

Tight Frame Graphs Arising as Line Graphs

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Abstract - Dual multiplicity graphs are those simple, undirected graphs that have a weighted Hermitian adjacency matrix with only two distinct eigenvalues. From the point of view of frame theory, their characterization can be restated as which graphs have a representation by a tight frame. In this paper, we classify certain line graphs that are tight frame graphs and improve a previous result on the embedding of frame graphs in tight frame graphs.

Keywords : tight frame graph; dual multiplicity graph; line graph; Gram matrix; Laplacian matrix

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1 Introduction

Much recent interest has concerned the inverse eigenvalue problem of a graph, namely, to determine all possible spectra of real symmetric (Hermitian) matrices whose off-diagonal pattern of zero/non-zero entries is given by the adjacencies of a graph (see [7] and the references therein). The special case of dual multiplicity graphs, or graphs that permit two distinct eigenvalues, has been thoroughly investigated (see [2], [3], [9], and [12]).

In [1] this problem was reintroduced from the point of view of frame theory. Frames in finite-dimensional spaces have received much attention from both pure and applied mathematicians and constitute a vast literature; see [11] for a thorough introduction or [14] for a development at the undergraduate level. In finite-dimensional spaces, a finite frame is a sequence of vectors whose span is the whole space; the redundancy of frames is their key advantage over orthonormal bases in applications such as image processing or data transmission. Tight frames [20] are especially important since they provide (non-unique) reconstruction of vectors similarly to orthonormal bases, without the requirement of orthogonality (or linear independence). Every simple graph is a frame graph, that is, has a vector representation by a frame $\{f_1, \dots, f_n\}$ with $\langle f_i, f_j \rangle \neq 0$ if and only if the vertices represented by f_i and f_j are connected by an edge. In the real case, this is also known as a faithful orthogonal representation, dating back to [17]. Dual multiplicity graphs are those which have a representation by a tight frame, and classifying them is a difficult question.

In this paper, we continue the line of investigation of [1], applying frame theoretic tools to a graph theoretic question. In the next section, we provide definitions and preliminary



material from both graph theory and frame theory, concluding with some key connections. In Section 3, we focus on tight frames and establish an improved result on frame graphs being embedded as induced subgraphs of tight frame graphs. Although a straightforward consequence of Proposition 2.1 of [10], Corollary 3.8 yields embeddings that are minimal in terms of the number of additional vectors required; we contrast our construction with the non-minimal embedding of Theorem 4.2 of [12]. Section 4 contains our main contribution, the natural recognition of frame graphs as line graphs and the classification of certain line graphs as tight frame graphs. Using what we refer to as the “Laplacian method,” we establish the line graph of the complete graph as a tight frame graph (Theorem 4.3) and give a new, constructive proof of the tightness of the complete graph (Theorem 4.4). We illustrate relationships between line graphs and root graphs in terms of whether or not they are tight frame graphs and end by pointing out some limitations of this approach.

2 Background

A *graph* $\Gamma = (V, E)$ is an ordered pair, where $V = V(\Gamma) = \{v_1, \dots, v_n\}$ is a non-empty set of *vertices* and $E = E(\Gamma)$ is a set of unordered pairs of distinct vertices, called *edges* (or *lines*) connecting vertices. Here we consider only undirected simple graphs, without loops, or edges connecting a vertex to itself, and without multiple edges between a pair of vertices. We say two vertices v_i, v_j are *adjacent* (or are *neighbors*) if $\{v_i, v_j\} \in E$, and two edges are *incident* if they share an endpoint. The *degree* of a vertex v , denoted $\deg(v)$, is the number of vertices adjacent to v . The *order* of a graph is $|V| = n$ and its *size* is $|E|$.

A *trail* is a sequence of distinct, incident edges connecting a sequence of vertices. If the vertices in a trail are distinct, the associated graph is a *path*, and if a trail begins and ends at the same vertex but all other vertices are distinct, the graph is a *cycle*. The path and cycle on n vertices are written P_n and C_n , respectively. An *induced subgraph* is a subset of vertices of a graph and all edges whose endpoints are both in that subset. If a graph has at least one induced path connecting any two vertices, the graph is *connected*. If the deletion of an edge from such a graph results in a disconnected graph, that edge is known as a *bridge*.

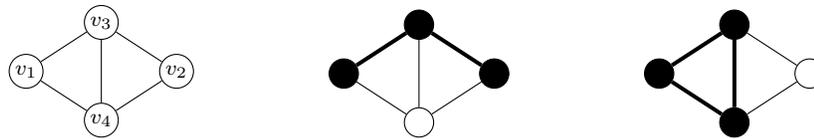


Figure 1: The diamond graph is a connected, bridgeless graph of order 4 and size 5; P_3 and C_3 as induced subgraphs

The *complete graph* K_n on n vertices has $\{v_i, v_j\} \in E$ for all $i \neq j$. Equivalently, the degree of every vertex of K_n is $n - 1$. The *star graph* S_n on n vertices has one vertex of degree $n - 1$ and $n - 1$ vertices of degree one. The *complete bipartite graph* $K_{m,n}$ has vertex set $V = V_1 \cup V_2$ ($|V_1| = m, |V_2| = n$) and edge set $E = \{\{v_1, v_2\} : v_1 \in V_1 \text{ and } v_2 \in V_2\}$. Note that S_n is the same graph as $K_{1,n-1}$. The *Cartesian product* of graphs Γ and Λ ,

denoted $\Gamma \square \Lambda$, is the graph with vertex set $V(\Gamma) \times V(\Lambda)$, such that vertices (u, u') and (v, v') are adjacent in $\Gamma \square \Lambda$ if and only if either $u = v$ and $\{u', v'\} \in E(\Lambda)$ or $u' = v'$ and $\{u, v\} \in E(\Gamma)$.

Both graphs and frames determine associated matrices, the entries of which may be over the real or complex field. In the case of frames, we consider the finite-dimensional Hilbert space \mathbb{H}^d , equipped with the standard inner product $\langle \cdot, \cdot \rangle$, where $\mathbb{H} = \mathbb{C}$ or \mathbb{R} . In what follows, we denote by I_n , 0_n , and J_n the $n \times n$ identity matrix, zero matrix, and all-ones matrix, respectively.

For a graph Γ on n vertices, the $n \times n$ Laplacian matrix L of Γ is defined by

$$L_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian can also be written as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix with the degree of each vertex of Γ along the diagonal and $A(\Gamma)$ is the *adjacency* matrix with zeros along the diagonal and off-diagonal zero/one pattern corresponding to adjacencies between vertices. More generally, to a graph Γ we associate the following subset of $n \times n$ Hermitian (or real symmetric) matrices:

$$\mathcal{H}(\Gamma) = \{M \in \mathbb{H}^{n \times n} : M = M^*, m_{ij} \neq 0 \iff \{v_i, v_j\} \in E(\Gamma) \text{ for } i \neq j\}$$

where $*$ denotes the conjugate transpose. All matrices in $\mathcal{H}(\Gamma)$ must have real diagonal entries, that are otherwise unrestricted, and real eigenvalues. The Spectral Theorem highlights the crucial connection between a Hermitian matrix and its eigenvalues and an orthonormal basis of eigenvectors:

Theorem 2.1 *A Hermitian matrix M may be diagonalized as $M = UDU^*$, where the eigenvalues of M are the entries of the real diagonal matrix D and U is a unitary matrix whose columns are a complete set of corresponding unit eigenvectors of M .*

The set of *positive semidefinite* Hermitian matrices associated with a graph Γ is defined by

$$\mathcal{H}^+(\Gamma) = \{M \in \mathcal{H}(\Gamma) : \langle Mx, x \rangle \geq 0 \text{ for all } x \in \mathbb{H}^n\}.$$

Clearly any $M \in \mathcal{H}^+(\Gamma)$ has nonnegative eigenvalues. The *minimum semidefinite rank* of a graph is

$$\text{msr}(\Gamma) = \min\{\text{rank}(M) : M \in \mathcal{H}^+(\Gamma)\}.$$

Although the minimum semidefinite rank is not field-independent [6], the named graphs that appear in this paper have equal minimum semidefinite rank over \mathbb{C} and \mathbb{R} .

The Laplacian matrix of a graph of order n is positive semidefinite with smallest eigenvalue 0; note that the all-ones vector $\mathbf{1}$ is an eigenvector of L corresponding to $\lambda = 0$. The rank of L is $n - k$, where k is the number of connected components of the graph [13]. Suppose Γ is a connected graph on n vertices. Then $\text{rank}(L(\Gamma)) = n - 1$ and $\text{msr}(\Gamma) \leq n - 1$. Now suppose $\text{msr}(\Gamma) = 1$. Then there exists $M \in \mathcal{H}^+(\Gamma)$ with only one



nonzero eigenvalue λ , and $M = u\lambda u^*$, where u is its corresponding unit eigenvector. If any $u_k = 0$, both the k th row and k th column of M contain only zeros, which contradicts the assumption that Γ is connected. Therefore, any connected graph Γ with $\text{msr}(\Gamma) = 1$ must be the complete graph K_n , and for $\Gamma \neq K_n$, $2 \leq \text{msr}(\Gamma) \leq n - 1$. In the remainder of this paper, we assume Γ is connected since a matrix associated with a disconnected graph can be written as the direct sum of matrices for each component.

Denote by $q(\Gamma)$ the minimum number of distinct eigenvalues for any $M \in \mathcal{H}(\Gamma)$. Clearly $1 \leq q(\Gamma) \leq n$. Now if $q(\Gamma) = 1$, then $M = UDU^* = \lambda I_n$ for some $M \in \mathcal{H}(\Gamma)$, so Γ is the *edgeless*, or totally disconnected, graph on n vertices. Therefore, connected graphs must have $q(\Gamma) \geq 2$. Of particular interest are *dual multiplicity* graphs where $q(\Gamma) = 2$. As a consequence of the last paragraph, K_n is a dual multiplicity graph.

We now turn our attention to the frame representation of a graph. A sequence of vectors $\mathcal{F} = \{f_i\}_{i=1}^n$ is a *finite frame* for a d -dimensional Hilbert space \mathbb{H}^d if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^n |\langle x, f_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathbb{H}^d. \quad (1)$$

The two constants A and B are called *frame bounds*. The largest lower frame bound and smallest upper frame bound are called the *optimal* frame bounds. If $A = B$, the frame is *tight* (or B -tight), and when $A = B = 1$, the frame is a *Parseval frame*. Inequality (1) is equivalent to $\text{span}\{f_i\}_{i=1}^n = \mathbb{H}^d$ (although this is not valid in infinite dimensions). The *synthesis operator* $F : \mathbb{H}^n \rightarrow \mathbb{H}^d$ is given by $F(e_i) = f_i$, where $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis for \mathbb{H}^n . Its adjoint, the *analysis operator* $F^* : \mathbb{H}^d \rightarrow \mathbb{H}^n$ is given by $F^*(x) = (\langle x, f_i \rangle)_{i=1}^n$. In what follows, we use the matrix representations of these operators,

$$F = \begin{bmatrix} | & & | \\ f_1 & \cdots & f_n \\ | & & | \end{bmatrix} \quad \text{and} \quad F^* = \begin{bmatrix} - & f_1^* & - \\ & \vdots & \\ - & f_n^* & - \end{bmatrix}$$

for the $d \times n$ synthesis matrix and $n \times d$ analysis matrix, respectively. The *frame operator* $S = FF^* : \mathbb{H}^d \rightarrow \mathbb{H}^d$ is defined by

$$FF^*(x) = \sum_{i=1}^n \langle x, f_i \rangle f_i = \sum_{i=1}^n f_i f_i^* x,$$

and the *Gram operator* (also *Gramian* or *Gram matrix*) $G = F^*F : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is given by $g_{ij} = \langle f_j, f_i \rangle$ for $1 \leq i, j \leq n$. Note that $A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|x\|^2$ for all $x \in \mathbb{H}^d$, and \mathcal{F} is a Parseval frame if and only if $S = I_d$. The matrix $S = FF^*$ is invertible, positive definite, and Hermitian. Its eigenvalues are the nonzero eigenvalues of the positive semidefinite, Hermitian matrix $G = F^*F$. The following result is well-known.

Lemma 2.2 *The frame \mathcal{F} is a Parseval frame if and only if $G^2 = G$. That is, a frame is Parseval if and only if its Gram matrix is an orthogonal projection.*



Proof. If $FF^* = I_d$, then $G^2 = G$. If $G^2 = G$, then left-multiplying both sides by F and right-multiplying both sides by F^* gives $S^3 = S^2$. Since S is invertible, it follows that $S = I_d$. \square

As investigated in [1], a frame $\mathcal{F} = \{f_i\}_{i=1}^n$ for \mathbb{H}^d and a graph $\Gamma = (V, E)$ with $|V| = n$ may be *associated* with each other provided there is a one-to-one correspondence between vectors f_i and vertices v_i such that $\{v_i, v_j\} \in E$ if and only if $\langle f_i, f_j \rangle \neq 0$. This is known as a *vector representation*, or for us a *frame representation*, of the graph. Through an abuse of notation, we may refer to the vertices v_i as the vectors f_i .

We end this section by pointing out the meanings of $\text{msr}(\Gamma)$ and $q(\Gamma)$ in the frame theoretic context and providing a specific example to illustrate the frame theoretic concepts encountered above.

Given a (connected) graph Γ with $\text{msr}(\Gamma) = k$, there exists some $M \in \mathcal{H}^+(\Gamma)$ with rank k , which can be decomposed as $M = UDU^*$ by Theorem 2.1. Letting \tilde{D} be the truncated matrix of only the nonzero eigenvalues of M and \tilde{U} the corresponding truncated matrix of eigenvectors, we recognize M as the Gram matrix of a frame:

$$M = \tilde{U}\sqrt{\tilde{D}}^* \sqrt{\tilde{D}}\tilde{U}^* = F^*F,$$

where $F = \sqrt{\tilde{D}}\tilde{U}^*$ is a $k \times n$ matrix whose columns form a frame for \mathbb{H}^k . Clearly the minimum semidefinite rank of a graph Γ is the lowest dimension of a Hilbert space in which a frame associates with Γ .

If a graph is associated with a tight frame, then it is known as a *tight frame graph*. Let $\mathcal{F} = \{f_i\}_{i=1}^n$ be a frame for \mathbb{H}^d associated with a connected graph Γ . The frame operator S has eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$, where the optimal lower frame bound of \mathcal{F} is $A = \lambda_1$ and optimal upper frame bound is $B = \lambda_d$ [11]. If \mathcal{F} is tight, $A = B$, the Gramian G must have only one nonzero eigenvalue, and $q(\Gamma) = 2$. In fact, a connected graph is a tight frame graph if and only if it is a dual-multiplicity graph ([1], Theorem 5.2).

Example 2.3 The diamond graph $K_4 - e$ in Figure 1 is a tight frame graph since

$$F = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{2}} & 2 & 1 \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 1 & 2 \end{bmatrix}$$

is the synthesis matrix of a Parseval frame for \mathbb{R}^2 associated with the diamond graph. Indeed, it is easy to check that $S = FF^* = I_2$ and $G = F^*F \in \mathcal{H}^+(K_4 - e)$. As mentioned in the introduction, a key advantage of frames over orthonormal bases is their redundancy, which makes frames robust to erasures. For example, any $x \in \mathbb{R}^2$ can be reconstructed from the coefficients $\langle x, f_i \rangle$ as $x = Sx = \sum_{i=1}^4 \langle x, f_i \rangle f_i$, where the f_i are the columns of F . Suppose that, in the course of data transmission, $\langle x, f_1 \rangle$ and $\langle x, f_3 \rangle$ are lost. Then $\tilde{\mathcal{F}} = \{f_2, f_4\}$ is still a spanning set for \mathbb{R}^2 and hence still a frame. If \tilde{S} is the frame operator of $\tilde{\mathcal{F}}$, then x can still be recovered as

$$x = \langle x, f_2 \rangle \tilde{S}^{-1}(f_2) + \langle x, f_4 \rangle \tilde{S}^{-1}(f_4).$$



We see in Figure 2 that the original frame is robust to the loss of any two vectors. On the other hand, the sequence $\mathcal{F}' = \{f_1, f_2, f_3, f_3\}$ is a frame for \mathbb{R}^2 that represents the diamond graph but which is robust to only one erasure, since reconstruction would be impossible in the event that both $\langle x, f_1 \rangle$ and $\langle x, f_2 \rangle$ were lost.

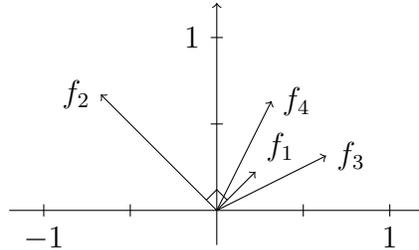


Figure 2: The column vectors of the synthesis matrix in Example 2.3

3 Tight Frames

In this section, we use a frame theoretic approach to give alternate proofs of some known results about tight frame graphs and to improve a result on embedding frame graphs in tight frame graphs.

Lemma 3.1 *A graph Γ is a tight frame graph if and only if it is a Parseval frame graph.*

Proof. Suppose Γ is a tight frame graph on n vertices represented in \mathbb{H}^d . Then there exists a tight frame $\mathcal{F} = \{f_i\}_{i=1}^n \subseteq \mathbb{H}^d$ with frame operator $S = FF^* = BI_d$ and Gramian $G = F^*F \in \mathcal{H}^+(\Gamma)$. Since $\frac{1}{\sqrt{B}}F\frac{1}{\sqrt{B}}F^* = I_d$ and $\frac{1}{B}G$ has the same off-diagonal zero/nonzero pattern as G , $\frac{1}{\sqrt{B}}\mathcal{F}$ is a Parseval frame representing Γ . Conversely, if Γ is a Parseval frame graph, then there exists a tight frame with frame bound $B = 1$ representing Γ . \square

The following simple but highly useful visual characterization of graphs that are not tight frame graphs was given as Corollary 3.4 of [12], Corollary 4.5 of [3], and Theorem 5.3 of [1]:

Proposition 3.2 *If Γ is a tight frame graph, any two non-adjacent vertices of Γ do not have exactly one common neighbor.*

Proof. By Lemma 3.1, we may assume that Γ is a Parseval frame graph, with corresponding Parseval frame \mathcal{F} and Gram matrix G . Assume non-adjacent vertices f_i and f_j have a unique common neighbor f_k . Then $\langle f_j, f_i \rangle = 0$, $\langle f_k, f_i \rangle \neq 0$, $\langle f_j, f_k \rangle \neq 0$, and $\langle f_l, f_i \rangle \langle f_j, f_l \rangle = 0$ for all $l \neq k$. By Lemma 2.2,

$$0 = \langle f_j, f_i \rangle = \sum_{l=1}^n \langle f_l, f_i \rangle \langle f_j, f_l \rangle = \langle f_k, f_i \rangle \langle f_j, f_k \rangle,$$



a contradiction. \square

As a consequence of the previous result, a connected tight frame graph with three or more vertices must be bridgeless. In fact, we can say more.

Corollary 3.3 *If Γ is a connected tight frame graph with at least three vertices, then every edge of Γ belongs to a 3-cycle or a 4-cycle.*

Proof. Suppose $\{u, v\} \in E(\Gamma)$. Since Γ is connected with $|V(\Gamma)| \geq 3$, we may assume there exists $w \in V(\Gamma)$ such that $\{v, w\} \in E(\Gamma)$. If u is adjacent to w , then $\{u, v\}$ belongs to a 3-cycle. Otherwise, if u is not adjacent to w , then $\{u, v\}$ belongs to a 4-cycle by Proposition 3.2. \square

The redundancy (overcompleteness) of frames is the feature that makes them desirable in many applications when compared to orthonormal bases. The following shows that redundancy is necessary for tight frames associated with connected (or merely not edgeless) graphs.

Proposition 3.4 *Suppose $\mathcal{F} = \{f_i\}_{i=1}^n$ is a tight frame for \mathbb{H}^d associated with a connected graph Γ . If $d \geq 2$, then $d < n$.*

Proof. By Lemma 3.1, assume F is the synthesis matrix of a Parseval frame for \mathbb{H}^d with Gramian $G = F^*F \in \mathcal{H}^+(\Gamma)$. Since $S = FF^* = I_d$, the rows of F are orthonormal. If $d = n$, then F is a unitary matrix, and $G = S = I_d$, which implies Γ is edgeless, contradicting the assumption that Γ is connected. \square

Corollary 3.5 *Suppose $\mathcal{F} = \{f_i\}_{i=1}^n$ is a tight frame for \mathbb{H}^d associated with a connected graph Γ . If $d \geq 2$, then zero is an eigenvalue of its Gramian G .*

Proof. If S is the frame operator of \mathcal{F} , then $\text{rank}(G) = \text{rank}(S) = d < n$, by Proposition 3.4. \square

Naimark's Theorem is a central result in the theory of Parseval frames [11]. A *Naimark complement* of a Parseval frame $\mathcal{F} = \{f_i\}_{i=1}^n$ for \mathbb{H}^d for $d < n$ can be viewed as a completion of the $d \times n$ synthesis matrix to an $n \times n$ unitary matrix [10]. As an explicit construction, begin with a Parseval frame $\mathcal{F} = \{f_i\}_{i=1}^n$ for \mathbb{H}^d with Gramian $G = F^*F$. By Theorem 2.1, we can write $G = UDU^*$ where U is unitary and $D = I_d \oplus 0_{n-d}$. Since

$$I_n - G = I_n - UDU^* = U(I_n - D)U^*,$$

we see that $I_n - G$ is the Gram matrix of the Parseval frame $\tilde{\mathcal{F}}$ for \mathbb{H}^{n-d} whose synthesis matrix is the last $n - d$ rows of U^* . And $I_n - G$ has the same off-diagonal zero/nonzero pattern as G , so \mathcal{F} and its complement $\tilde{\mathcal{F}}$ have the same associated graph (see also Theorem 2.1 of [12]). By Lemma 3.1, we may refer to the *frame complement* in the case of a tight frame graph Γ and note that $\text{msr}(\Gamma) \leq \lfloor \frac{n}{2} \rfloor$.



Proposition 3.6 K_n is the only connected graph on n vertices represented by a tight frame for \mathbb{H}^{n-1} .

Proof. If \mathcal{F} is a tight frame for \mathbb{H}^{n-1} with Gramian $G \in \mathcal{H}^+(\Gamma)$, then the frame complement is a tight frame for \mathbb{H} , but the only connected graph with $\text{msr}(\Gamma) = 1$ is $\Gamma = K_n$. \square

Although it is proved in [12] that any graph may occur as an induced subgraph of a tight frame graph, with the same minimum semidefinite rank, the following frame theoretic approach often gives a smaller such embedding.

Theorem 3.7 ([10], Proposition 2.1) Let $\mathcal{F} = \{f_j\}_{j=1}^n$ be a frame for \mathbb{H}^d with synthesis matrix F . Suppose $S = FF^*$ has orthonormal eigenvectors $\{x_i\}_{i=1}^d$ corresponding to eigenvalues $A = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k < \lambda_{k+1} = \dots = \lambda_d = B$. Then $\{f_j\}_{j=1}^n \cup \{h_i\}_{i=1}^k$ is a B -tight frame for \mathbb{H}^d where

$$h_i = (B - \lambda_i)^{\frac{1}{2}} x_i$$

for $1 \leq i \leq k$.

Corollary 3.8 Any connected frame graph is an induced subgraph of a connected tight frame graph, represented by vectors in the same Hilbert space. For a specific frame, the number of additional vectors required for the embedding is minimal.

Proof. Let Γ be a connected graph represented by the frame $\mathcal{F} = \{f_j\}_{j=1}^n \subseteq \mathbb{H}^d$, and let $\{h_i\}_{i=1}^k$ be defined as in Theorem 3.7. Since \mathcal{F} is a spanning set for \mathbb{H}^d and each $h_i \neq 0$, it is impossible for $\langle h_i, f_j \rangle = 0$ for all $1 \leq j \leq n$, which ensures that the corresponding tight frame graph, represented by $\{f_j\}_{j=1}^n \cup \{h_i\}_{i=1}^k$, is connected. As noted in [10], this construction adds a minimal number of vectors needed to complete \mathcal{F} to a tight frame. Indeed, suppose the eigenvalues of $S = FF^*$ are $A = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k < \lambda_{k+1} = \dots = \lambda_d = B$ and there exist vectors $h'_i \in \mathbb{H}^d$ for $1 \leq i \leq k' < k$ such that writing H' for the matrix with columns $\{h'_i\}_{i=1}^{k'}$ yields $[F \ H'] [F \ H']^* = FF^* + H'H'^* = CI_d$ for some constant $C > 0$. Since $B = \lambda_d$ is the optimal upper frame bound of \mathcal{F} , Inequality (1) implies $B \leq C$. By a theorem of Weyl ([16], Corollary 4.3.5), $C = \lambda_1(FF^* + H'H'^*) \leq \lambda_{1+k'}(FF^*) < B$, a contradiction. \square

Remark 3.9 For a given graph, we can contrast the the worst-case number of additional vectors required by the construction in Corollary 3.8 with that of Theorem 4.2 of [12]. By Corollary 3.8, the largest possible value for the additional number of vectors is $d - 1$, in the case when all eigenvalues of S are distinct. On the other hand, the construction in the proof of Theorem 4.2 of [12] is a two-step process: first, columns are adjoined to F to yield a matrix with orthogonal rows; then more columns are adjoined to result in equal-norm rows, which may then be normalized. The entire process could result in up to $\frac{(d-1)(d+2)}{2}$ additional vectors.

We illustrate the difference between the two constructions compared in Remark 3.9. If we take the frame in Example 2.3 and delete a single vector to create the frame \mathcal{F}' , the



rows of the corresponding synthesis matrix F' are no longer orthogonal, so the frame is no longer tight. Then \mathcal{F}' is a frame for \mathbb{H}^2 representing a truncated version of Figure 1 (either P_3 or C_3), and its frame operator S has two distinct eigenvalues $A < B$. (Although C_3 is a tight frame graph, both constructions may proceed for a non-tight frame representation of it.) By Theorem 3.7, only one additional vector is needed to complete \mathcal{F}' to a B -tight frame. That is to say the truncated graph occurs as an induced subgraph of some tight frame graph on four vertices. On the other hand, applying the construction method from Theorem 4.2 of [12] to \mathcal{F}' , one vector is added to make the rows of F' orthogonal and one additional vector must then be added to normalize the rows. The embedding is therefore not minimal for this frame. Note that the frame given in Example 6.10 of [12] corresponds to the same graph, and if either of the first two columns is deleted, then the construction method in Theorem 4.2 of [12] will only add a single vector; however, if either of the second two columns is deleted, then two additional vectors will be added.

For larger tight frame graphs, if a single frame vector is deleted (resulting in a non-tight frame), the embedding in Theorem 4.2 of [12] may add significantly more vectors to the frame than necessary to embed in a tight frame graph. Consider the tight frame graph $\Gamma = K_3 \square K_2$ with tight frame given in Figure 2.1 of [12]. The deletion of any single vertex necessarily results in a non-tight frame graph. Applying Theorem 4.2 of [12] adds between 3 and 5 vectors to the truncated frame while Theorem 3.7 appends only one vector. Figure 3 below shows the tight frame graph created by deleting the first vector of the tight frame in Figure 2.1 of [12] and then applying the construction techniques in Theorem 4.2 of [12] and Theorem 3.7, respectively. The highlighted vertices and edges constitute the truncated frame graph represented by the frame with synthesis matrix

$$F' = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

and frame bounds $A = 3/5$ and $B = 1$; the two algorithms construct Parseval frames with synthesis matrices

$$F_1 = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 & 1 & \sqrt{5} & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 & \frac{1}{\sqrt{5}} & 0 & \frac{2\sqrt{30}}{5} \end{bmatrix} \text{ and } F_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & 1 \end{bmatrix},$$

respectively.

Remark 3.10 If F' does not have a scaling factor of $\frac{1}{\sqrt{5}}$, the construction technique from Theorem 4.2 of [12] appends only a single vector to make the rows orthogonal and equal norm. But even without scaling, deleting any of the last three vectors from the original frame in Figure 2.1 of [12] results in three vectors being appended to achieve orthogonality, using the technique of [12].



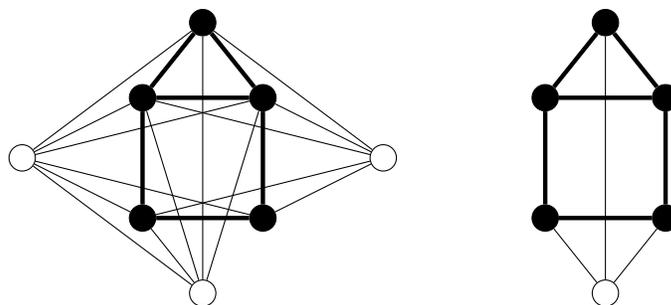


Figure 3: The graphs represented by tight frames with synthesis matrices F_1 (obtained via Theorem 4.2 of [12]) and F_2 (obtained via Theorem 3.7), respectively

4 Line Graphs

It is natural to try to construct frames for graphs using the Laplacian matrix since the resultant frame has a graph theoretic interpretation. This approach, however, requires us to consider the Laplacian of a graph and the vector representation of its line graph.

Given a graph Γ , the (*unoriented*) *incidence matrix* $B^u(\Gamma)$ is the $|V| \times |E|$ matrix with $b_{ij}^u = 1$ if vertex v_i is an endpoint of the j -th edge and 0 otherwise. We may assign an *orientation* to Γ by associating each edge $\{v_i, v_k\}$ arbitrarily with one of the ordered pairs (v_k, v_i) (in which case, we say the edge is negatively incident to v_k and positively incident to v_i) or (v_i, v_k) ; the *oriented incidence matrix* $B(\Gamma)$ has $b_{ij} = 1 = -b_{kj}$ if the j -th edge $\{v_i, v_k\}$ is associated to (v_k, v_i) .

For a connected graph Γ with n vertices, the Laplacian matrix $L = L(\Gamma)$ is a real, positive semidefinite matrix of rank $n - 1$. The eigenvalue $\lambda_0 = 0$ corresponds to the one-dimensional eigenspace spanned by the all-ones eigenvector $x_0 = \mathbf{1}$. Now, $L(\Gamma)$ can be decomposed as $L(\Gamma) = B(\Gamma)B(\Gamma)^*$, where $B = B(\Gamma)$ is an oriented incidence matrix of Γ . Since the frame operator must be full rank, the authors of [5] create an $(n - 1) \times (n - 1)$ matrix L_0 by restricting B to the $(n - 1)$ -dimensional subspace spanned by the orthonormal eigenvectors x_1, \dots, x_{n-1} of L excluding x_0 . They let $X = [x_1 \ x_2 \ \dots \ x_{n-1}]$ and consider $F = X^*B$ and $L_0 = FF^*$. Then L_0 is the (full rank) frame operator of the frame for \mathbb{H}^{n-1} with synthesis matrix F , whose spectrum consists of the nonzero eigenvalues of L ([5], Lemma 5). We will refer to a frame constructed in this manner as being constructed by the *Laplacian method*.

The only connected graph which yields a tight frame constructed by the Laplacian method is K_n , by Proposition 3.6. However, this frame will not represent K_n in the sense used in this paper; that is, the Gram matrix of the frame will not belong to $\mathcal{H}^+(K_n)$. To establish the relationship between the frame and the graph, we instead are motivated by the well-known connection between the unoriented incidence matrix of a graph and the adjacency matrix of its line graph. Given a graph P (pronounced ‘‘Rho’’), called the *root graph*, its *line graph* $\Gamma = \mathcal{L}(P)$ is the graph created by assigning a vertex in $V(\Gamma)$ to each edge in $E(P)$, so $|V(\Gamma)| = |E(P)|$, and calling two vertices in Γ adjacent if and only if their corresponding edges are incident in P . Every connected line graph, with the



exception of K_3 , determines a unique connected root graph (up to isomorphism) [21].

Proposition 4.1 *Let P be a graph with oriented incidence matrix $B = B(P)$, and let $\Gamma = \mathcal{L}(P)$. Then $BB^* = L(P)$ and $B^*B \in \mathcal{H}^+(\Gamma)$.*

Proof. As noted above, for any graph with oriented incidence matrix B , the Laplacian $L = BB^*$. Since any off-diagonal entry of B^*B is 1 or -1 if two edges share an endpoint and 0 otherwise, $B^*B \in \mathcal{H}^+(\Gamma)$, by the definition of the line graph. (Note that for the unoriented incidence matrix $B^u(P)$, $B^{u*}B^u - 2I$ is the adjacency matrix of Γ .) \square

The previous result allows us to view a given line graph Γ of order m in terms of its root graph P of order $n \leq m + 1$ (and for a root graph with many edges, n can be significantly less than m) and build a frame corresponding to Γ with rank $n - 1$.

Lemma 4.2 *Let Γ be a connected line graph with m vertices, and let $B = B(P)$ be an $n \times m$ oriented incidence matrix of its root graph P . Suppose $F = X^*B$ is a frame for \mathbb{H}^{n-1} constructed by the Laplacian method applied to the graph P . Then $F^*F \in \mathcal{H}^+(\Gamma)$.*

Proof. It is easy to see that the orthogonal projection XX^* of \mathbb{H}^n onto the eigenspace spanned by $\{x_1, x_2, \dots, x_{n-1}\}$ satisfies $XX^*f = f - \frac{1}{n}\langle f, x_0 \rangle x_0$ for any $f \in \mathbb{H}^n$ where $x_0 = \mathbf{1}$. For any columns b_i, b_j of B ,

$$\langle X^*b_i, X^*b_j \rangle = \langle b_i, XX^*b_j \rangle = \langle b_i, b_j - \frac{1}{n}\langle b_j, x_0 \rangle x_0 \rangle = \langle b_i, b_j \rangle.$$

By Proposition 4.1, $B^*B \in \mathcal{H}^+(\Gamma)$, so $F^*F \in \mathcal{H}^+(\Gamma)$. \square

The Laplacian method allows us to demonstrate the tightness of a new family of graphs, the line graphs of complete graphs. We know that $\text{msr}(\mathcal{L}(K_n)) = n - 2$ by [19]. The Laplacian method generates a tight frame for \mathbb{H}^{n-1} , and a modification of this frame allows us to obtain a tight frame for \mathbb{H}^{n-2} , corresponding to $\mathcal{L}(K_n)$.

Theorem 4.3 *For $n \geq 3$, if $P = K_n$, then $\Gamma = \mathcal{L}(P)$ is a tight frame graph for \mathbb{H}^{n-1} and \mathbb{H}^{n-2} .*

Proof. Let $P = K_n$ and $\Gamma = \mathcal{L}(P)$. Apply the Laplacian method to construct the frame $F = X^*B$ for \mathbb{H}^{n-1} , where $B = B(P)$ is an oriented incidence matrix of P . By Lemma 4.2, $F^*F \in \mathcal{H}^+(\Gamma)$. Since $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = n$ are the nonzero eigenvalues of $L = BB^*$ (see, for example, [13]),

$$L_0 = FF^* = X^*BB^*X = X^*LX = nI_{n-1}.$$

Thus F is a tight frame associated to Γ in dimension $n - 1$.

To show that $\Gamma = \mathcal{L}(K_n)$ is in fact a tight frame graph in dimension $n - 2$, we use the construction from [4] in the proof that the minimum rank of Γ is $n - 2$ (also employed in [19] to show $\text{msr}(\Gamma) = n - 2$). Let D denote an oriented incidence matrix of K_{n-1} ,



and let $C = I_{n-1} - \frac{1}{n-1}J_{n-1}$. Consider the $(n-1) \times \frac{n(n-1)}{2}$ matrix $M = [C \ D]$. Easy computations show that $C^*C = C^2 = C$ and $C^*D = D$. The matrix

$$M^*M = \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} = \begin{bmatrix} C & D \\ D^* & D^*D \end{bmatrix}$$

has nonzero entries in the top left corner of the partition, corresponding to all edges between each vertex of K_{n-1} and the n th vertex of K_n , which are all incident with one another; has nonzero entries in the top right (and bottom left) corner of the partition, corresponding to all edges of K_{n-1} incident with the edges connecting the vertices of K_{n-1} to the n th vertex of K_n ; and has nonzero entries in the bottom right corner of the partition, corresponding to all edge-incidences of K_{n-1} (by Proposition 4.1). That is, $M^*M \in \mathcal{H}^+(\Gamma)$.

The eigenvector $x_0 = \mathbf{1}$ of $L(K_{n-1})$ is also an eigenvector of C , corresponding to the same eigenvalue $\lambda_0 = 0$. For $1 \leq j \leq n-2$, set $x_j(j) = -1$, $x_j(n-1) = 1$, and $x_j(i) = 0$ for $i \neq j, n-1$; the eigenvectors x_1, x_2, \dots, x_{n-2} of $L(K_{n-1})$ are also eigenvectors of C , corresponding to $\lambda_j = n-1$ and $\lambda_j = 1$ for all j , respectively. Since

$$MM^* = CC^* + DD^* = C + L(K_{n-1}),$$

$\lambda = 0$ is an eigenvalue of multiplicity 1 of MM^* and $\lambda = n$ is an eigenvalue of multiplicity $n-2$. Let $\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-2}\}$ be a corresponding orthonormal set of eigenvectors of MM^* . The Laplacian method now can proceed exactly as it would for a Laplacian matrix: let $\tilde{X} = [\tilde{x}_1 \ \tilde{x}_2 \ \cdots \ \tilde{x}_{n-2}]$ and $\tilde{F} = \tilde{X}^*M$. This time,

$$\tilde{F}\tilde{F}^* = \tilde{X}^*MM^*\tilde{X} = nI_{n-2},$$

and \tilde{F} is a tight frame for \mathbb{H}^{n-2} . As in the proof of Lemma 4.2, $\tilde{X}\tilde{X}^*f = f - \langle f, \tilde{x}_0 \rangle \tilde{x}_0$ for any $f \in \mathbb{H}^{n-1}$. If m_j is a column of M , then it is a column of C or a column of D . In either case, it is easy to see that $\langle m_j, \tilde{x}_0 \rangle = 0$. Therefore, for any columns m_i, m_j of M ,

$$\langle \tilde{X}^*m_i, \tilde{X}^*m_j \rangle = \langle m_i, \tilde{X}\tilde{X}^*m_j \rangle = \langle m_i, m_j - \langle m_j, \tilde{x}_0 \rangle \tilde{x}_0 \rangle = \langle m_i, m_j \rangle,$$

and $M^*M \in \mathcal{H}^+(\Gamma)$ implies $\tilde{F}^*\tilde{F} \in \mathcal{H}^+(\Gamma)$. □

By Lemma 5.1 of [2], if a graph on n vertices is a tight frame graph for \mathbb{H}^2 , then it is a tight frame graph for \mathbb{H}^d for all $2 \leq d \leq \lfloor \frac{n}{2} \rfloor$. It remains an open question (see [2]) whether the implication holds for graphs Γ with $\text{msr}(\Gamma) > 2$. In particular, since $|\mathcal{L}(K_n)| = \frac{1}{2}n(n-1)$, it remains to be determined whether $\mathcal{L}(K_n)$ is a tight frame graph for \mathbb{H}^d whenever $n \leq d \leq \lfloor \frac{1}{4}n(n-1) \rfloor$. Although it is well known that the complete graph on n vertices is a tight frame graph in all possible dimensions, we provide an algorithmic proof here that utilizes a slight variant of the Laplacian method.

Theorem 4.4 *For $n \geq 2$, if $P = S_{n+1}$, then $\Gamma = \mathcal{L}(P) = K_n$ is a tight frame graph for \mathbb{H}^d for $1 \leq d \leq n-1$.*



Proof. Let $P = S_{n+1}$, $\Gamma = \mathcal{L}(P) = K_n$, and B be an $(n+1) \times n$ oriented incidence matrix of P ; then $L = BB^*$ is the Laplacian matrix of P , and $B^*B \in \mathcal{H}^+(\Gamma)$, by Proposition 4.1. To illustrate,

$$B = \begin{bmatrix} -1 & -1 & \dots & \dots & -1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} n & -1 & -1 & \dots & \dots & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Recall that L has eigenvalue $\lambda_0 = 0$ corresponding to the eigenvector $x_0 = \mathbf{1}$. It is easy to see that $\lambda_n = n+1$ is an eigenvalue with eigenvector $x_n = (n, -1, \dots, -1)^*$. The remaining eigenvalues are $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 1$ since the columns of the $(n+1) \times (n-1)$ matrix

$$\tilde{X} = \begin{bmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ \sqrt{\frac{n-1}{n}} & 0 & 0 & \dots & \dots & 0 \\ \frac{-\sqrt{\frac{n-1}{n}}}{n-1} & \sqrt{\frac{n-2}{n-1}} & \ddots & \ddots & \dots & 0 \\ \frac{-\sqrt{\frac{n-1}{n}}}{n-1} & \frac{-\sqrt{\frac{n-2}{n-1}}}{n-2} & \sqrt{\frac{n-3}{n-2}} & \ddots & 0 & 0 \\ \vdots & \frac{-\sqrt{\frac{n-2}{n-1}}}{n-2} & \frac{-\sqrt{\frac{n-3}{n-2}}}{n-3} & \ddots & 0 & 0 \\ \vdots & \vdots & \frac{-\sqrt{\frac{n-3}{n-2}}}{n-3} & \ddots & \sqrt{\frac{2}{3}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{-\sqrt{\frac{2}{3}}}{2} & \sqrt{\frac{1}{2}} \\ \frac{-\sqrt{\frac{n-1}{n}}}{n-1} & \frac{-\sqrt{\frac{n-2}{n-1}}}{n-2} & \frac{-\sqrt{\frac{n-3}{n-2}}}{n-3} & \dots & \frac{-\sqrt{\frac{2}{3}}}{2} & -\sqrt{\frac{1}{2}} \end{bmatrix}$$

form an orthonormal eigenbasis corresponding to eigenvalue $\lambda = 1$.

Let \tilde{X}_d be the matrix consisting of d columns of \tilde{X} , with any $n-1-d$ columns of \tilde{X} deleted except for the first. Then $F_d = \tilde{X}_d^* B$ is the synthesis matrix of a Parseval frame for \mathbb{H}^d since its frame operator is the identity:

$$F_d F_d^* = \tilde{X}_d^* B B^* \tilde{X}_d = \tilde{X}_d^* L \tilde{X}_d = I_d.$$

To show that the Gramian $F_d^* F_d \in \mathcal{H}^+(\Gamma)$, we claim that $\langle r_i, r_j \rangle \neq 0$ for any distinct rows r_i, r_j of \tilde{X}_d with $2 \leq i, j \leq n+1$; we may assume $i < j$. Let $J \subset \{1, 2, \dots, n-1\}$ be the subset of indices for the columns of \tilde{X}_d and let $J' = J \cap \{1, \dots, i-1\}$. If $i-1 \notin J'$, then $i \geq 3$ and

$$\langle r_i, r_j \rangle = \sum_{k \in J'} \frac{1}{(n-k)(n-k+1)} > 0.$$

If $i-1 \in J'$, then, allowing the first sum in the expressions below to be empty when $i = 2$,

$$\langle r_i, r_j \rangle = \sum_{\substack{k \in J' \\ k \neq i-1}} \frac{1}{(n-k)(n-k+1)} - \frac{1}{n-i+2} \leq \sum_{k=1}^{i-2} \frac{1}{(n-k)(n-k+1)} - \frac{1}{n-i+2} < 0,$$



where the last inequality follows from a straightforward inductive argument for $i \geq 3$. So for any columns f_i, f_j of F_d with $1 \leq i, j \leq n$ and $i \neq j$,

$$\langle f_i, f_j \rangle = \langle \tilde{X}_d^* b_i, \tilde{X}_d^* b_j \rangle = \langle -r_1 + r_{i+1}, -r_1 + r_{j+1} \rangle = \langle r_{i+1}, r_{j+1} \rangle \neq 0.$$

□

Remark 4.5 If the last $n - d - 1$ columns of \tilde{X} were deleted in the proof of Theorem 4.4, then the resulting frame would have its last vector repeated $n - d$ times, so it is better to delete alternating columns and have fewer identical vectors in the frame.

Although both line graphs in Theorems 4.3 and 4.4 are tight frame graphs, only one of the root graphs (K_n) is. The question of whether or not a line graph is a tight frame graph is independent of whether or not its root graph is a tight frame graph. We illustrate this with the next example.

Example 4.6 The table below is illustrative, not comprehensive. The reference column refers to the numbering in this paper or a citation. Let $\Gamma = \mathcal{L}(P)$.

	Γ tight			Γ not tight		
	P	Γ	Ref.	P	Γ	Ref.
P tight	K_n	$\mathcal{L}(K_n)$	4.4, 4.3	$K_2 \square K_3$	$\mathcal{L}(K_2 \square K_3)$	4.14, 4.7
	C_3, C_4	C_3, C_4	[3]			
P not tight	$S_{n+1} (n \geq 3)$	K_n	3.2, 4.4	$C_n (n \geq 5)$	C_n	3.2
	$O_n (n \geq 4)$	$\mathcal{L}(O_n)$	3.2, 4.12	$P_n (n \geq 4)$	P_{n-1}	3.2
	$K_{2,n} (n \geq 3)$	$K_2 \square K_n$	4.14			

The necessary condition from Proposition 3.2 for a frame graph to be tight can be converted to a necessary condition on the root graph of a tight frame line graph. We state this precisely in the next result, used to construct a tight frame root graph with a line graph that is not a tight frame graph in Example 4.6.

Lemma 4.7 *If $\Gamma = \mathcal{L}(P)$ and P contains an induced path on 4 vertices, then Γ is not a tight frame graph.*

Proof. If P_4 is an induced subgraph of P , then Γ has P_3 as an induced subgraph, such that this P_3 is a subgraph of no C_4 in Γ . So there are two non-adjacent vertices in $V(\Gamma)$ that have a unique common neighbor and, by Proposition 3.2, Γ cannot be a tight frame graph. □

Conversely, certain easily observed features of the line graph serve as necessary conditions for the root graph to be associated with a tight frame. In a connected graph, let *pendant vertex* denote a vertex of degree 1, and let *pendant triangle* denote a K_3 subgraph such that exactly two of the three vertices have degree 2.



Observation 4.8 *If $\Gamma = \mathcal{L}(P)$ has a pendant vertex, then P is not a tight frame graph.*

Proof. If there is a degree-one vertex in Γ , then there is an edge $\{u, v\} \in E(P)$ that is incident with exactly one other edge $\{v, w\}$ in P . Then v is the unique neighbor of the non-adjacent vertices u and w , and P is not a tight frame graph, by Proposition 3.2. \square

Observation 4.9 *If $\Gamma = \mathcal{L}(P)$ has a pendant triangle, then P is not a tight frame graph.*

Proof. The root graph of K_3 may be either K_3 or $K_{1,3}$. However, if Γ has a pendant triangle, then P must contain a claw ($K_{1,3}$) as an induced subgraph. The two degree-two vertices of the pendant triangle in Γ must correspond to the two edges of the claw that are not incident to any other edges in P . It follows that each of these two edges has an endpoint that is a degree-one vertex; that is, P contains two non-adjacent vertices with a unique neighbor (the central vertex of the claw). By Proposition 3.2, P is not a tight frame graph. \square

When the root graph has the same number of vertices as, or one more than, its corresponding line graph, that is a tight frame graph, we can give a complete characterization of the root graph. We first cite two preliminary results.

Lemma 4.10 ([12], Theorem 6.3) *Let Γ be a tight frame graph on n vertices. The graph created by duplicating one vertex in Γ (that is, introducing a new vertex adjacent to one vertex u in Γ and to all the neighbors of u) is a tight frame graph.*

Lemma 4.11 ([3], Proposition 4.7) *The graph on $n \geq 4$ vertices created by deleting a single edge of K_n is a tight frame graph.*

We only utilize the previous result in the case $n = 4$. The proof of Proposition 4.7 of [3] for that case contains an error, resulting in the construction of C_4 instead of K_4 minus an edge. However, the result remains true, by Example 2.3. Lemmas 4.10 and 4.11 imply the following:

Corollary 4.12 *Each graph in the sequence in Figure 4, obtained by iteratively applying Lemma 4.10 to the tight frame graph of Example 2.3, is a tight frame graph.*

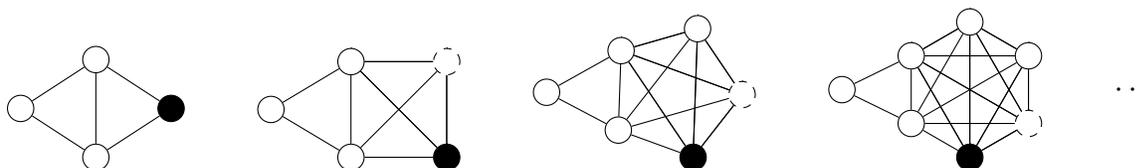


Figure 4: Duplicating a vertex

Let O_n be the graph on $n \geq 3$ vertices constructed by taking the star graph S_n and connecting two of its degree-one vertices with an edge (note $O_3 = C_3 = K_3$). Then $\mathcal{L}(O_n)$ is a tight frame graph by Corollary 4.12.





Figure 5: O_6 and $\mathcal{L}(O_6)$

Proposition 4.13 *Suppose $\Gamma = \mathcal{L}(P)$ is a tight frame graph on $n \geq 3$ vertices with connected root graph P .*

(a) *If $|V(P)| = n$, then $P \in \{C_3, C_4, O_n\}$.*

(b) *If $|V(P)| = n + 1$, then $P = S_{n+1}$.*

Proof. (a) Suppose $P \notin \{C_3, C_4, O_n\}$ has $|E(P)| = n \geq 3$ and $|V(P)| = n$. Then P has a single, induced cycle on $k \geq 3$ vertices. We show that P must contain an induced path on four vertices, from which the result follows by Lemma 4.7.

Case 1: If $k = 3$, since $P \notin \{O_n, C_3\}$, $n \geq 5$, and there must be two non-adjacent vertices adjacent to vertices on the induced cycle, such that these connecting edges are not incident with one another, or two vertices adjacent to each other, such that only one is adjacent to a vertex on the induced cycle. In either situation, P contains an induced path on four vertices.

Case 2: If $k = 4$, since $P \neq C_4$, $n \geq 5$, and there must exist some vertex not on the induced cycle adjacent to a vertex on the cycle, from which a path of length four can be induced in P .

Case 3: If $k \geq 5$, there exists an induced path of length four in P by taking the subgraph induced by any four vertices on the cycle.

(b) Now assume $|V(P)| = n + 1 \geq 4$. Then P must be a tree. If $P = P_{n+1}$, then P clearly contains an induced path of length 4, so assume there exists at least one vertex u with degree 3 or more. Suppose $P \neq S_{n+1}$. Then $|V(P)| \geq 5$, and there exists a vertex w that is not adjacent to u but that has a common neighbor v with u . Now choose any other vertex adjacent to u , say t , and induce the (unique) path of length four from t to w . Lemma 4.7 again concludes the argument. \square

The minimum number of distinct eigenvalues is well-studied for complete multipartite graphs [2]. We tie one such result to the line graph of certain complete bipartite graphs.

Proposition 4.14 *If $P = K_{2,n}$ for $n \geq 3$, then P is not a tight frame graph and $\Gamma = \mathcal{L}(P) = K_2 \square K_n$ is a tight frame graph.*

Proof. Corollary 6.5 of [3] is the first statement. Corollary 6.8 of [3] implies the second statement. As a direct demonstration, we note that

$$F = [J_n - I_n \quad J_n - (n - 1)I_n]$$

is a tight frame associated with Γ . \square



In general, $K_{m,n}$ is a tight frame graph if and only if $m = n$ ([3], Corollary 6.5). For $\mathcal{L}(K_{m,n}) = K_m \square K_n$, it is known ([9], Proposition 3.1) that $q(\mathcal{L}(K_{m,n})) \leq 3$. To the best of the authors' knowledge, however, it remains open whether or not the Cartesian product of two complete graphs is a tight frame graph. Note that every two non-adjacent vertices in $\mathcal{L}(K_{m,n})$ have exactly two common neighbors [15], so $\mathcal{L}(K_{m,n})$ satisfies the necessary condition in Proposition 3.2 for being a tight frame graph.

We conclude by noting the limitations of the line graph approach. Line graphs are characterized as those graphs that do not contain any of nine forbidden graphs as induced subgraphs [8]. For reference, we list these graphs in Figure 6, using Beineke's numbering, G1 through G9.

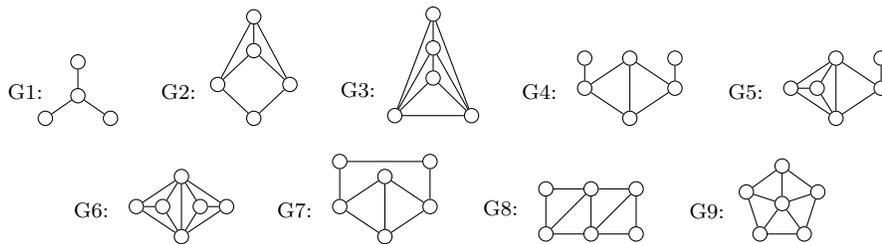


Figure 6: The nine forbidden subgraphs of [8]

Families known to be tight frame graphs are often not line graphs. For example, recall that if Γ is obtained from a complete graph ($n > 3$) by deleting a single edge, then Γ is a tight frame graph. For $n = 4$, $K_n - e$ is a line graph; however, for $n > 4$, $K_n - e$ contains G3 as an induced subgraph and is therefore not a line graph. The hypercube Q_n , defined recursively as $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$, is another example: Q_n is a tight frame graph for each $n \geq 1$ ([3], Corollary 6.9), but for $n \geq 3$, Q_n contains the claw G1 as an induced subgraph and is therefore not a line graph.

The *join* of two graphs Γ and Λ , denoted $\Gamma \vee \Lambda$, is the graph with vertex set $V(\Gamma) \cup V(\Lambda)$ and edge set $E(\Gamma) \cup E(\Lambda) \cup \{\{u, v\} : u \in V(\Gamma), v \in V(\Lambda)\}$. For two connected graphs Γ and Λ on n vertices, $\Gamma \vee \Lambda$ is a tight frame graph ([18], Theorem 5.2). Most members of this large set of examples are not line graphs.

Proposition 4.15 *If Γ and Λ are two connected graphs on $n \geq 3$ vertices, not both the complete graph, then $\Gamma \vee \Lambda$ is not a line graph.*

Proof. Suppose $\Gamma \neq K_n$ and consider three vertices $u, v, w \in \Gamma$ and three vertices $a, b, c \in \Lambda$.

Case 1: Assume the subgraph induced by $\{u, v, w\}$ is three isolated vertices. Then $\{u, v, w, a\}$ induces the claw graph G1 as a subgraph in $\Gamma \vee \Lambda$.

Case 2: Assume the subgraph induced by $\{u, v, w\}$ is the disjoint union of P_2 and an isolated vertex, where w is the isolated vertex. Due to the commutativity of the join operation, we need to consider only the cases when the subgraph induced by $\{a, b, c\}$ is the disjoint union of P_2 and an isolated vertex (c being the isolated vertex), P_3 , or C_3 . In



the first case, $\{u, v, w, b, c\}$ induces G2, in the second case, $\{u, v, a, b, c\}$ induces G3, and in the third case, $\{v, w, a, b, c\}$ induces G3 in $\Gamma \vee \Lambda$.

Case 3: Assume the subgraph induced by $\{u, v, w\}$ is P_3 . Again, the commutativity of join allows us to consider only the cases when the subgraph induced by $\{a, b, c\}$ is P_3 (with b as the degree-two vertex) or C_3 . In both cases, $\Gamma \vee \Lambda$ contains G3 as an induced subgraph, induced by $\{u, v, w, a, b\}$.

If every three vertices of Γ induce a copy of C_3 , then Γ must be the complete graph on n vertices, which contradicts our assumption. Cases 1 through 3 are therefore exhaustive and always contain one of the nine forbidden induced subgraphs; hence, $\Gamma \vee \Lambda$ cannot be a line graph. \square

Finally, we note that of the nine graphs listed in Figure 6, G2, G3, and G6 are tight frame graphs (apply Lemma 4.10 once to the tight frame graph C_4 and to the diamond graph of Example 2.3 to obtain G2 and G3, respectively, and apply twice to the diamond graph to obtain G6). Graphs G1, G4, G5, G7, and G8 fail to be tight frame graphs by Proposition 3.2. The wheel graph G9 is not a tight frame graph over symmetric real matrices ([12], Example 6.13), and, to the best of the authors' knowledge, whether it is a tight frame graph over complex matrices remains an open question. The occurrence of these nine forbidden graphs as induced subgraphs of tight frame graphs is, of course, guaranteed by Corollary 3.8.

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