The Power Mean Test

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Abstract - We devise a new test for convergence or divergence of an infinite series the Power Mean Test. We explore the strength of this test relative to that of the Ratio and Root Tests and provide a family of series where the Power Mean Test is the most useful of the three tests.

Keywords : convergence; divergence; series; power mean

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1 Introduction

The question of convergence of an infinite series has long been pondered by many mathematicians. Several tests have been devised to determine convergence such as the Ratio Test and the Root Test. In this paper, we construct a new test which can be useful for some series. Suppose we have the series $\sum_{i=0}^{\infty} a_i$, $a_i \neq 0$.

For p real, we recall the *power mean* M_p on sets of positive real numbers to be

$$\left(\frac{1}{n}\sum_{i=0}^{n-1}(a_i)^p\right)^{\frac{1}{p}},$$

where n is the number of elements in the set. Also recall the following useful properties of power means: M_1 is the arithmetic mean, and $M_0 = \lim_{p\to 0} M_p$ is exactly the geometric mean. Additionally, the power mean is increasing in p by the generalized mean inequality, i.e., if $p < q$, then $M_p(x_1, \ldots, x_n) \leq M_q(x_1, \ldots, x_n)$, with equality occurring only when $x_1 = \ldots = x_n$. These facts are proven in pages 202 – 215 of [2].

Using this, we will find a convergence test that is stronger than the ratio test. While not stronger than the root test, for some series, it is simpler to apply.

We define the partial power mean of the series $\sum_{i=0}^{\infty} a_i$ as $M_p^n\left(\sum_{i=0}^{\infty} a_i\right) = M_p\left(\left|\frac{a_1}{a_0}\right|, \left|\frac{a_2}{a_1}\right|, \ldots, \left|\frac{a_n}{a_{n-1}}\right|\right)$ which we will refer generally to as M_p^n. This is the power mean of the first n *consecutive ratios* of the series a_i.

Cruz-Uribe considers this approach in his paper [1], specifically with $p = 1$. He shows that $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}} \cdots \frac{a_1}{a_0}\right|^{1/n}$, meaning that the root test could also be calculated by finding the geometric mean of the consecutive ratios of the series. He then goes on to say that by the arithmetic geometric mean inequality, if the arithmetic mean of the consecutive ratios is less than one, the series will converge. He denotes this test the *Arithmetic Mean Test*.

This brings us to our first theorem, which is a generalization of the Arithmetic Mean Test.

2 Results

Theorem 2.1 (Power Mean Test) *Let $s < 0 < t$. If*

$$\liminf_{n\to\infty} M_s^n \left(\sum_{i=0}^{\infty} a_i\right) > 1,$$

the series $\sum_{i=0}^{\infty} a_i$ diverges, and if

$$\limsup_{n\to\infty} M_t^n \left(\sum_{i=0}^{\infty} a_i\right) < 1,$$

the series $\sum_{i=0}^{\infty} a_i$ converges.

Proof. $M_s^n \leq M_0^n \leq M_t^n$ by the generalized mean inequality. Take the appropriate limit, and recall that $\lim_{n\to\infty} M_0^n$ is exactly the root test. □

Corollary 2.2 *If $\lim_{n\to\infty} M_p^n \left(\sum_{i=0}^{\infty} a_i\right) = L$, then the series $\sum_{i=0}^{\infty} a_i$ converges if $L < 1$ and $p > 0$, and diverges if $L > 1$ and $p < 0$.*

Proof. This follows easily from Theorem 2.1 and definitions of lim sup and lim inf. □

The strength of this test exists through the comparison between the power mean and the geometric mean. However, the power mean is increasing in p. Thus this *Power Mean Test* is a weaker version of Root Test since the geometric mean of a sequence could be less than 1, while a power mean on that sequence could be greater than 1. Because of this, it may seem as though the usefulness of the Power Mean Test decreases dramatically with large p, however, this is not the case. As we go on to explore its connection with the Ratio Test, we will find that the Power Mean Test is actually stronger than the Ratio Test even with large p.

Theorem 2.3 *Let $s < 0 < t$.*

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty} M_s^n \left(\sum_{i=0}^{\infty} a_i\right)$$

$$\limsup_{n\to\infty} M_t^n \left(\sum_{i=0}^{\infty} a_i\right) \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Proof. For the sake of brevity, we will only show the proof of the second inequality, as the first follows a very similar argument.

Set $L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L = +\infty$, there is nothing to prove. If L is finite, then let $\varepsilon > 0$. There must be some $N > 0$ such that for all $n > N$, $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon/2$. There must also be some $N' > N$ such that for all $n > N'$, $M_t\left(\left|\frac{a_1}{a_0}\right|, \ldots, \left|\frac{a_{N+1}}{a_N}\right|, L+\varepsilon/2, L+\varepsilon/2, \ldots, L+\varepsilon/2\right)$ is the power mean of n items and is less than $(L+\varepsilon/2)+\varepsilon/2$. Now, for all $n > N'$, $M_t\left(\left|\frac{a_1}{a_0}\right|, \ldots, \left|\frac{a_{n+1}}{a_n}\right|\right) \le M_t\left(\left|\frac{a_1}{a_0}\right|, \ldots, \left|\frac{a_{N+1}}{a_N}\right|, L+\varepsilon/2, L+\varepsilon/2, \ldots, L+\varepsilon/2\right) < L+\varepsilon$. Hence $\limsup_{n\to\infty} M_t\left(\left|\frac{a_1}{a_0}\right|, \ldots, \left|\frac{a_{n+1}}{a_n}\right|\right) < L+\varepsilon$. Since this is true for any $\varepsilon > 0$, in fact $\limsup_{n\to\infty} M_t^n \le L$. □

This means that if the Power Mean Test is inconclusive, the Ratio Test will be as well. And if the Ratio Test shows either convergence or divergence, the Power Mean Test will give the same result.

Example 2.4

One may wonder in what instance the Power Mean Test would be preferable to use. In some cases, the Ratio Test will be inconclusive, and the Power Mean Test will be easier to apply than the Root Test. Take the following series for example.

$$a_0 = 1, \qquad a_k = a_{k-1}\left(\frac{4}{3}\left|\sin\left(\frac{\pi}{6}+\frac{k\pi}{2}\right)\right| + \frac{1}{k^2+5k+6}\right)^r$$

where r is a positive real constant. Let $T_k = \left|\dfrac{a_k}{a_{k-1}}\right|$ be the kth ratio of consecutive terms in a_k. We find that $\limsup_{k\to\infty} T_k = \left(\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\right)^r = \left(\frac{2}{\sqrt{3}}\right)^r > 1$ and $\liminf_{k\to\infty} T_k = \left(\frac{4}{3}\cdot\frac{\sqrt{1}}{2}\right)^r = \left(\frac{2}{3}\right)^r < 1$. Therefore the Ratio Test is indeterminate. We could apply the Root Test by finding the geometric mean of consecutive ratios, however due to the rational term, an easier option would be to use the Power Mean Test with $p = 1/r$. This yields

$$\limsup_{n\to\infty} M_{1/r}^n\left(\sum_{k=0}^{\infty} a_k\right) = \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(T_k)^{1/r}\right)^r =$$

$$\limsup_{n\to\infty}\left(\frac{1}{2n}\sum_{k=1}^{2n}\left(\left(\frac{4}{3}\left|\sin\left(\frac{\pi}{6}+\frac{k\pi}{2}\right)\right| + \frac{1}{k+2} - \frac{1}{k+3}\right)^r\right)^{1/r}\right)^r =$$

$$\limsup_{n\to\infty}\left(\frac{1}{2n}\left(\frac{2}{\sqrt{3}}n + \frac{2}{3}n + \frac{1}{3} - \frac{1}{2n+3}\right)\right)^r =$$

$$\limsup_{n\to\infty}\left(\frac{1}{2}\left(\frac{2\sqrt{3}+2}{3} + \frac{1}{3n} - \frac{1}{2n^2+3n}\right)\right)^r = \left(\frac{\sqrt{3}+1}{3}\right)^r < 1.$$

Therefore the series converges by the Power Mean Test. Notice we could have done a similar calculation with an $r < 0$, except the Power Mean Test would show divergence.

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References

[1] D. Cruz-Uribe, The Relation Between the Root and Ratio Tests, *Math. Mag.*, **70** (1997), 214 – 215.

[2] P.S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, 2003.

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