the* n*-cells to* X^{n-1} *to build* X^n.
- $X = \bigcup_i X^i$.
- X *and each* X^i *has the weak topology.*

We call each subset X^n *the* n**-skeleton** *of* X. *Elements of* X^0 *are called* **vertices** *or* 0**-cells**. *A CW complex is called* **finite** *if it contains finitely many cells, and* **infinite** *otherwise. If* $X = X^n$ *for some integer* n, *we say that the CW complex is* **finite dimensional**, *and the least such* n *is its* **dimension**. *For cells* e^m *and* e^n, *we say that* e^m *is a* **face** *of* e^n *if* $e^m \subseteq \mathrm{Cl}(e^n)$.

Example A.2 (n**-Sphere)** The n-sphere can be given a CW structure such that there are two cells: a 0-cell and an n-cell. Then the k-skeletons of S^n, for $0 \leq k < n$, contain a single point, and the n-skeleton is S^n.

We note that a CW structure is not unique and there are other CW structures that can be put on the n-sphere.

Example A.3 (Cellular homology of the 2-Sphere) Let the two-sphere, S^2, have the CW structure consisting of one vertex, v, and one 2-cell, F. The chain groups are then $C_0(S^2) = \langle v \rangle \cong \mathbb{Z}$, $C_1(S^2) = \{0\}$, $C_2(S^2) = \langle F \rangle \cong \mathbb{Z}$, and $C_k(S^2) = \{0\}$ for $k > 2$. Since $C_1(S^2) = \{0\}$, we have $\ker \partial_2 = C_2(S^2) \cong \mathbb{Z}$. By definition, $\ker \partial_0 = C_0(S^2) \cong \mathbb{Z}$. All other images and kernels are $\{0\}$. Thus, the homology groups for S^2 are

$$H_0(S^2) = \frac{\ker \partial_0}{\mathrm{im}\, \partial_1} = \frac{\langle v \rangle}{\{0\}} = \langle v \rangle \cong \mathbb{Z},$$
$$H_2(S^2) = \frac{\ker \partial_2}{\mathrm{im}\, \partial_3} = \frac{\langle F \rangle}{\{0\}} = \langle F \rangle \cong \mathbb{Z},$$

and all other homology groups are $\{0\}$. Thus, S^2 is path connected with a single 3-dimensional void and no voids of any other dimension.

Example A.4 (Cellular homology of the 2-sphere with $\mathbb{Z}_2$ coefficients) Here we repeat Example A.3 with $\mathbb{Z}_2$-coefficients. The chain groups are then $C_0(S^2) = \langle v \rangle \cong \mathbb{Z}_2$, $C_2(S^2) = \langle F \rangle \cong \mathbb{Z}_2$, and $C_k(S^2) = \{0\}$ for $k = 1$ and $k > 2$. The computations in Example A.3 give $\ker \partial_2 = C_2(S^2) \cong \mathbb{Z}_2$, $\ker \partial_0 = C_0(S^2) \cong \mathbb{Z}_2$, and all other images and kernels are $\{0\}$. Thus, the homology groups for S^2 with $\mathbb{Z}_2$-coefficients are

$$H_0(S^2; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$
$$H_2(S^2; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

and all other homology groups are $\{0\}$.

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[9] Torus image, available online at the URL: `https://urldefense.com/v3/__https://commons.wikimedia.org/wiki/File:Toruscycles1.svg__;!!P7nkOOY!68S_OmmG6WnGWB0qIE4EN-70pAM0GfZvGhWEG4FR1MP5evojGv_tuEsM3Fz0ZQE$`

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