Sequences, q-multinomial Identities, Integer Partitions with Kinds, and Generalized Galois Numbers

A. Avalos and M. Bly

Abstract - Using sequences of finite length with positive integer elements and the inversion statistic on such sequences, a collection of binomial and multinomial identities are extended to their q-analog form via combinatorial proofs. Using the major index statistic on sequences, a connection between integer partitions with kinds and finite differences of the coefficients of generalized Galois numbers is established.

Keywords : q-analog; inversion statistic; generating functions; multinomial identities; major index statistic; integer partitions with kinds; generalized Galois numbers

Mathematics Subject Classification (2010) **:** 05A05; 05A15; 05A17; 05A19.

1 Introduction

Broadly speaking, the primary topic of study within this paper is q-analogs. Introduced in the 19th century [16], q-analogs are found within many areas of mathematics and related fields, including hypergeometric series [4, 11, 22, 24], elliptic integrals [18, 29], complex nonlinear dynamics [1, 25], quantum calculus [9, 17], and string theory [6, 20]. Given an expression, a q-analog is simply a corresponding expression parameterized by q such that the limit as q approaches 1 yields the original expression. A specific example that is central to this paper is the q-multinomial coefficient, which is a particular q-analog of multinomial coefficients. Stated explicitly in binomial form

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k}.$$

These expressions can occur quite naturally: when considering the k-dimensional subspaces of an n-dimensional vector space over a finite field with q elements, the number of such subspaces is the q-binomial coefficient $\binom{n}{k}_q$.

Our approach utilizes sequences of finite length with positive integer entries to provide an accessible treatment of q-multinomial coefficients. We will focus on the enumerative combinatorics of such sequences using a pair of elementary discrete statistics: the inversion statistic in Sections 1-2 and the major index statistic in Section 3. This viewpoint yields an intuitive understanding of q-multinomial coefficients to facilitate smooth navigation of the results of this paper.

Two additional key elements in this paper are Galois numbers and integer partitions. The Galois numbers, originally introduced by Goldman and Rota [12], are obtained from the q-multinomial coefficients. Imagine Pascal's q-triangle (see Figure 1), a triangular array containing the expressions $\binom{n}{k}_q$ rather than the conventional numbers $\binom{n}{k}$. The n^{th} Galois number is simply the sum of the expressions in the n^{th} row of Pascal's q-Triangle. On the other hand, an integer partition is a finite series of nonincreasing positive integers. They are pervasive in combinatorics [13, 14, 19, 26] and also arise in the study of: number theory [8, 27, 30], symmetric polynomials [3, 28], and group representation theory [2, 5].

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0}_q & & & \\
 & & \binom{1}{0}_q & & \binom{1}{1}_q & & \\
 & \binom{2}{0}_q & & \binom{2}{1}_q & & \binom{2}{2}_q & \\
\binom{3}{0}_q & & \binom{3}{1}_q & & \binom{3}{2}_q & & \binom{3}{3}_q
\end{array}
$$

Figure 1: Rows 0 through 3 of Pascal's q-Triangle

Throughout this paper, we will refer to the set $\{1, 2, \ldots, m\}$ as $[m]$ and the set sequences of length n whose elements include k_1 1's, …, k_m m's from the set $[m]$ as $\mathcal{S}_n^m(k_1, \ldots, k_m)$. In Section 1, we will establish some essential definitions and ideas. In Section 2, we will use the inversion statistic to concisely develop a collection of classical and other less so classical q-multinomial identities. In Section 3, we will introduce the major index statistic and use it to demonstrate a connection between generalizations of integer partitions and Galois numbers.

1.1 Inversion Statistic

To begin, we will introduce the well-known inversion statistic and a couple observations relevant to results in this paper. A general treatment of the inversion statistic from the literature can be found in [31].

Definition 1.1 *Let n, m be nonnegative integers, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\operatorname{inv}(\sigma) := |\{(a, b) \mid a < b \text{ and } \sigma_a > \sigma_b\}|.$$

*If a particular σ_a is fixed, ordered pairs of the form (a, b) that are accounted for by $\operatorname{inv}(\sigma)$ shall be referred to as the **inversions induced by $\boldsymbol{\sigma_a}$** or simply $\mathrm{i}(\sigma_a)$. Should a particular σ_b be fixed, ordered pairs of the form (a, b) that are accounted for by $\operatorname{inv}(\sigma)$ shall be referred to as the **inversions received by $\boldsymbol{\sigma_b}$** or simply $\mathrm{r}(\sigma_b)$.*

Figure 2 contains some examples.

2211 $\text{inv}(\sigma) = 4$	2121 $\text{inv}(\sigma) = 3$	2112 $\text{inv}(\sigma) = 2$
1221 $\text{inv}(\sigma) = 2$	1212 $\text{inv}(\sigma) = 1$	1122 $\text{inv}(\sigma) = 0$

Figure 2: All sequences of length 4 with two 2s and two 1s.

Proposition 1.2 *Let n, m be nonnegative integers, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\text{inv}(\sigma) = \sum_{a \in [n]} \mathrm{i}(\sigma_a) = \sum_{b \in [n]} \mathrm{r}(\sigma_b) .$$

Proof. Observe the unions expressed below are disjoint.

$$\{ (a,b) \mid a < b \} = \bigcup_{a \in [n]} \{ (a,b) \mid a < b \} = \bigcup_{b \in [n]} \{ (a,b) \mid a < b \} .$$

The result follows from the above statement of equality, and the definitions of: inversions, induced inversions, and received inversions. □

Corollary 1.3 *Let n, m be nonnegative integers, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\text{inv}(\sigma) = \sum_{\sigma_a \geq 2} \mathrm{i}(\sigma_a) = \sum_{\sigma_b \leq m-1} \mathrm{r}(\sigma_b) .$$

Proof. When σ_a is equal to 1 the value of $\mathrm{i}(\sigma_a)$ equals zero. Similarly, when σ_b is equal to m the value of $\mathrm{r}(\sigma_b)$ equals zero. □

1.2 q-binomial and q-multinomial Coefficients

We will now introduce a formal definition of the q-binomial coefficient. The following definition, though somewhat unconventional, is inspired by [21]. It will lead to an intuitive interpretation in terms of the inversion statistic, allowing for a smooth transition to their multinomial counterparts.

Definition 1.4 *Let n, k be nonnegative integers such that $n \geq k$, and let q be an indeterminate. Then*

$$\binom{n}{k}_q := \sum_{\substack{E \subseteq [n] \\ |E| = k}} q^{\left(\sum_{i=1}^{k} (n - e_i) - (k - i) \right)} .$$

where $E = \{ e_1, \ldots, e_k \}$ with $e_i < e_{i+1}$ for every $1 \leq i \leq k - 1$.

Noting that the number of subsets of $[n]$ of cardinality k is exactly $\binom{n}{k}$, we can see that letting $q \to 1$ yields the corresponding standard binomial coefficient.

Figure 3 contains some examples. Observe the parallelism between Figures 2 and 3.

$\{1,2\}$ q^4	$\{1,3\}$ q^3	$\{1,4\}$ q^2
$\{2,3\}$ q^2	$\{2,4\}$ q^1	$\{3,4\}$ q^0

Figure 3: The sets associated with the terms of $\binom{4}{2}_q = q^4 + q^3 + 2q^2 + q + 1$.

Proposition 1.5 *If n, k are nonnegative integers such that $n \geq k$ and q is an indeterminate, then*

$$\binom{n}{k}_q = \sum_{\sigma \in \mathcal{S}_n^2(k,n-k)} q^{\mathrm{inv}(\sigma)}.$$

Proof. Let $E \subseteq [n]$ be of cardinality k, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be the sequence in $\mathcal{S}_n^2(k, n-k)$ such that σ_a is 2 precisely when $a \in E$. Fix some $a \in E$ and consider σ_a. The ordered pairs (a, b) accounted for by $\mathrm{inv}(\sigma)$ correspond to indices b such that σ_b is 1 and $b > a$. Notice that $n - e_i$ equals $n - a$ and counts the number of indices j such that $j > a$. Also notice that $k - i$ counts the numbers of elements σ_j such that $j > a$ and σ_j is 2. Hence, $(n - e_i) - (k - i)$ counts all ordered pairs (a, b) of interest. The result follows from observing that every $\sigma \in \mathcal{S}_n^2(k, n-k)$ can be attained similarly by some $E \subseteq [n]$. $\square$

In other words, the polynomial $\binom{n}{k}_q$ is the generating function for the statistic of inversions on the set $\mathcal{S}_n^2(k, n-k)$, a standard result which can be found in [31]. The following definition and proposition provides the well-known q-multinomial generalization.

Definition 1.6 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then*

$$\binom{n}{k_1, \ldots, k_m}_q := \binom{n}{k_m}_q \binom{n-k_m}{k_{m-1}}_q \cdots \binom{n-k_m-\cdots-k_2}{k_1}_q.$$

Proposition 1.7 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then*

$$\binom{n}{k_1, \ldots, k_m}_q = \sum_{\sigma \in \mathcal{S}_n^m(k_1, \ldots, k_m)} q^{\mathrm{inv}(\sigma)}.$$

Proof. Fix a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathcal{S}_n^m(k_1, \ldots, k_m)$. Note that the inversions induced by all σ_a for which σ_a equals m correspond to ordered pairs (a, b) such that σ_b is less than m. By Proposition 1.5, it follows that $\binom{n}{k_m}_q$ corresponds precisely to inversions induced by all σ_a equal to m.

Further observe that inversions induced by all σ_a for which σ_a equals $m-1$ correspond to ordered pairs (a,b) such that σ_b is less than $m-1$. In particular, no such (a,b) will correspond to a σ_b equal to m. As such, Proposition 1.5 applies to the subsequence of σ containing the $n-k_m$ elements of σ that do not equal m, and it follows that $\binom{n-k_m}{k_{m-1}}_q$ corresponds precisely to inversions induced by all σ_a equal to $m-1$.

A similar argument holds for the remaining elements of σ. □

1.3 Fundamental Sequences

We will introduce an additional definition, inspired by [35] and named after P.A. MacMahon [23], that will be especially helpful in establishing the results of Section 3.

Definition 1.8 *Let m,n be nonnegative integers. Define the* ***fundamental Mahonian set $\mathcal{F}_n^m$*** *to be*

$$\mathcal{F}_n^m := \left\{ (F_1, \ldots, F_m) \;\middle|\; |F_1| + \cdots + |F_m| = n \ \text{ and } \ \forall\, x \in F_{j+1}\,,\ x \in [K_j] \right\},$$

where each F_i is a (possibly empty) multiset of nonnegative integers and K_j is equal to $|F_1| + \cdots + |F_j|$. Elements within $\mathcal{F}_n^m$ shall be referred to as ***fundamental sequences****.*

Figure 4 contains some examples. Observe the parallelism between Figures 2 and 4.

$$(\{0,0\}, \{2,2\}) \qquad (\{0,0\}, \{2,1\}) \qquad (\{0,0\}, \{2,0\})$$

$$(\{0,0\}, \{1,1\}) \qquad (\{0,0\}, \{1,0\}) \qquad (\{0,0\}, \{0,0\})$$

Figure 4: Fundamental sequences in $\mathcal{F}_4^2$ for which $|F_2| = 2$ and $|F_1| = 2$.

Proposition 1.9 *If m,n are nonnegative integers, then*

$$|\,\mathcal{S}_n^m\,| = |\,\mathcal{F}_n^m\,|\,.$$

Proof. Define the function $\varphi\colon \mathcal{S}_n^m \to \mathcal{F}_n^m$ by the assignment $\sigma \mapsto (F_1, \ldots, F_m)$, where each F_j is the (possibly empty) multiset $\{\, \mathrm{i}\,(\sigma_a) \mid a \in [n] \ \text{ and } \ \sigma_a = j \,\}$.

We will first show that φ maps $\mathcal{S}_n^m$ into $\mathcal{F}_n^m$. Since each σ in $\mathcal{S}_n^m$ contains n elements, it follows that $|F_1| + \cdots + |F_m|$ is equal to n. Also observe that if σ_a is equal to j, then the value of $\mathrm{i}(\sigma_a)$ is at most $|\{\, b \in [n] \mid \sigma_b < \sigma_a \,\}|$.

We will now show injectivity. Assume σ^1, σ^2 are sequences in $\mathcal{S}_n^m$ such that $\varphi(\sigma^1)$ and $\varphi(\sigma^2)$ are both equal to $(F_1, \ldots, F_m)$. Observe that the elements of F_m force the set $\{\, a \in [n] \mid \sigma_a^i = m \,\}$ to be equal for $i = 1,2$. Subsequently observe that the elements of F_{m-1} force the sets $\{\, a \in [n] \mid \sigma_a^i = m-1 \,\}$ to be equal for $i = 1,2$, and so on. Hence, φ is injective.

To show surjectivity, fix a fundamental sequence $(F_1, \ldots, F_m)$ in $\mathcal{F}_n^m$. We will construct a sequence σ in $\mathcal{S}_n^m$ that maps to $(F_1, \ldots, F_m)$ via φ. Let $f_1^m \le \ldots \le f_{|F_m|}^m$

be the elements of F_m, and let σ^0 be the sequence in $\mathcal{S}_n^m$ containing only 1's. Find an $a \in [n]$ such that the cardinality of $\{\, b \in [n] \mid b > a \text{ and } \sigma_b^0 = 1 \,\}$ equals f_1^m. Note, such an a exists because of the restriction on the elements of F_m in the definition of $\mathcal{F}_n^m$. Replace σ_a^0 with m, and call this new sequence σ^1. Next, find an $a \in [n]$ such that σ_a^1 is 1 and the cardinality of $\{\, b \in [n] \mid b > a \text{ and } \sigma_b^1 = 1 \,\}$ equals f_2^m. Replace σ_a^1 with m, and call this new sequence σ^2. Continuing similarly for the remaining elements of F_m and for the elements of $F_{m-1}, \ldots, F_2$ respectively, observe that the multiset $\{\, \mathrm{i}(\sigma_a^{n-|F_1|}) \mid a \in [n] \text{ and } \sigma_a^{n-|F_1|} = j \,\}$ is equal to F_j for all j in $[m]$. As such, $\sigma^{n-|F_1|}$ is our desired σ. □

The following definition and corollary can help us clarify the result of Proposition 1.9.

Definition 1.10 *If m, n, k are nonnegative integers, define $\boldsymbol{\mathcal{F}_n^m(k)}$ in the following manner:*

$$\mathcal{F}_n^m(k) := \left\{ (F_1, \ldots, F_m) \in \mathcal{F}_n^m \;\middle|\; \sum_{i=1}^{m} \sum_{x \in F_i} x = k \right\}.$$

Figure 5 contains some examples.

$(\{0,0\},\{2\},\emptyset)$ $(\{0,0\},\emptyset,\{2\})$ $(\{0\},\{1,1\},\emptyset)$ $(\{0\},\{1\},\{1\})$ $(\{0\},\emptyset,\{1,1\})$

Figure 5: Fundamental sequences in $\mathcal{F}_3^3(2)$ for which $0 \notin F_2$ and $0 \notin F_3$.

Corollary 1.11 *If m, n, k are nonnegative integers, then*

$$|\,\mathcal{F}_n^m(k)\,| = |\,\{\, \sigma \in \mathcal{S}_n^m \mid \mathrm{inv}(\sigma) = k \,\}\,|.$$

Proof. Restrict φ from Proposition 1.9 so its domain is $\{\, \sigma \in \mathcal{S}_n^m \mid \mathrm{inv}(\sigma) = k \,\}$. □

1.4 Integer Partitions with Kinds

Rather than using the conventional definition of integer partitions as finite series of non-increasing positive integers, we have chosen a logically-equivalent and widely-utilized alternative which generalizes nicely to introduce integer partitions with kinds [15].

Definition 1.12 *Let k be a nonnegative integer. An **integer partition of k** is a multiset of positive integers whose elements add to k. Let m also be a nonnegative integer. An **integer partition of k with m kinds** is*

$$(P_1, \ldots, P_m) \quad \text{such that} \quad \sum_{i=1}^{m} \sum_{x \in P_i} x = k,$$

where each P_i is a (possibly empty) multiset of positive integers. The set of all integer partitions of k with m kinds shall be denoted $\boldsymbol{\mathcal{P}_k^m}$.

$$(\{2\},\emptyset) \qquad (\emptyset,\{2\}) \qquad (\{1,1\},\emptyset) \qquad (\{1\},\{1\}) \qquad (\emptyset,\{1,1\})$$

Figure 6: The integer partitions of 2 with 2 kinds.

Figure 6 contains some examples. Observe the parallelism between Figures 5 and 6.

Proposition 1.13 *Let m, n be positive integers and k be a nonnegative integer such that $k < n$. Then*

$$|\mathcal{P}_k^m| = \left|\{(F_1,\ldots,F_{m+1}) \in \mathcal{F}_n^{m+1}(k) \mid 0 \notin F_2, \ldots, 0 \notin F_{m+1}\}\right|.$$

Proof. Define $\varphi\colon \mathcal{P}_k^m \to \{(F_1,\ldots,F_{m+1}) \in \mathcal{F}_n^{m+1}(k) \mid 0 \notin F_2, \ldots, 0 \notin F_{m+1}\}$ via

$$(P_1,\ldots,P_m) \mapsto (F_1, P_1, \ldots, P_m),$$

where F_1 is a multiset of cardinality $n - |P_1| - \cdots - |P_m|$ containing only zeros. Note that since each P_i contains only positive integers, the value of $|P_1| + \cdots + |P_m|$ is at most k and $|F_1|$ is positive.

We will now show that φ maps into the codomain. Observe that: the elements of $(F_1, P_1, \ldots, P_m)$ are $m+1$ multisets; the sum $|F_1|+|P_1|+\cdots+|P_m|$ equals n; the elements of $P_1,\ldots,P_m$ are nonzero; and the elements of $F_1, P_1, \ldots, P_m$ add to k. It remains to show that $(F_1, P_1, \ldots, P_m)$ satisfies the restriction on elements from the definition of $\mathcal{F}_n^{m+1}(k)$. Let p equal $|P_1| + \cdots + |P_m|$. Since $P_1 \cup \cdots \cup P_m$ contains p positive integers that sum to k, observe any element of P_i could be at most $k-(p-1)$ in value. Since k is less than n, it follows that

$$k-(p-1) = (k+1)-p \leq n-p.$$

As such, every element in $P_1 \cup \cdots \cup P_m$ is in $[K_1]$, where K_1 is equal to $|F_1|$.

Finally, observe that bijectivity of φ follows naturally from its rule of assignment. ☐

It may be worth reflecting on the set $\{\sigma \in \mathcal{S}_n^{m+1} \mid \mathrm{inv}(\sigma) = k \text{ and } \sigma_n = 1\}$ within the context of Propositions 1.9 and 1.13, respectively.

2 q-multinomial Identities

This section will focus on developing q-analogs of both classical and lesser-known multinomial identities. Our motivation is to provide a uniform treatment of these q-analogs utilizing the intuitive nature of the inversion statistic. It is the viewpoint of the authors that this approach is not only concise but also enables a deep understanding.

2.1 Symmetry

We will begin with the well-known q-analog to binomial symmetry, namely $\binom{n}{k} = \binom{n}{n-k}$.

Proposition 2.1 *If n, k are nonnegative integers such that $n \geq k$, then*

$$\binom{n}{k}_q = \binom{n}{n-k}_q .$$

Proof. Let $\mathcal{S}_n^2(k, n-k)$ be the set of sequences of length n whose elements are in $[2]$ with k 2's, and refer to an arbitrary sequence in $\mathcal{S}_n^2(k, n-k)$ by $\sigma = (\sigma_1, \ldots, \sigma_n)$. For every $x \in [2]$, say that $\overline{x}$ equals 1 when x is 2 and $\overline{x}$ equals 2 when x is 1 . Define a map

$$\varphi \colon \mathcal{S}_n^2(k, n-k) \to \mathcal{S}_n^2(n-k, k) \ \textit{by} \ (\sigma_1, \ldots, \sigma_n) \mapsto (\overline{\sigma_n}, \ldots, \overline{\sigma_1}) .$$

Fix some $a \in [n]$ and consider σ_a . If σ_a is 2 and $\mathrm{i}(\sigma_a)$ is c , then the number of 1's that follow σ_a in σ must be c . By the definition of φ , notice the number of 2's preceding $\overline{\sigma_a}$ in $\varphi(\sigma)$ is also c . Hence, the numbers $\mathrm{i}(\sigma_a)$ and $\mathrm{r}(\overline{\sigma_a})$ are equal. Should σ_a be 1 , observe that $\mathrm{i}(\sigma_a)$ and $\mathrm{r}(\overline{\sigma_a})$ are both zero. Further observing that φ is bijective, the desired result follows from Proposition 1.2. □

We will now generalize to the well-known q-analog of multinomial symmetry.

Proposition 2.2 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$ and π is a permutation of $[m]$, then*

$$\binom{n}{k_1, \ldots, k_m}_q = \binom{n}{k_{\pi(1)}, \ldots, k_{\pi(m)}}_q .$$

Proof. Refer to an arbitrary sequence in $\mathcal{S}_n^m(k_1, \ldots, k_m)$ by $\sigma = (\sigma_1, \ldots, \sigma_n)$, and define a map

$$\theta \colon \mathcal{S}_n^m(k_1, \ldots, k_i, k_{i+1}, \ldots, k_m) \to \mathcal{S}_n^m(k_1, \ldots, k_{i+1}, k_i, \ldots, k_m)$$

such that $\theta(\sigma)_a$ equals σ_a when σ_a is neither i nor $i+1$. It follows that

$$\sum_{\sigma_a > i+1} \mathrm{i}(\sigma_a) = \sum_{\theta(\sigma)_a > i+1} \mathrm{i}\left(\theta(\sigma)_a\right) , \qquad \sum_{\sigma_a < i} \mathrm{i}(\sigma_a) = \sum_{\theta(\sigma)_a < i} \mathrm{i}\left(\theta(\sigma)_a\right) .$$

For the subsequence of σ for which σ_a is equal to i or $i+1$, let θ act on that subsequence analogously to φ in Proposition 2.1. It follows that

$$\sum_{\sigma_a \in \{i, i+1\}} \mathrm{i}(\sigma_a) = \sum_{\theta(\sigma)_a \in \{i, i+1\}} \mathrm{i}\left(\theta(\sigma)_a\right) .$$

By Proposition 1.2, we have that $\mathrm{inv}(\sigma)$ equals $\mathrm{inv}(\theta(\sigma))$.

Observe that this Proposition has been shown for π that are of the form of a simple transposition. Given that any permutation is a composition of simple transpositions, we have our desired result for any permutation π . □

2.2 Pascal's Identity

We will now consider Pascal's Identity in order to develop a well-known q-analog,

$$\binom{n}{k_1,\ldots,k_m} = \binom{n-1}{k_1-1,\ldots,k_m} + \binom{n-1}{k_1,k_2-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,\ldots,k_m-1}.$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $\mathcal{S}_n^m(k_1,\ldots,k_m)$, then $\binom{n-1}{k_1-1,\ldots,k_m}$ counts such sequences that end in a 1, $\binom{n-1}{k_1,k_2-1,\ldots,k_m}$ counts such sequences that end in a 2, and so on.

Proposition 2.3 *If $m,n,k_1,\ldots,k_m$ are nonnegative integers such that $k_1+\cdots+k_m=n$, then $\binom{n}{k_1,\ldots,k_m}_q$ is equal to*

$$q^{k_2+\cdots+k_m}\binom{n-1}{k_1-1,\ldots,k_m}_q + q^{k_3+\cdots+k_m}\binom{n-1}{k_1,k_2-1,\ldots,k_m}_q + \cdots + \binom{n-1}{k_1,\ldots,k_m-1}_q.$$

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $\mathcal{S}_n^m(k_1,\ldots,k_m)$. For such sequences that end in a 1, note that $k_2+\cdots+k_m$ inversions will be received by that 1. Thus, the product of $q^{k_2+\cdots+k_m}$ and $\binom{n-1}{k_1-1,\ldots,k_m}_q$ accounts precisely for the inversions of sequences that end in a 1. The argument is similar for the remaining terms of our desired sum. □

Note that applying Proposition 2.2 to Proposition 2.3 yields $m!$ different articulations of the q-analog to Pascal's Identity. For the case $m=2$, Figure 7 contains the resulting $2!$ articulations.

$$q^{k_2}\binom{n-1}{k_1-1,k_2}_q + \binom{n-1}{k_1,k_2-1}_q \qquad\qquad \binom{n-1}{k_1-1,k_2}_q + q^{k_1}\binom{n-1}{k_1,k_2-1}_q$$

Figure 7: The two articulations of $\binom{n}{k_1,k_2}_q$ via the q-analog of Pascal's Identity.

2.3 A Useful Lemma

We will now introduce a lemma that will supply a framework of thinking to address a number of the Propositions that follow within this section.

Lemma 2.4 *Let m,n,t be nonnegative integers; let $p_1,\ldots,p_t$ be nonnegative integers such that $p_1+\cdots+p_t=n$; for all (i,j) in $[t]\times[m]$, let $\ell_{i,j}$ be nonnegative integers such that $\ell_{i,1}+\cdots+\ell_{i,m}=p_i$; for all i in $[t]$, let d_i equal $p_1+\cdots+p_i$; for any σ in $\mathcal{S}_n^m$ and for all $u\in[t]$, let $s_u(\sigma)$ be $\left(\sigma_{d_{u-1}+1},\ldots,\sigma_{d_u}\right)$; and let*

$$T \coloneqq \left\{\, \sigma\in\mathcal{S}_n^m \;\middle|\; s_u(\sigma)\in S_{p_u}^m(\ell_{u,1},\ldots,\ell_{u,m}),\ \forall\, u\in[t] \,\right\}.$$

Then,

$$\sum_{\sigma \in T} q^{\mathrm{inv}(\sigma)} \;=\; q^{\left(\sum_{u=1}^{t}\sum_{v=2}^{m} \ell_{u,v}\left(\sum_{i=u+1}^{t}\sum_{j=1}^{v-1} \ell_{i,j}\right)\right)} \binom{p_1}{\ell_{1,1},\ldots,\ell_{1,m}}_q \cdots \binom{p_t}{\ell_{t,1},\ldots,\ell_{t,m}}_q .$$

Proof. Let X_1 be equal to $[d_1]$; for all $2 \le u \le t$, let X_u be equal to $[d_u] \setminus [d_{u-1}]$; and interpret the left-hand side of the equality in the lemma statement as the generating function for inversions on T.

Observe that for a sequence σ in T, ordered pairs (a,b) associated with $\mathrm{inv}(\sigma)$ are of exactly one of the following forms: a,b are both in the same X_u; and a,b are not both in the same X_u. Note that for all u in $[t]$, $\binom{p_u}{\ell_{u,1},\ldots,\ell_{u,m}}_q$ accounts for ordered pairs (a,b) associated with inversions such that a,b are both in X_u. It remains to consider ordered pairs (a,b) associated with inversions such that a,b are not both in the same X_u.

Fix some u in $[t]$ and some v in $[m]\setminus\{1\}$. Observe that $\ell_{u,v}$ is the number of elements σ_a such that a is in X_u and σ_a equals v. Furthermore, $\sum\sum \ell_{i,j}$ accounts for the number of elements σ_b such that b is greater than d_u and σ_b is less than v. Hence, $\ell_{u,v}\left(\sum\sum \ell_{i,j}\right)$ accounts for ordered pairs (a,b) associated with inversions where a is in X_u, σ_a equals v, and b in not in X_u. Summing over all $\ell_{u,v}$, the factor of $q^{\sum(\cdots)}$ accounts for ordered pairs (a,b) associated with inversions such that a,b are not both in the same X_u. □

2.4 Diagonal Sum Identity

We will now consider the Diagonal Sum Identity in order to develop a well-known q-analog,

$$\binom{n}{k_1,\ldots,k_m} = \sum_{i=0}^{k_1}\sum_{j=2}^{m} \binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}.$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $\mathcal{S}_n^m(k_1,\ldots,k_m)$, then the expression $\binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}$ counts such sequences that end in a j followed by i 1's.

Proposition 2.5 *If $m,n,k_1,\ldots,k_m$ are nonnegative integers such that $k_1+\cdots+k_m=n$, then*

$$\binom{n}{k_1,\ldots,k_m}_q = \sum_{i=0}^{k_1}\sum_{j=2}^{m} q^{\left((n-k_1)i+\sum_{v=j+1}^{m} k_v\right)} \binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}_q .$$

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $\mathcal{S}_n^m(k_1,\ldots,k_m)$. We will apply Lemma 2.4 to achieve our desired result on the terms of the right-hand side. We will do so by letting: p_1 equal $n-i-1$; $\ell_{1,1}$ equal k_1-i; $\ell_{1,j}$ equal k_j-1; $\ell_{1,v}$ equal k_v for all v in $[m]\setminus\{1,j\}$; p_2 and $\ell_{2,j}$ equal 1; p_3 and $\ell_{3,1}$ equal i.

Since this is our first application of Lemma 2.4, we will provide explicit details regarding the Lemma's implementation. Note that: $\binom{n-i-1}{k_1-i,\ldots,k_m}_q$ accounts for ordered pairs (a,b) associated with inversions such that a,b are both in the same X_u ; there are $(n-k_1-1)i+\sum k_v$ ordered pairs (a,b) associated with inversions such that a is in X_1 and b is not in X_1 ; and there are i ordered pairs (a,b) associated with inversions such that a is in X_2 and b is not in X_2 . Hence, our factor of $q^{(\cdots)}$ corresponds precisely with that in Lemma 2.4 . ☐

2.5 Vandermonde's Identity

Next, we will consider Vandermonde's Identity to develop a well-known q-analog,

$$\binom{n_1+n_2}{k_1,\ldots,k_m} = \sum_{\substack{r_1+\cdots+r_m=n_1 \\ 0\le r_i\le k_i}} \binom{n_1}{r_1,\ldots,r_m}\binom{n_2}{k_1-r_1,\ldots,k_m-r_m}.$$

Interpreting $\binom{n_1+n_2}{k_1,\ldots,k_m}$ as the number of sequences in $\mathcal{S}^m_{n_1+n_2}(k_1,\ldots,k_m)$, then each term of the sum accounts for the sequences whose first n_1 elements contains exactly r_1 1's, …, r_m m's.

Proposition 2.6 *If $m,n_1,n_2,k_1,\ldots,k_m$ are nonnegative integers such that $k_1+\cdots+k_m$ equals n_1+n_2 , then*

$$\binom{n_1+n_2}{k_1,\ldots,k_m}_q = \sum_{\substack{r_1+\cdots+r_m=n_1 \\ 0\le r_i\le k_i}} q^{\left(\sum_{v=2}^{m} f(r_v)\right)} \binom{n_1}{r_1,\ldots,r_m}_q \binom{n_2}{k_1-r_1,\ldots,k_m-r_m}_q$$

where $f(r_v) = r_v \sum_{j=1}^{v-1}(k_j-r_j)$ *for every* $v\in[m]$.

Proof. Interpret $\binom{n_1+n_2}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $\mathcal{S}^m_{n_1+n_2}(k_1,\ldots,k_m)$. We will apply Lemma 2.4 to achieve our desired result on the terms of the right-hand side. We will do so by letting: p_1 equal n_1 ; $\ell_{1,v}$ equal r_v for all $v\in[m]$; p_2 equal n_2 ; and $\ell_{2,v}$ equal k_v-r_v for all $v\in[m]$.

Note that: the q-multinomial coefficients on the right-hand side account for ordered pairs (a,b) associated with inversions such that a,b are both in the same X_u ; and $q^{\sum f(r_v)}$ accounts for ordered pairs (a,b) associated with inversions such that a,b are not both in the same X_u . ☐

For a well-known generalization of Vandermonde's Identity, we will provide a q-analog.

Proposition 2.7 *If $m,n_1,\ldots,n_t,k_1,\ldots,k_m$ are nonnegative integers such that $k_1+\cdots+k_m$ is equal to $n_1+\cdots+n_t$, then*

$$\binom{n_1+\cdots+n_t}{k_1,\ldots,k_m}_q = \sum_{\substack{r_{u,1}+\cdots+r_{u,m}=n_u \\ r_{1,v}+\cdots+r_{t,v}=k_v \\ 0\le r_{u,v}}} q^{\left(\sum_{u=1}^{t}\sum_{v=2}^{m} f(r_{u,v})\right)} \binom{n_1}{r_{1,1},\ldots,r_{1,m}}_q \cdots \binom{n_t}{r_{t,1},\ldots,r_{t,m}}_q$$

where $f(r_{u,v}) = r_{u,v} \sum_{i=u+1}^{t} \sum_{j=1}^{v-1} r_{i,j}$ *for every* $(u,v) \in [t] \times [m]$.

Proof. Consider $\mathcal{S}^m_{n_1+\cdots+n_s}(k_1,\ldots,k_m)$, and interpret $\binom{n_1+\cdots+n_s}{k_1,\ldots,k_m}_q$ as the generating function for inversions on this set of sequences. Letting p_i equal n_i for all $i \in [t]$ and $\ell_{u,v}$ equal $r_{u,v}$ for all (u,v) in $[t] \times [m]$, the application of Lemma 2.4 is direct. □

2.6 Chu Shih-Chieh (Zhu Shijie)'s Identity

We will now consider the well-known Chu Shih-Chieh's Identity,

$$\binom{n}{k_1,\ldots,k_m} = \sum_{r=0}^{n-k_1} \sum_{\substack{r_2+\cdots+r_m=r \\ 0\le r_j\le k_j}} \binom{r}{0,r_2,\ldots,r_m}\binom{n-r-1}{k_1-1,k_2-r_2,\ldots,k_m-r_m}.$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $\mathcal{S}^m_n(k_1,\ldots,k_m)$, then each term of the sum accounts for the sequences $(\sigma_1, \ldots, \sigma_n)$ such that: σ_{r+1} equals 1; $(\sigma_1, \ldots, \sigma_r)$ is a sequence with r_2 2's, …, r_m m's; and $(\sigma_{r+2}, \ldots, \sigma_n)$ is a sequence with $k_1 - 1$ 1's, $k_2 - r_2$ 2's, …, $k_m - r_m$ m's.

A lesser-known, albeit natural, generalization follows.

Proposition 2.8 *If* $m, n, k_1, \ldots, k_m$ *are nonnegative integers such that* $k_1 + \cdots + k_m$ *is equal to* n *, then*

$$\binom{n}{k_1,\ldots,k_m} = \sum_{\substack{E\subseteq[n] \\ |E|=k_1}} \sum_{\substack{r_{i,2}+\cdots+r_{i,m}=n_i \\ r_{1,j}+\cdots+r_{s,j}=k_j \\ 0\le r_{i,j}}} \binom{n_1}{0,r_{1,2},\ldots,r_{1,m}}\cdots\binom{n_s}{0,r_{s,2},\ldots,r_{s,m}}$$

where $E = \{e_1, \ldots, e_{k_1}\}$ *with* $e_i < e_{i+1}$ *for every* $1 \le i \le k_1 - 1$ *;* s *is equal to* $k_1 + 1$ *;* n_1 *equals* $e_1 - 1$ *;* n_i *equals* $e_i - e_{i-1} - 1$ *for every* $2 \le i \le k_1$ *; and* n_s *equals* $n - e_{k_1}$ *.*

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $\mathcal{S}^m_n(k_1,\ldots,k_m)$. Each term of the sum accounts for the sequences $(\sigma_1, \ldots, \sigma_n)$ such that: σ_i equals 1 if and only if i is in E ; the subsequence $(\sigma_1, \ldots, \sigma_{e_1-1})$ contains $r_{1,2}$ 2's, …, $r_{1,m}$ m's; for every $2 \le i \le k_1$, the subsequence $(\sigma_{e_{i-1}+1}, \ldots, \sigma_{e_i-1})$ contains $r_{i,2}$ 2's, …, $r_{i,m}$ m's; and the subsequence $(\sigma_{e_{s-1}+1}, \ldots, \sigma_n)$ contains $r_{s,2}$ 2's, …, $r_{s,m}$ m's. □

A q-analog of Proposition 2.8 follows.

Proposition 2.9 *If* $m, n, k_1, \ldots, k_m$ *are nonnegative integers such that* $k_1 + \cdots + k_m$ *is equal to* n *, then* $\binom{n}{k_1,\ldots,k_m}_q$ *is equal to*

$$\sum_{\substack{E\subseteq[n] \\ |E|=k_1}} \sum_{\substack{r_{u,2}+\cdots+r_{u,m}=n_u \\ r_{1,v}+\cdots+r_{s,v}=k_v \\ 0\le r_{u,v}}} q^{\left(\sum_{u=1}^{s}\sum_{v=2}^{m} f(r_{u,v})\right)} \binom{n_1}{0,r_{1,2},\ldots,r_{1,m}}_q \cdots \binom{n_s}{0,r_{s,2},\ldots,r_{s,m}}_q$$

where $E = \{ e_1, \ldots, e_{k_1} \}$ *with* $e_i < e_{i+1}$ *for every* $1 \le i \le k_1 - 1$ *;* s *is equal to* $k_1 + 1$ *;* n_1 *equals* $e_1 - 1$ *;* n_i *equals* $e_i - e_{i-1} - 1$ *for every* $2 \le i \le k_1$ *;* n_s *equals* $n - e_{k_1}$ *; and* $f(r_{u,v}) = r_{u,v}\left(k_1 - u + 1 + \sum_{i=u+1}^{s} \sum_{j=2}^{v-1} r_{i,j} \right)$ *for every* $(u,v) \in [s] \times [m]$ *.*

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $\mathcal{S}_n^m(k_1,\ldots,k_m)$. We will apply Lemma 2.4 to achieve our desired result on the sequences σ associated with the terms of the right-hand side. We will do so by letting: p_{2k} equal 1 for all k in $[k_1]$; $\ell_{2k,1}$ equal 1 for all k in $[k_1]$; p_{2k-1} equal n_k for all k in $[s]$; $\ell_{2k-1,1}$ equal zero for all k in $[s]$; and $\ell_{2k-1,v}$ equal $r_{k,v}$ for all k in $[s]$ and for all v in $[m] \setminus 1$.

Observe that: the q-multinomial coefficients on the right-hand side account for ordered pairs (a,b) associated with inversions such that a,b are both in X_{2k-1} for some k in $[s]$; when a is in X_{2k} for some k in $[k_1]$, then σ_a equals 1 and no ordered pairs (a,b) are associated with inversions; $r_{u,v}(k_1 - u + 1)$ accounts for ordered pairs (a,b) associated with inversions such that a is in X_{2k-1} for some k in $[s]$, σ_a equals v, and b is in X_{2j} for some j in $[k_1]$; and $r_{u,v}(\sum\sum r_{i,j})$ accounts for ordered pairs (a,b) associated with inversions such that a is in X_{2k-1} for some k in $[s]$, σ_a equals v, and b is in X_{2j-1} for some j in $[s]$ where j is greater than that k. □

2.7 "Apartment Complex" Identity

The following identity was inspired from an identity contained in [32]. Consider a hypothetical scenario with an apartment complex whose buildings will contain exactly one unit per floor. Assume there are to be n_1 buildings, with n_2 of them receiving a second floor. Exactly k of the units will be rented.

$$\binom{n_1}{n_2}\binom{n_1+n_2}{k} = \sum_{k_1+k_2=k} \binom{n_1}{k_1}\binom{n_1}{n_1-n_2, k_2, n_2-k_2}.$$

The complex owner could first choose which n_2 of the n_1 buildings will receive a second floor, and then k tenants could choose which of the n_1+n_2 units to rent. Alternatively, for all k_1 in between 0 and k, the owner could first rent out k_1 of the n_1 first floor units, and then of the n_1 buildings: $n_1 - n_2$ buildings could receive no second floor; k_2 of them could receive a second floor that is rented; and $n_2 - k_2$ could receive a second floor that is unrented. This naturally generalizes as follows.

Proposition 2.10 *If* $n_1, \ldots, n_j, k$ *are nonnegative integers such that* $n_j \le \cdots \le n_1$ *and* $k \le n_1 + \cdots + n_j$ *, then*

$$\left(\prod_{i=2}^{j} \binom{n_{i-1}}{n_i} \right) \binom{n_1+\cdots+n_j}{k} = \sum_{k_1+\cdots+k_j=k} \binom{n_1}{k_1} \prod_{i=2}^{j} \binom{n_{i-1}}{n_{i-1}-n_i, n_i-k_i, k_i}.$$

Proof. For every $2 \le i \le j$, let S_{i-1} be the set $S^2_{n_{i-1}}(n_{i-1}-n_i, n_i)$. Also let S_j be the set $S^2_{n_1+\cdots+n_j}(n_1+\cdots+n_j-k, k)$. For every $k_1+\cdots+k_j$ equal to k, let: $T_1^{(k_1,\ldots,k_j)}$ be the set $S^2_{n_1}(n_1-k_1, k_1)$; and for every $2 \le i \le j$, let $T_i^{(k_1,\ldots,k_j)}$ be the set $S^3_{n_{i-1}}(n_{i-1}-n_i, n_i-k_i, k_i)$.

Define

$$\varphi\colon \prod_{i=1}^{j} S_i \to \coprod_{k_1+\cdots+k_j=k} \left(\prod_{i=1}^{j} T_i^{(k_1,\ldots,k_j)} \right) \text{ via } (\sigma^1,\ldots,\sigma^j) \mapsto (\tau^1,\ldots,\tau^j)$$

in the following way. For every $1 \leq i \leq j-1$, let $C_i = \{ s \in [n_i] \mid \sigma_s^i = 2 \}$. Express C_i as $\{ c_{i,1}, \ldots, c_{i,n_{i+1}} \}$ where $c_{i,p} < c_{i,p+1}$ for every $1 \leq p \leq n_{i+1}-1$. Further, let N_i be equal to $n_1+\cdots+n_i$. Finally, for every $1 \leq i \leq j-1$, let

$$\tau_s^1 = \sigma_s^j ,$$

$$\tau_s^{i+1} = \begin{cases} 1 \text{ if } \sigma_s^i = 1 , \\ 2 \text{ if } \sigma_s^i = 2 \text{ and } \sigma_{N_i+p}^j = 1 \text{ where } s = c_{i,p} , \\ 3 \text{ if } \sigma_s^i = 2 \text{ and } \sigma_{N_i+p}^j = 2 \text{ where } s = c_{i,p} . \end{cases}$$

The desired result follows from observing that φ is bijective. □

Our q-analog of Proposition 2.10 follows.

Proposition 2.11 *If $n_1,\ldots,n_j,k$ are nonnegative integers such that $n_j \leq \cdots \leq n_1$ and $k \leq n_1+\ldots+n_j$, then*

$$\left(\prod_{i=2}^{j} \binom{n_{i-1}}{n_i}_q \right) \binom{n_1+\cdots+n_j}{k}_q = \sum_{k_1+\cdots+k_j=k} q^{f(K)} \binom{n_1}{k_1}_q \prod_{i=2}^{j} \binom{n_{i-1}}{n_{i-1}-n_i, n_i-k_i, k_i}_q$$

where $f(K) = \sum_{u=1}^{j-1} k_u \left(\sum_{i=u+1}^{j} (n_i-k_i) \right)$ for every K equal to $(k_1,\ldots,k_j)$.

Proof. We will utilize the notation of Proposition 2.10 and interpret the q-analogs within this identity as generating functions for the inversion statistic on sequences.

We will begin by accounting for the inversions associated with $\binom{n_1+\cdots+n_j}{k}_q$, namely the inversions associated with σ^j. We will apply Lemma 2.4 by letting (for all i in $[j]$): p_i equal n_i; $\ell_{i,1}$ equal n_i-k_i; and $\ell_{i,2}$ equal k_i. Note that for every i in $[j]$, the ordered pairs (a,b) associated with $\mathrm{inv}(\sigma^j)$ such that a,b are both in X_i are accounted for by

$$\mathrm{inv}\left(\tau^1\right) , \text{ when } i=1;$$

$$\sum_{\tau_s^i=2} \mathrm{r}\left(\tau_s^i\right) , \text{ when } i \geq 2.$$

Also note that $q^{f(K)}$ accounts for ordered pairs (a,b) associated with $\mathrm{inv}(\sigma^j)$ such that a,b are not both in X_i for some i in $[j]$.

We will now account for inversions associated with $\prod \binom{n_{i-1}}{n_i}_q$. Observe that for every $2 \leq i \leq j$,

$$\mathrm{inv}\left(\sigma^{i-1}\right) = \sum_{\sigma_s^{i-1}=1} \mathrm{r}\left(\sigma_s^{i-1}\right) = \sum_{\tau_s^i=1} \mathrm{r}\left(\tau_s^i\right) .$$

The desired result follows as an application of Corollary 1.3. □

Notice that developing a deep enumerative understanding of the original "apartment complex" identity in terms of sequences enabled us to develop a corresponding q-analog. It is the viewpoint of the authors that a complete grasp of the enumerative combinatorics of any binomial or multinomial identity supplements the development of a q-analog generalization.

3 Integer Partitions and Galois Numbers

In this section, we will introduce the major index statistic and generalized Galois numbers. Ultimately, we will develop a theorem that reveals a connection between the coefficients of generalized Galois numbers and integer partitions with kinds.

3.1 Major Index Statistic

We will now formally define the well-known major index statistic. From our experience, the major index statistic more naturally and elegantly conveys the results sought in this section, which were proving to be a cumbersome endeavor using the inversion statistic.

Definition 3.1 *If m, n are nonnegative integers and $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a sequence whose elements are in $[m]$, then*

$$\mathrm{maj}(\sigma) := \sum_{\substack{a \in [n-1] \\ \sigma_a > \sigma_{a+1}}} a\,.$$

*The value of $\mathrm{maj}(\sigma)$ shall be referred to as the **major index of $\boldsymbol{\sigma}$**.*

Figure 8 contains some examples. Observe the parallelism between Figures 2 and 8.

2211 $\mathrm{maj}(\sigma) = 2$	2121 $\mathrm{maj}(\sigma) = 4$	2112 $\mathrm{maj}(\sigma) = 1$
1221 $\mathrm{maj}(\sigma) = 3$	1212 $\mathrm{maj}(\sigma) = 2$	1122 $\mathrm{maj}(\sigma) = 0$

Figure 8: All sequences of length 4 with two 2s and two 1s.

The observed parallelism between Figures 2 and 8 is in fact not a coincidence. MacMahon showed in [23] that when considering the set of sequences $\mathcal{S}_n^m(k_1, \ldots, k_m)$, the generating functions for major index and inversions are equal: a fact which is now well-known. Stated precisely, if $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then

$$\binom{n}{k_1, \ldots, k_m}_q = \sum_{\sigma \in \mathcal{S}_n^m(k_1,\ldots,k_m)} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n^m(k_1,\ldots,k_m)} q^{\mathrm{maj}(\sigma)} . \tag{1}$$

The following two lemmas and corollary will develop additional familiarity with the major index statistic while also proving vital in the later theorem.

Lemma 3.2 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$,*

$$\begin{aligned} \mathcal{M}_n^m(k) &\coloneqq \{ \sigma \in \mathcal{S}_n^m \mid \mathrm{maj}(\sigma) = k \} , \\ A_i &= \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_{n-m+i} = \sigma_{n-m+i+1} \} \text{ when } 1 \leq i \leq m-1 , \\ A_m &= \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_n = m+1 \} . \end{aligned}$$

Then,

$$\mathcal{M}_n^{m+1}(k) \setminus \bigcup_{i \in [m]} A_i = \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_{k+1} = 1 \text{ and } \forall\, j \in [m] ,\ \sigma_{n-m+j} = j \} .$$

Proof. Let σ be in $\mathcal{M}_n^{m+1}(k) \setminus \cup A_i$ and ω be the subsequence of σ containing its last m elements, namely $(\sigma_{n-m+1}, \ldots, \sigma_n)$. Since $n - m + 1$ must be at least $k + 1$ in value and $\mathrm{maj}(\sigma)$ is equal to k , the subsequence ω must be nondecreasing. In addition, since σ is not in $\cup A_i$, the subsequence ω must be strictly increasing and not end in $m + 1$. Given that the length of ω is m , it is forced that $\omega = (1, 2, \ldots, m)$. The desired inclusion follows from observing that for every $k + 1 \leq j \leq n - m + 1$, the value of σ_j must be 1 or else the major index of σ would be greater than k .

The reverse inclusion follows by the definitions of the A_i's and $\mathcal{M}_n^{m+1}(k)$. □

Corollary 3.3 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$. Also let $A_1, \ldots, A_m$ be as in Lemma 3.2. Then,*

$$\left| \mathcal{M}_n^{m+1}(k) \setminus \bigcup_{i \in [m]} A_i \right| = \left| \{ \sigma \in \mathcal{M}_{k+1}^{m+1}(k) \mid \sigma_{k+1} = 1 \} \right| .$$

Proof. The result follows from observing that for every σ in $\mathcal{M}_n^{m+1}(k) \setminus \cup A_i$, the value of elements $\sigma_{k+2}, \ldots, \sigma_n$ are fixed and $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma')$, where σ' is equal to $(\sigma_1 , \ldots , \sigma_{k+1})$. □

Lemma 3.4 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$. Also let $A_1, \ldots, A_m$ be as in Lemma 3.2. If J is a subset of $[m]$ with $|J| = i$, then*

$$\left| \bigcap_{j \in J} A_j \right| = \left| \mathcal{M}_{n-i}^{m+1}(k) \right| .$$

Proof. Let $\varphi\colon \cap A_j \to \mathcal{M}_{n-i}^{m+1}(k)$ via $\sigma \mapsto \overline{\sigma}$, where σ is $(\sigma_1, \ldots, \sigma_{n-m}, \omega_1, \ldots, \omega_m)$ and $\overline{\sigma}$ is the subsequence of σ with ω_j removed for every $j \in J$. Note that $\overline{\sigma}$ is of the proper length for the expressed codomain of φ. Also note that the elements of σ whose indices are accounted for by $\operatorname{maj}(\sigma)$ are unaffected by φ: when $|J| < m$, the values of $n - m$ is at least k; when $|J| = m$, every ω_i equals $m+1$. As such, the values of $\operatorname{maj}(\sigma)$ and $\operatorname{maj}(\overline{\sigma})$ are equal. Hence, the image of φ is contained within the desired codomain.

To show surjectivity, observe that each A_j in $\cap A_j$ induces a loss of one degree of freedom in the expression of any σ from $\mathcal{M}_n^{m+1}(j)$. Viewing this loss as being induced on the element ω_j, the map φ results in $\overline{\sigma}$ being free from the adjacent element equality that is forced by the A_j's.

To show injectivity, consider σ^1, σ^2 in $\cap A_j$ such that σ^1 and σ^2 are unequal. Let a be the largest index of element such that σ_a^1 differs from σ_a^2. If a is greater than $n - m$, the result follows from observing that σ_a^1 and σ_a^2 are necessarily not among the ω_j removed by φ. Should a be at most $n-m$, the result follows given that such σ_a^1 and σ_a^2 are unaffected by φ. □

3.2 The Insertion Method

We will now describe a construction, called The Insertion Method, that provides a bridge between fundamental sequences and the major index statistic. This construction, first developed by Carlitz [7] and later clarified by Wilson [35], will be an essential component of the proof for the upcoming theorem.

Let m, n be nonnegative integers, and let $(F_1, \ldots, F_m)$ be a fundamental sequence in $\mathcal{F}_n^m$. For every v in $[m]$, list the elements of F_v in nonincreasing order, labeling them as $f_{v,1} \geq \cdots \geq f_{v,k_v}$ where k_v equals $|F_v|$. The sequence $(f_{1,1}, f_{1,2}, \ldots, f_{m,k_m})$ will be referred to as $\tau = (\tau_1, \ldots, \tau_n)$. Also define the value function $v\colon [n] \to [m]$ such that $v(i)$ equals j, where τ_i corresponds to its respective $f_{j,k}$. We will build a sequence σ in $\mathcal{S}_n^m$ inductively using τ and v.

Let $\sigma^1 = (v(1))$. For every $2 \leq i \leq n$, there is some $a \in [i]$ such that σ_a^i equals $v(i)$. Moreover, the sequence σ^i shall be of the form

$$\sigma_b^i = \begin{cases} v(i) & \text{when } b = a, \\ \sigma_b^{i-1} & \text{when } 1 \leq b < a, \\ \sigma_{b-1}^{i-1} & \text{when } a < b \leq i. \end{cases}$$

The value a shall be determined by the following process:

1. Label σ_i^i with a zero.
2. Working greatest to least among j in $[i-2]$, for every $\sigma_j^{i-1} > \sigma_{j+1}^{i-1}$ label σ_{j+1}^i with successively increasing positive integers $1, 2, 3, \ldots, d$.
3. Working least to greatest among j in $[i-1]$, if σ_j^i is currently unlabeled, label σ_j^i with successively positive integers $d+1, d+2, \ldots, i-1$.

4. Find the σ^i_j labeled with a τ_i , and let a equal j .

Example 3.5 Consider the fundamental sequence

$$\big(F_1, F_2, F_3, F_4\big) \;=\; \big(\,\{0,0\}\,,\,\{1\}\,,\,\{2,3\}\,,\{1,5\}\,\big)\,.$$

Note the contents of Figure 9.

i	τ_i	$v(i)$	Labeling for σ^i	σ^i	$\mathrm{maj}(\sigma^i)$
1	0	1		(1)	0
2	0	1	(1,0)	(1,1)	0
3	1	2	(1,2,0)	(2,1,1)	1
4	3	3	(2,1,3,0)	(2,1,3,1)	4
5	2	3	(3,2,4,1,0)	(2,3,1,3,1)	6
6	5	4	(3,4,2,5,1,0)	(2,3,1,4,3,1)	11
7	1	4	(4,5,3,6,2,1,0)	(2,3,1,4,3,4,1)	12

Figure 9: The construction of σ for Example 3.5

Proof of the facts $\mathrm{maj}(\sigma^i) \;=\; \mathrm{maj}(\sigma^{i-1}) + \tau_i$ and that The Insertion Method provides a bijection from $\mathcal{F}^m_n$ to $\mathcal{S}^m_n$ are omitted here as they are contained in [7].

Proposition 3.6 *Let m, n, k be nonnegative integers, and consider the following sets*

$$(i)\;\;\{\,(F_1,\ldots,F_m)\in\mathcal{F}^m_n(k)\;\mid\;0\notin F_2\,,\ldots,\,0\notin F_m\,\}\,;$$
$$(ii)\;\;\{\,\sigma\in\mathcal{S}^m_n\;\mid\;\mathrm{maj}(\sigma)=k\;\text{ and }\;\sigma_n=1\,\}\,.$$

When restricted to (i) , The Insertion Method provides a bijection onto (ii) .

Proof. Consider the following two biconditional statements:

$$\forall\, v(i) > 1\,,\;\; \tau_i \neq 0 \quad\Longleftrightarrow\quad \sigma^i_i = 1\,,\;\;\forall\, i\in[n] \quad\Longleftrightarrow\quad \sigma_n = 1\,;$$

$$\mathrm{maj}(\sigma^i) \;=\; \mathrm{maj}(\sigma^{i-1}) + \tau_i \quad\Longleftrightarrow\quad \mathrm{maj}(\sigma) \;=\; \tau_1 + \cdots + \tau_n\,.$$

It follows that The Insertion Method maps (i) into (ii) and does so bijectively. □

3.3 Generalized Galois Numbers

We will now define a generalized Galois number, which can be found in [34] and whose coefficients will be the central object of our theorem.

Definition 3.7 *If m, n are nonnegative integers, then*

$$G^m_n \;:=\; \sum_{\substack{k_1+\cdots+k_m=n\\ k_i\geq 0}} \binom{n}{k_1,\ldots,k_m}_q\,.$$

This polynomial can be referred to as the ***generalized Galois number of (m,n)*** .

$$G_1^3 = 3 \qquad G_2^3 = 3q + 6$$

$$G_3^3 = q^3 + 8q^2 + 8q + 10 \qquad G_4^3 = 3q^5 + \cdots + 18q^3 + 21q^2 + 15q + 15$$

$$G_5^3 = 3q^8 + \cdots + 45q^3 + 39q^2 + 24q + 21 \qquad G_6^3 = q^{12} + \cdots + 82q^3 + 62q^2 + 35q + 28$$

Figure 10: Generalized Galois numbers $G_1^3, \ldots, G_6^3$.

The generalized Galois numbers of $(2, n)$ are precisely the Galois numbers that were defined by Goldman and Rota [12]. Figure 10 contains examples of generalized Galois numbers of $(3, n)$ that were calculated using a recursive relation from [34].

Proposition 3.8 *If m, n are nonnegative integers, then*

$$G_n^m = \sum_{\sigma \in \mathcal{S}_n^m} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n^m} q^{\mathrm{maj}(\sigma)}.$$

Proof. The result follows from Definition 3.7 and Equation (1). □

One final well-known definition is needed to concisely state the theorem.

Definition 3.9 *Let $f: \mathbb{Z} \to \mathbb{Z}$ be a function, and define* ***the finite difference of f*** *to be*

$$\nabla f : \mathbb{Z} \to \mathbb{Z} \quad \text{via} \quad n \mapsto f(n) - f(n-1).$$

Inductively defining the m^{th}-finite difference of f to be $\nabla^m f := \nabla(\nabla^{m-1} f)$ for any positive integer $m \geq 2$, a well-known result follows

$$\nabla^m f(n) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(n-i). \tag{2}$$

Letting $f_k^3(n)$ be the coefficient of q^k in the simplified polynomial G_n^3, Figure 11 contains some example finite difference computations using the polynomials in Figure 10.

$$\nabla^2 f_2^3(3) = 8 \qquad \nabla^2 f_2^3(4) = 5 \qquad \nabla^2 f_2^3(5) = 5 \qquad \nabla^2 f_2^3(6) = 5$$

$$\nabla^2 f_3^3(3) = 1 \qquad \nabla^2 f_3^3(4) = 16 \qquad \nabla^2 f_3^3(5) = 10 \qquad \nabla^2 f_3^3(6) = 10$$

Figure 11: Sample finite difference computations using $f_k^3(n)$.

Observe that $\nabla^2 f_2^3(4)$, $\nabla^2 f_2^3(5)$, $\nabla^2 f_2^3(6)$ are equal to the number of integer partitions of 2 with 2 kinds (from Figure 6).

Theorem 3.10 *Let m, n, k be nonnegative integers such that $n \geq m + k$. Then,*

$$\nabla^m f_k^{m+1}(n) \;=\; |\,\mathcal{P}_k^m\,|\,,$$

where $f_k^{m+1}(n)$ evaluates to the coefficient of q^k in the simplified polynomial G_n^{m+1}.

Proof. By the definition of $\mathcal{M}_n^{m+1}(k)$ in Lemma 3.2, observe that $f_k^{m+1}(n-i)$ is equal to $\left|\,\mathcal{M}_{n-i}^{m+1}(k)\,\right|$. Applying this observation and Equation 2, we have that

$$\nabla^m f_k^{m+1}(n) \;=\; \sum_{i=0}^{m}(-1)^i\binom{m}{i}\,|\,\mathcal{M}_{n-i}^{m+1}(k)\,|\,.$$

Note that the assumed relation $n \geq m + k$ satisfies the similar assumption of Lemma 3.2 and Lemma 3.4. Applying these two lemmas and the Principle of Inclusion and Exclusion, the following equality is yielded

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\,|\,\mathcal{M}_{n-i}^{m+1}(k)\,| \;=\; \left|\,\mathcal{M}_n^{m+1}(k)\;\setminus\;\bigcup_{i\in[m]} A_i\,\right|,$$

where $A_1, \ldots, A_m$ are as defined in Lemma 3.2. Letting $\mathcal{T}$ be $\{\,\sigma \in \mathcal{M}_{k+1}^{m+1}(k)\;\mid\;\sigma_{k+1} = 1\,\}$ and applying Corollary 3.3, it follows that

$$\nabla^m f_k^{m+1}(n) \;=\; |\,\mathcal{T}\,|\,.$$

Since the rightmost element of every sequence in $\mathcal{T}$ is 1, Proposition 3.6 applies to $\mathcal{T}$ and it follows that

$$\nabla^m f_k^{m+1}(n) \;=\; \left|\,\{\,(F_1, \ldots, F_{m+1}) \in \mathcal{F}_{k+1}^{m+1}(k)\;\mid\;0 \notin F_2\,, \ldots,\,0 \notin F_{m+1}\,\}\,\right|\,.$$

In the case that m is greater than 0, further applying Proposition 1.13 achieves the desired result. Should m equal 0, the desired result occurs from observing that both sides of the equality in the theorem statement equal zero when $k \neq 0$ and equal 1 when $k = 0$. □

Simply stated, Theorem 3.10 expresses that as n grows the m^{th} finite difference of $f_k^{m+1}(n)$ is eventually constant, and the resulting constant is precisely the number of integer partitions of k with m kinds. Reflecting back to Figure 11, we can observe that the sample computations of $\nabla^2 f_k^3(n)$ become constant when n is at least $k+2$ in value.

Corollary 3.11 *If n, k are nonnegative integers such that $n \geq k$, then*

$$\frac{d^k}{dq^k}\left(\frac{G_{n+1}^2 - G_n^2}{k!}\right)\Bigg|_{q=0} \;=\; \operatorname{part}(k)\,,$$

where $\frac{d}{dq}$ is the conventional derivative operator and $\operatorname{part}(k)$ is the number of integer partitions.

Proof. Follows directly from Theorem 3.10 and Taylor's Theorem. □

4 Concluding Remarks

This research culminated in Theorem 3.10, which may be interesting to prove using the inversion statistic rather than the major index statistic. Also, it may be insightful to use analytical methods relating to the calculus of finite differences to achieve Theorem 3.10. Moreover, it may be worth investigating information buried within expressions of q-multinomial coefficients different from the Galois numbers. Furthermore, q-multinomial coefficients can be generalized to p, q-binomial coefficients [10], and many of this paper's results may be able to be extended.

Acknowledgments

The authors would like to extend gratitude to the editors and reviewers of the PUMP Journal for their thoughtful advice during the revision process. We would also like to extend thanks to Alex Foster and Jonathan Winter for generously offering their computer programming skills in assistance of this project, with an additional mention of thanks to Alex for his pointing us to the Online Encyclopedia of Integer Sequences [33].

References

[1] T. Abdeljawad, D. Baleanu, E. Gündoğdu, F. Jarad, On the Mittag-Leffler stability of q-fractional nonlinear dynamical systems, *Proc. Ram. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.*, **12** (2011), 309–314.

[2] G.E. Andrews, J.B. Olsson, Partition identities with an application to group representation theory, *J. Reine Angew. Math.*, **413** (1991), 198–212.

[3] L.C. Biedenharn, J.D. Louck, Inhomogeneous basis set of symmetric polynomials defined by tableaux, *Proc. Nat. Acad. Sci. U.S.A.*, **87** (1990), 1441–1445.

[4] K. Bringmann, A. Folsom, R.C. Rhoades, Partial theta functions and mock modular forms as q-hypergeometric series, *Ramanujan J.*, **29** (2012), 295–310.

[5] S. Capparelli, A combinatorial proof of a partition identity related to level 3 representations of a twisted affine Lie algebra, *Comm. Algebra*, **23** (1995), 2959–2969.

[6] N. Caporaso, M. Cirafici, L. Griguolo, S. Pasquetti, D. Seminara, R.J. Szabo, Topological string and large N phase transitions: nonchiral expansion of q-deformed Yang-Mills theory, *J. High Energy Phys.*, (2006), 797–863.

[7] L. Carlitz, A combinatorial property of q-Eulerian numbers, *Amer. Math. Monthly*, **82** (1975), 51–54.

[8] H.H. Chan, P.C. Toh, New analogues to Ramanujan's partition identities, *J. Number Theory*, **130** (2010), 1898–1913.

[9] P. Cheung, V. Kac, *Quantum Calculus*, Springer-Verlag, 2002.

[10] R.B. Corcino, On p, q-binomial coefficients, *Integers*, **8** (2008), 1–16.

[11] I. Gessel, D. Stanton, Applications of q-Lagrange inversion to basic hypergeometric series, *Tras. Amer. Math. Soc.*, **277** (1983), 173–201.

[12] J. Goldman, G.C. Rota, The number of subspaces of a vector space, *Proc. Third Waterloo Conf. on Combinatorics*, (1968), 75–83.

[13] E. Goles, M. Morvan, H.D. Phan, Sandpiles and order structure of integer partitions, *Discrete Appl. Math.*, **117** (2002), 51–64

[14] C. Greene, G.J. Kleitman, Longest chains in the lattice of integer partitions ordered by majorization, *European J. Combin.*, **7** (1986), 1–10.

[15] H. Gupta, C.E. Gwyther, J.C.P. Miller, *Royal Society Mathematical Tables: Volume 4, Tables of Partitions*, Cambridge University Press, 1958.

[16] E. Heine, Über die Riehne $1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}\,x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}\,x^2+\cdots$, *J. Reine Angew. Math*, **32** (1846), 210–212.

[17] W. Heping, V. Gupta, The rate of convergence of q-Durrmeyer operators for $0<q<1$, *Math. Methods Appl. Sci.*, **31** (2008), 1946–1955.

[18] T. Kamiya, S. Takeuchi, Complete (p,q)-elliptic integrals with application to a family of means, *J. Class. Anal.* **10** (2017), 15–25.

[19] S. Kanade, M.C. Russell, Staircases to analytic sum-sides for many new integer partition identities of Roger-Ramanujan type, *Electron. J. Combin.*, **26** (2019), 1–33.

[20] E. Kiritsis, B. Pioline, On R^4 threshold corrections in type IIB string theory and (p,q)-string instantons, *Nuclear Phys.* **508** (1997), 509–534.

[21] D.E. Knuth, Subspaces, subsets, and partitions, *J. Combin. Theory*, **10** (1971), 178–180.

[22] C. Krattenthaler, H.M. Srivastava, Summations for basic hypergeometric series involving a q-analogue of the digamma function, *Comput. Mat. Appl.*, **32** (1996), 73–91.

[23] P.A. MacMahon, The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of an assemblage of objects, *Amer. J. Math.*, **35** (1913), 281–322.

[24] S.C. Milne, A q-analog of the Gauss summation theorem for hypergeometric series in $U(n)$, *Adv. in Math.*, **72** (1988), 59–131.

[25] S. Malek, On complex singularity analysis for linear partial q-difference-differential equations using nonlinear differential equations, *J. Dyn. Control Syst.*, **19** (2013), 69–93.

[26] G. Moshkovitz, A. Shapira, Ramsey theory, integer partitions and a new proof of the Erdős-Szekeres theorem, *Adv. Math.*, **262** (2014), 1107–1129.

[27] K. Ono, Distribution of the partition function modulo m, *Ann. of Math.*, **151** (2000), 293–307.

[28] H.J. Schmidt, J. Schnack, Partition functions and symmetric polynomials, *Amer. J. Phys.*, **70** (2002), 53–57.

[29] R.N. Siddiqi, M.A. Rakha, q-beta and elliptic type integrals, *Int. J. Appl. Math.*, **10** (2002), 385–401.

[30] A.V. Sills, Compositions, partitions, and Fibonacci numbers, *Fibonacci Quart.*, **49** (2011), 348–354.

[31] R.P. Stanley, *Enumerative Combinatorics*, Cambridge University Press, 1997.

[32] P. Tardy, Sopra alcunae formole relativa ai coefficienti binomiali, *G. Mat. Battaglini*, **3** (1865), 1–3.

[33] The On-Line Encyclopedia of Integer Sequences, available online at the URL: `https://oeis.org/`.

[34] C.R. Vinroot, An enumeration of flags in finite vector spaces, *Electron. J. Combin.*, **19** (2012), 1–9.

[35] A.T. Wilson, An extension of MacMahon's equidistribution theorem to ordered multiset partitions, *Electron. J. Combin.*, **23** (2016), 1–22.

Adrian Avalos
Coastal Carolina University
100 Chanticleer Drive East
Conway, SC 29528
E-mail: alavalos@coastal.edu

Mark Bly
Coastal Carolina University
100 Chanticleer Drive East
Conway, SC 29528
E-mail: mbly@coastal.edu

Received: October 24, 2019 **Accepted**: June 18, 2020
Communicated by Scott Annin