Partition Problems and a Pattern of Vertical Sums

E. Krinsky, S. Raianu, and A. Wittmond

Abstract - We give a possible explanation for the mystery of a missing number in the statement of a problem that asks for the non-negative integers to be partitioned into three subsets. Based on a pattern of sums of certain elements in the three sets, we find a more standard solution to the problem, using only congruence modulo five. We also show that the original statement plays a special role among all statements that satisfy the same pattern of the sums.

Keywords : congruence modulo an integer; partitions of sets; partitions of integers

Mathematics Subject Classification (2010) **:** 11A07; 11P81; 05A15; 05A17; 05A18

1 The problem and its two solutions

In this paper we study Problem 2.1.18 of [1], whose statement is given in Figure 1.

2.1.18 **The non-negative integers are divided into three groups as follows:**

$$A = \{0,3,6,8,9,\ldots\}, \quad B = \{1,4,7,11,14,\ldots\},$$

$$C = \{2,5,10,13,\ldots\}.$$

Explain.

Figure 1: Scan from [1, p. 24]

We note first that there are infinitely many ways to complete the partition required by the problem. Perhaps the easiest way to see that is to take the complement of the given set of numbers, order it increasingly, then use it to fill in all the empty positions in various serpentine patterns. These solutions can be applied to any similar statement involving arbitrary given numbers. The solution given in the Instructor's Guide for [1] says: "The first set contains only numbers whose numerals are drawn exclusively with curves; the second contains "linear" numerals; the third contains the mixed numerals." A quick inspection of the statement shows that 12 does not appear, while 13 and 14 do.

According to the solution, 12 should belong to C, and so we would expect C to be given as $C = \{2, 5, 10, 12, 13, \dots\}$, or at least as $C = \{2, 5, 10, 12, \dots\}$. The other thing that looks like a clue is that there are five elements given in A and B, but only four in C; this seems to indicate that we should first find the fifth element of C. In this section we show that these two hints (the missing 12 and the number of elements of the three sets) lead to another solution to the problem; that solution can be found (starting with the seventh elements of each set) using a standard congruence classes argument. We begin by writing the sets A, B and C one on top of the other

$$\begin{array}{rcllllll} A &=& \{0, & 3, & 6, & 8, & 9, & \dots\} \\ B &=& \{1, & 4, & 7, & 11, & 14, & \dots\} \\ C &=& \{2, & 5, & 10, & 13, & & \dots\}. \end{array}$$

We see that in our font 7 looks like it should belong to C. However, in a different font, like the one used on the display of a a microwave oven, not just 7, but all numbers are "linear", and thus belong to B. The solution we find in this section will not be font-dependent.

Definition 1.1 *We will call the sums of elements on the same position in each set* vertical sums. *The first, second, third, and fourth vertical sums are: 3, 12, 23, and 32. The differences between the first consecutive four vertical sums are 9, 11, and 9. We assume that this is a pattern, and we call it the* pattern of vertical sums.

We find another solution to the problem in Figure 1 based on the assumption that the pattern of vertical sums continues.

We denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x, and we use it to define an important concept that will play a central role for the rest of this section.

Definition 1.2 *The* **standard partition** *of the non-negative integers is given by*

$$SP_1 = \{3n - \left\lfloor \frac{n}{2} \right\rfloor - 2 \mid n \geq 1\}$$
$$SP_2 = \{3n - \left\lfloor \frac{n}{2} \right\rfloor - 1 \mid n \geq 1\}$$
$$SP_3 = \{5n - 5 \mid n \geq 1\}$$

Note that the sum of the elements on position n in each set is $S(n) = 11n - 2\left\lfloor \frac{n}{2} \right\rfloor - 8$.

Remark 1.3 The formulas for the elements on position $2k+1$, $2k+2$, and $2k+3$ ($k \geq 0$) in the standard partition can be also written as:

$$\begin{array}{ccc} 5k+1 & 5k+3 & 5k+6 \\ 5k+2 & 5k+4 & 5k+7 \\ 5(2k) & 5(2k+1) & 5(2k+2), \end{array}$$

This shows that SP_1 consists of the numbers with remainder 1 or 3 (mod 5), the elements of SP_2 are congruent to 2 or 4 (mod 5), while SP_3 is the set of multiples of 5.

It also shows that the standard partition also has alternating differences between vertical sums of 9 and 11:

$$S(2k+2) - S(2k+1) = (3-1) + (4-2) + 5 = 9,$$

while

$$S(2k+3) - S(2k+2) = (6-3) + (7-4) + 5 = 11$$

for $k \geq 0$. These can also be written as

$$S(n+1) - S(n) = 10 + (-1)^n$$

for all $n \geq 1$.

Definition 1.4 *Two partitions of the non-negative integers, given by* $A = \{a_k \mid k \geq 1\}$, $B = \{b_k \mid k \geq 1\}$, $C = \{c_k \mid k \geq 1\}$, *and* $D = \{d_k \mid k \geq 1\}$, $E = \{e_k \mid k \geq 1\}$, $F = \{f_k \mid k \geq 1\}$, *are said to be* **equivalent** *if there exists an integer* N *such that*

$$\{a_k \mid k > N\} = \{d_k \mid k > N\}, \{b_k \mid k > N\} = \{e_k \mid k > N\}, \{c_k \mid k > N\} = \{f_k \mid k > N\}.$$

In particular, this will happen if for all $k > N$ *we have* $a_k = d_k$, $b_k = e_k$, *and* $c_k = f_k$. *Note that this implies that the sets* $\{a_k, b_k, c_k \mid 1 \leq k \leq N\}$ *and* $\{d_k, e_k, f_k \mid 1 \leq k \leq N\}$ *are permutations of each other.*

Next we prove our first result:

Proposition 1.5 *There exists a solution to the problem in Figure 1 that is equivalent to the standard partition.*

Proof. We assume that the pattern of the vertical sums described in Definition 1.1 continues, so that the fifth sum is $32+11 = 43$, and the fifth number in C is $43-9-14 = 20$. The sixth sum is then $43+9 = 52$, and we try the first two unused numbers, 12 and 15, as the sixth elements of A and B, respectively. We get that the sixth element in C is $52-12-15 = 25$. The next two unused numbers are then 16 and 17. Putting them on the seventh position in A and B, and assuming the seventh sum is $52+11 = 63$, we get that the seventh element of C is $63-16-17 = 30$. For $n \geq 8$ we use the following algorithm:

> The n-th elements of A and B are the first two integers that do not appear so far. We subtract their sum from $S(n)$ to find the n-th element of C.

In view of Remark 1.3, the standard partition can be constructed using this algorithm for $n \geq 2$ (one easy way to see this is to use induction on n: check first that for $n = 2$ and $n = 3$ the algorithm produces columns 2 and 3 of the standard partition; then, after columns $2k$ and $2k+1$, show that the algorithm produces columns $2k+3$ and $2k+4$ of the standard partition). Now we take $N = 6$ in Definition 1.4. The set of the first six elements in each of the three sets of the solution constructed above is a permutation of the set of the first six elements in the three sets of the standard partition. Hence, from the seventh element on, the solution constructed above and the standard partition have the same elements, because they are both constructed using the same algorithm and the first two unused numbers are the same at each step. □

As an alternative proof of Proposition 1.5, we show now that we can start with the standard partition and change it by a permutation in order to get the statement in Figure 1.

We start now with the standard partition

$$\begin{array}{rcl} SP_1 & = & \{1, \; 3, \; 6, \; 8, \; 11, \; 13, \; \ldots\} \\ SP_2 & = & \{2, \; 4, \; 7, \; 9, \; 12, \; 14, \; \ldots\} \\ SP_3 & = & \{0, \; 5, \; 10, \; 15, \; 20, \; 25, \; \ldots\}. \end{array}$$

We notice that if we look at the second, third and fourth columns only, the numbers in the first set are drawn using curved lines only, the numbers in the third row use both curves and straight lines, while 4 and 7 are drawn with straight lines only (remember to keep Fig. 1 in mind when we talk about 7). We permute the numbers on the first column accordingly, and we get that the description now fits the first four columns:

$$\begin{array}{rcl} SP'_1 & = & \{0, \; 3, \; 6, \; 8, \; 11, \; 13, \; \ldots\} \\ SP'_2 & = & \{1, \; 4, \; 7, \; 9, \; 12, \; 14, \; \ldots\} \\ SP'_3 & = & \{2, \; 5, \; 10, \; 15, \; 20, \; 25, \; \ldots\}. \end{array}$$

Moving on to the fifth column, we see that 8 and 15 fit the above descriptions, but 9 does not, it should belong to the first set. We now trade 9 in the second set for the first available "linear" number, which is 11, and we get

$$\begin{array}{rcl} SP''_1 & = & \{0, \; 3, \; 6, \; 8, \; 9, \; 13, \; \ldots\} \\ SP''_2 & = & \{1, \; 4, \; 7, \; 11, \; 12, \; 14, \; \ldots\} \\ SP''_3 & = & \{2, \; 5, \; 10, \; 15, \; 20, \; 25, \; \ldots\}. \end{array}$$

Now the description fits the first five columns, but the fourth sum is broken, so we must trade 15 for 13 to repair it:

$$\begin{array}{rcl} SP'''_1 & = & \{0, \; 3, \; 6, \; 8, \; 9, \; 15, \; \ldots\} \\ SP'''_1 & = & \{1, \; 4, \; 7, \; 11, \; 12, \; 14, \; \ldots\} \\ SP'''_1 & = & \{2, \; 5, \; 10, \; 13, \; 20, \; 25, \; \ldots\}. \end{array}$$

The fourth sum is now repaired, but both the fifth and the sixth ones are broken, the fifth one is down two, and the sixth one is up two. Consequently, they can be both repaired by switching the 12 and the 14:

$$\begin{array}{rcl} A & = & \{0, \; 3, \; 6, \; 8, \; 9, \; 15, \; \ldots\} \\ B & = & \{1, \; 4, \; 7, \; 11, \; 14, \; 12, \; \ldots\} \\ C & = & \{2, \; 5, \; 10, \; 13, \; 20, \; 25, \; \ldots\}. \end{array}$$

All we need to do now is erase 15, 12, 20 and 25 and ask the question from Figure 1.

In the next section we will show other ways of obtaining partitions equivalent to the standard partition by performing certain permutations.

2 Related partition problems

After completing the work on the first section we were told by Paul Zeitz [2] that the initial omission of 12 may have been just a typo, and therefore the pattern of the alternating differences between vertical sums may have been a fluke. In this section we find all possible statements for the problem in Section 1 that preserve the same pattern of the vertical sums, and we try to find out how likely it is to obtain a partition equivalent to the standard one. More precisely, we answer the following questions:

1. In how many ways can we fill in the 14 given elements of the sets A, B, and C in Figure 1, such that the first four vertical sums are 3, 12, 23, and 32, and the fifth vertical sum can be 43?

2. For each of these statements we use the algorithm described in the proof of Proposition 1.5 to produce partitions. How many of these partitions are different up to equivalence?

3. How many of these partitions are equivalent to the standard partition?

We start answering the first question by showing that there are only 36 different possibilities for the first five elements of A, B, and C, such that the sums are 3, 12, 23, 32, and 43, if we order the k^{th} elements of A, B, and C increasingly for $k = 1, 2, 3, 4, 5$. Note that filling in the 5th element of the third set cuts down the number of possible statements by two thirds, while the number of non-equivalent solutions remains the same. The total number of different statements, i.e. the answer to the first question above, is then easily seen to be $36 \cdot (3!)^5 = 6^7 = 279936$.

In order to list the 36 possibilities described above we notice first that there are only two ways to write 23 as a sum of three numbers greater than or equal to 6: $6+7+10$, and $6+8+9$. There are six ways to write 32 as a sum of three numbers greater than or equal to 7, such that the set of three numbers is disjoint from one of the sets $\{6,7,10\}$ and $\{6,8,9\}$: $7+10+15$, $7+11+14$, $7+12+13$, $8+9+15$, $8+11+13$, and $9+11+12$. The elements on positions 3 and 4 in the three sets can therefore be one of the following six combinations:

$$\begin{array}{rcccccc} A = & \{6, \ 8\} & \{6, \ 8\} & \{6, \ 9\} & \{6, \ 7\} & \{6, \ 7\} & \{6, \ 7\} \\ B = & \{7, \ 9\} & \{7, \ 11\} & \{7, \ 11\} & \{8, \ 10\} & \{8, \ 11\} & \{8, \ 12\} \\ C = & \{10, \ 15\} & \{10, \ 13\} & \{10, \ 12\} & \{9, \ 15\} & \{9, \ 14\} & \{9, \ 13\} \end{array}$$

Finally, there are 25 ways to write 43 as a sum of three numbers greater than or equal to 8, such that the set of three numbers is disjoint from at least one of the 6 sets of 6 numbers listed above: $8+13+22$, $8+14+21$, $8+15+20$, $\ldots$ $13+14+16$.

We list all of them as columns:

8	8	8	8	8	9	9	9	9	9	9	10	10	10	10	10	10
13	14	15	16	17	11	12	13	14	15	16	11	12	13	14	15	16
22	21	20	19	18	23	22	21	20	19	18	22	21	20	19	18	17

11	11	11	11	12	12	12	13
12	13	14	15	13	14	15	14
20	19	18	17	18	17	16	16

We list now all 36 possibilities. Since for the first two elements in each set the choice is unique, we will only list elements on positions 3, 4, and 5. Moreover, in each group of six choices, the elements on positions 3 and 4 are the same, so we only list them once. We assign numbers to these 36 choices from left to right and up to down so the first group contains choices 1 through 6, the second one 7 through 12, and so on. The number assigned to each choice is listed in boldface on top of the elements on position 5 in that choice. The 36 possible choices follow:

			1	**2**	**3**	**4**	**5**	**6**
$A =$	{6,	8,	11}	11}	11}	12}	12}	13}
$B =$	{7,	9,	12}	13}	14}	13}	14}	14}
$C =$	{10,	15,	20}	19}	18}	18}	17}	16}

			7	**8**	**9**	**10**	**11**	**12**
$A =$	{6,	8,	9}	9}	9}	9}	12}	12}
$B =$	{7,	11,	12}	14}	15}	16}	14}	15}
$C =$	{10,	13,	22}	20}	19}	18}	17}	16}

			13	**14**	**15**	**16**	**17**	**18**
$A =$	{6,	9,	8}	8}	8}	8}	8}	13}
$B =$	{7,	11,	13}	14}	15}	16}	17}	14}
$C =$	{10,	12,	22}	21}	20}	19}	18}	16}

			19	**20**	**21**	**22**	**23**	**24**
$A =$	{6,	7,	11}	11}	11}	12}	12}	13}
$B =$	{8,	10,	12}	13}	14}	13}	14}	14}
$C =$	{9,	15,	20}	19}	18}	18}	17}	16}

			25	**26**	**27**	**28**	**29**	**30**
$A =$	{6,	7,	10}	10}	10}	10}	12}	12}
$B =$	{8,	11,	12}	13}	15}	16}	13}	15}
$C =$	{9,	14,	21}	20}	18}	17}	18}	16}

			31	**32**	**33**	**34**	**35**	**36**
$A =$	{6,	7,	10}	10}	10}	10}	11}	11}
$B =$	{8,	12,	11}	14}	15}	16}	14}	15}
$C =$	{9,	13,	22}	19}	18}	17}	18}	17}

If we eliminate the choices whose sets of first five elements are a permutation of each other (this eliminates the whole fourth group, choices 19 through 24, because its first two columns are a permutation of the first two columns in the first group, then keep 2 and discard 9, keep 3 and discard 27, keep 4 and discard 33, keep 7 and discard 13 and 31, keep 8 and discard 26, keep 14 and discard 25, keep 29 and discard 35) and we also leave out the standard partition (choices 1 and 15), we end up with 20 possibilities.

Recall that the algorithm described in the proof of Proposition 1.5 puts the first two unused numbers as the next elements of A and B, then uses the fact that the sum of the n-th elements is $S(n) = 11n - 2\left\lfloor \frac{n}{2} \right\rfloor - 8$ to find the next element of C.

We wrote a Haskell program (shown in Figures 2 and 3, starting on the next page) which applies the algorithm from the proof of Proposition 1.5 to the 20 options, and we list the first 46 elements in each set in Figures 4, 5, and 6 (the sets A, B, and C are listed vertically).

We see from the figures that we obtain 20 partitions, and only one of them, partition number 7 from Figure 4, seems to be equivalent to the standard partition. We will prove that this is actually true, and we count the number of equivalence classes in the following result.

Proposition 2.1 *The 20 partitions in Figures 4, 5, and 6 contain eight equivalence classes:*
Class 1: partitions 1 and 18.
Class 2: partitions 2, 3, 4, 5, 9, 10, 14, 15, 16, 17, 19, and 20.
Class 3: partition 6.
Class 4: partition 7 (it is equivalent to the standard partition).
Class 5: partition 8.
Class 6: partition 11.
Class 7: partition 12.
Class 8: partition 13.

Proof.

We will describe a partition in each equivalence class by explaining how they differ from the standard partition.

Class 1: A partition in this class coincides with the standard partition after the 11th elements, with the following exceptions:

On positions $2^k \cdot 10 + 1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 25 + 1, 2^k \cdot 25 + 2, \text{ and } 2^k \cdot 50,$$

while a partition in Class 1 has elements

$$2^k \cdot 25, 2^k \cdot 25 + 2, \text{ and } 2^k \cdot 50 + 1.$$

We will only explain this case in detail, all the other ones are similar.

We use induction on k. The case $k = 0$ is clear. We assume that the assertion is true for elements up to position $2^k \cdot 10 + 1$. (The proof continues on page 13, after the tables.)

```
module Main where

import System.Environment
import Data.Char (isSpace)
import Data.List.Split
import Data.List
import System.IO
import Text.Read
import Data.Maybe (fromJust)

--Data type representing the grid, the first int is the number of rows
data Grid = Grid Int [[Maybe Int]]

--Converts the grid to a string
instance Show Grid where
   show (Grid rows grid) =
      let showData Nothing = ""
         showData (Just a) = show a
         showRow i (col:rest) = (showData (col !! i)) ++ "," ++ showRow i rest
         showRow i [] = []
         doShow i
            | i<rows = (reverse $ drop 1 (reverse $ showRow i grid)) ++ "\n" ++ doShow (i+1)
            | otherwise = []
      in doShow 0

--gets the multiple for the modulo partition from the number of rows
multiple :: Grid -> Int
multiple (Grid rows _) = 2*rows-1

--converts a string (read in from a file) to a grid
toGrid :: String -> Grid
toGrid s = let list = (map (map readMaybe . (splitOn ",")) (splitOn "\n" s)) -- splits csv file on newlines a commas
         in let l = length list
           in if ((l `mod` 2) == 0) then -- make sure the number of rows is odd then convert to column major
             Grid (l-1) (transpose(reverse (drop 1 ( reverse list))))
              else
             Grid l (transpose list)

--the formula for what the sum of a column should be
getColSum :: Int -> Int -> Int
getColSum m n = (6*m*n + m^2*(2*n+1) - 2*(m-1)*(mod n 2) - 1 ) `div` 8

--fills in the empty spots in a grid up to the passed row number
--uses the next unused integer when at liberty to, otherwise uses the number that
--gives the column the correct sum
fillGrid :: Grid-> Int ->Grid
fillGrid (Grid rows list) columns =
   let numbers = [0,1..] -- an infinite list of integers
      remove a (x:xs) -- tries pull the requested integer off of the list and returns nothing on failure
            | x < a  = (:) <$> (Just x) <*> (remove a xs)
            | x == a = Just xs
```

Figure 2: The Haskell program, part 1

```
| x > a  = Nothing
      fillCol index ((Just a):xs) (Just nums) col       =  fillCol index xs (remove a nums) (Just a:col)
      fillCol index (Nothing:xs)  (Just num) col
         | (length col) < rows - 1  = fillCol index xs (Just (tail num)) (Just (head num):col)
         | (length col) == rows - 1 = let val = (getColSum ((rows*2) -1) index) - (fromJust (foldl (\x y -> (+) <$> x <*> y) (Just
0) col))
                              in fillCol index [] (remove val num) (Just val:col)
      fillCol index [] (Just num) col
         | (length col) < rows - 1  = fillCol index [] (Just (tail num)) (Just (head num):col)
         | (length col) == rows - 1 = let val = (getColSum ((rows*2) -1) index) - (fromJust (foldl (\x y -> (+) <$> x <*> y) (Just
0) col))
                              in fillCol index [] (remove val num) (Just val:col)
         | (length col) == rows     =  (reverse col, Just num )
      fillCol index original Nothing col = ((reverse col) ++ original, Nothing)
      doFill index (x:xs) (Just nums)
         |  index < columns =
            let (col, newNums) = fillCol index x (Just nums) []
            in col : doFill (index+1) xs newNums
         |  index == columns  = []
      doFill index [] (Just nums)
         | index < columns   =
            let (col, newNums) = fillCol index [] (Just nums) []
            in col : doFill (index+1) [] newNums
         | index == columns  = []
      doFill a b Nothing = b
   in Grid rows (doFill 0 list (Just numbers))

--the entry point of the program
--the program expects the name of a comma delimited csv file and the number of columns to fill in
--there is an option -t to transpose the grid and -d to print out the default partition for 3 rows
--depends on the split library
main :: IO ()
main = do args <- getArgs
        let justToString (Just a) = show a
            justToString Nothing = "Nothing"
        let showAlt (Grid rows lists) = foldl (\s list -> s ++ "\n" ++ (foldl (\str ls -> str ++ (justToString ls) ++ ",") "" list)) ""
lists
        let argv     = if (head args == "-t") then tail args else args
            toString = if (head args == "-t") then showAlt else show
        if(head argv == "-d") then
          withFile (head (tail argv)) WriteMode (\handle ->
                hPutStr handle (toString (fillGrid (Grid 3 [[Just 1]]) (read (argv!!2))))
              )
        else do
            infile <- openFile (head argv) ReadMode
            inString <- hGetContents infile --binds the contents of the file to lazy string
            let outPath = (reverse $ (reverse "outfile.csv")++(drop 4 (reverse $ head argv)))
                grid = toGrid inString
            let (Grid rows list) = grid
            print ("saving to " ++ outPath ++ "\n")
            withFile outPath WriteMode (\handle -> hPutStr handle (toString transformedGrid) ) --writes the file
            hClose infile
```

Figure 3: The Haskell program, part 2

1			2			3			4			5			6			7		
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
3	4	5	3	4	5	3	4	5	3	4	5	3	4	5	3	4	5	3	4	5
6	7	10	6	7	10	6	7	10	6	7	10	6	7	10	6	7	10	6	7	10
8	9	15	8	9	15	8	9	15	8	9	15	8	9	15	8	11	13	8	11	13
11	13	19	11	14	18	12	13	18	12	14	17	13	14	16	9	12	22	9	14	20
12	14	26	12	13	27	11	14	27	11	13	28	11	12	29	14	15	23	12	15	25
16	17	30	16	17	30	16	17	30	16	18	29	17	18	28	16	17	30	16	17	30
18	20	34	19	20	33	19	20	33	19	20	33	19	20	33	18	19	35	18	19	35
21	22	40	21	22	40	21	22	40	21	22	40	21	22	40	20	21	42	21	22	40
23	24	45	23	24	45	23	24	45	23	24	45	23	24	45	24	25	43	23	24	45
25	27	51	25	26	52	25	26	52	25	26	52	25	26	52	26	27	50	26	27	50
28	29	55	28	29	55	28	29	55	27	30	55	27	30	55	28	29	55	28	29	55
31	32	60	31	32	60	31	32	60	31	32	60	31	32	60	31	32	60	31	32	60
33	35	64	34	35	63	34	35	63	34	35	63	34	35	63	33	34	65	33	34	65
36	37	70	36	37	70	36	37	70	36	37	70	36	37	70	36	37	70	36	37	70
38	39	75	38	39	75	38	39	75	38	39	75	38	39	75	38	39	75	38	39	75
41	42	80	41	42	80	41	42	80	41	42	80	41	42	80	40	41	82	41	42	80
43	44	85	43	44	85	43	44	85	43	44	85	43	44	85	44	45	83	43	44	85
46	47	90	46	47	90	46	47	90	46	47	90	46	47	90	46	47	90	46	47	90
48	49	95	48	49	95	48	49	95	48	49	95	48	49	95	48	49	95	48	49	95
50	52	101	50	51	102	50	51	102	50	51	102	50	51	102	51	52	100	51	52	100
53	54	105	53	54	105	53	54	105	53	54	105	53	54	105	53	54	105	53	54	105
56	57	110	56	57	110	56	57	110	56	57	110	56	57	110	56	57	110	56	57	110
58	59	115	58	59	115	58	59	115	58	59	115	58	59	115	58	59	115	58	59	115
61	62	120	61	62	120	61	62	120	61	62	120	61	62	120	61	62	120	61	62	120
63	65	124	64	65	123	64	65	123	64	65	123	64	65	123	63	64	125	63	64	125
66	67	130	66	67	130	66	67	130	66	67	130	66	67	130	66	67	130	66	67	130
68	69	135	68	69	135	68	69	135	68	69	135	68	69	135	68	69	135	68	69	135
71	72	140	71	72	140	71	72	140	71	72	140	71	72	140	71	72	140	71	72	140
73	74	145	73	74	145	73	74	145	73	74	145	73	74	145	73	74	145	73	74	145
76	77	150	76	77	150	76	77	150	76	77	150	76	77	150	76	77	150	76	77	150
78	79	155	78	79	155	78	79	155	78	79	155	78	79	155	78	79	155	78	79	155
81	82	160	81	82	160	81	82	160	81	82	160	81	82	160	80	81	162	81	82	160
83	84	165	83	84	165	83	84	165	83	84	165	83	84	165	84	85	163	83	84	165
86	87	170	86	87	170	86	87	170	86	87	170	86	87	170	86	87	170	86	87	170
88	89	175	88	89	175	88	89	175	88	89	175	88	89	175	88	89	175	88	89	175
91	92	180	91	92	180	91	92	180	91	92	180	91	92	180	91	92	180	91	92	180
93	94	185	93	94	185	93	94	185	93	94	185	93	94	185	93	94	185	93	94	185
96	97	190	96	97	190	96	97	190	96	97	190	96	97	190	96	97	190	96	97	190
98	99	195	98	99	195	98	99	195	98	99	195	98	99	195	98	99	195	98	99	195
100	102	201	100	101	202	100	101	202	100	101	202	100	101	202	101	102	200	101	102	200
103	104	205	103	104	205	103	104	205	103	104	205	103	104	205	103	104	205	103	104	205
106	107	210	106	107	210	106	107	210	106	107	210	106	107	210	106	107	210	106	107	210
108	109	215	108	109	215	108	109	215	108	109	215	108	109	215	108	109	215	108	109	215
111	112	220	111	112	220	111	112	220	111	112	220	111	112	220	111	112	220	111	112	220
113	114	225	113	114	225	113	114	225	113	114	225	113	114	225	113	114	225	113	114	225

Figure 4: Partitions one through seven

8			9			10			11			12			13			14		
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
3	4	5	3	4	5	3	4	5	3	4	5	3	4	5	3	4	5	3	4	5
6	7	10	6	7	10	6	7	10	6	8	9	6	7	10	6	7	10	6	7	10
8	11	13	8	11	13	8	11	13	7	11	14	9	11	12	9	11	12	9	11	12
9	16	18	12	14	17	12	15	16	10	12	21	8	16	19	8	17	18	13	14	16
12	14	26	9	15	28	9	14	29	13	15	24	13	14	25	13	14	25	8	15	29
15	17	31	16	18	29	17	18	28	16	17	30	15	17	31	15	16	32	17	18	28
19	20	33	19	20	33	19	20	33	18	19	35	18	20	34	19	20	33	19	20	33
21	22	40	21	22	40	21	22	40	20	22	41	21	22	40	21	22	40	21	22	40
23	24	45	23	24	45	23	24	45	23	25	44	23	24	45	23	24	45	23	24	45
25	27	51	25	26	52	25	26	52	26	27	50	26	27	50	26	27	50	25	26	52
28	29	55	27	30	55	27	30	55	28	29	55	28	29	55	28	29	55	27	30	55
30	32	61	31	32	60	31	32	60	31	32	60	30	32	61	30	31	62	31	32	60
34	35	63	34	35	63	34	35	63	33	34	65	33	35	64	34	35	63	34	35	63
36	37	70	36	37	70	36	37	70	36	37	70	36	37	70	36	37	70	36	37	70
38	39	75	38	39	75	38	39	75	38	39	75	38	39	75	38	39	75	38	39	75
41	42	80	41	42	80	41	42	80	40	42	81	41	42	80	41	42	80	41	42	80
43	44	85	43	44	85	43	44	85	43	45	84	43	44	85	43	44	85	43	44	85
46	47	90	46	47	90	46	47	90	46	47	90	46	47	90	46	47	90	46	47	90
48	49	95	48	49	95	48	49	95	48	49	95	48	49	95	48	49	95	48	49	95
50	52	101	50	51	102	50	51	102	51	52	100	51	52	100	51	52	100	50	51	102
53	54	105	53	54	105	53	54	105	53	54	105	53	54	105	53	54	105	53	54	105
56	57	110	56	57	110	56	57	110	56	57	110	56	57	110	56	57	110	56	57	110
58	59	115	58	59	115	58	59	115	58	59	115	58	59	115	58	59	115	58	59	115
60	62	121	61	62	120	61	62	120	61	62	120	60	62	121	60	61	122	61	62	120
64	65	123	64	65	123	64	65	123	63	64	125	63	65	124	64	65	123	64	65	123
66	67	130	66	67	130	66	67	130	66	67	130	66	67	130	66	67	130	66	67	130
68	69	135	68	69	135	68	69	135	68	69	135	68	69	135	68	69	135	68	69	135
71	72	140	71	72	140	71	72	140	71	72	140	71	72	140	71	72	140	71	72	140
73	74	145	73	74	145	73	74	145	73	74	145	73	74	145	73	74	145	73	74	145
76	77	150	76	77	150	76	77	150	76	77	150	76	77	150	76	77	150	76	77	150
78	79	155	78	79	155	78	79	155	78	79	155	78	79	155	78	79	155	78	79	155
81	82	160	81	82	160	81	82	160	80	82	161	81	82	160	81	82	160	81	82	160
83	84	165	83	84	165	83	84	165	83	85	164	83	84	165	83	84	165	83	84	165
86	87	170	86	87	170	86	87	170	86	87	170	86	87	170	86	87	170	86	87	170
88	89	175	88	89	175	88	89	175	88	89	175	88	89	175	88	89	175	88	89	175
91	92	180	91	92	180	91	92	180	91	92	180	91	92	180	91	92	180	91	92	180
93	94	185	93	94	185	93	94	185	93	94	185	93	94	185	93	94	185	93	94	185
96	97	190	96	97	190	96	97	190	96	97	190	96	97	190	96	97	190	96	97	190
98	99	195	98	99	195	98	99	195	98	99	195	98	99	195	98	99	195	98	99	195
100	102	201	100	101	202	100	101	202	101	102	200	101	102	200	101	102	200	100	101	202
103	104	205	103	104	205	103	104	205	103	104	205	103	104	205	103	104	205	103	104	205
106	107	210	106	107	210	106	107	210	106	107	210	106	107	210	106	107	210	106	107	210
108	109	215	108	109	215	108	109	215	108	109	215	108	109	215	108	109	215	108	109	215
111	112	220	111	112	220	111	112	220	111	112	220	111	112	220	111	112	220	111	112	220
113	114	225	113	114	225	113	114	225	113	114	225	113	114	225	113	114	225	113	114	225

Figure 5: Partitions eight through fourteen

15			16			17			18			19			20		
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
3	4	5	3	4	5	3	4	5	3	4	5	3	4	5	3	4	5
6	8	9	6	8	9	6	8	9	6	8	9	6	8	9	6	8	9
7	11	14	7	11	14	7	11	14	7	12	13	7	12	13	7	12	13
10	16	17	12	13	18	12	15	16	10	14	19	10	16	17	11	15	17
12	13	27	10	15	27	10	13	29	11	15	26	11	14	27	10	14	28
15	18	30	16	17	30	17	18	28	16	17	30	15	18	30	16	18	29
19	20	33	19	20	33	19	20	33	18	20	34	19	20	33	19	20	33
21	22	40	21	22	40	21	22	40	21	22	40	21	22	40	21	22	40
23	24	45	23	24	45	23	24	45	23	24	45	23	24	45	23	24	45
25	26	52	25	26	52	25	26	52	25	27	51	25	26	52	25	26	52
28	29	55	28	29	55	27	30	55	28	29	55	28	29	55	27	30	55
31	32	60	31	32	60	31	32	60	31	32	60	31	32	60	31	32	60
34	35	63	34	35	63	34	35	63	33	35	64	34	35	63	34	35	63
36	37	70	36	37	70	36	37	70	36	37	70	36	37	70	36	37	70
38	39	75	38	39	75	38	39	75	38	39	75	38	39	75	38	39	75
41	42	80	41	42	80	41	42	80	41	42	80	41	42	80	41	42	80
43	44	85	43	44	85	43	44	85	43	44	85	43	44	85	43	44	85
46	47	90	46	47	90	46	47	90	46	47	90	46	47	90	46	47	90
48	49	95	48	49	95	48	49	95	48	49	95	48	49	95	48	49	95
50	51	102	50	51	102	50	51	102	50	52	101	50	51	102	50	51	102
53	54	105	53	54	105	53	54	105	53	54	105	53	54	105	53	54	105
56	57	110	56	57	110	56	57	110	56	57	110	56	57	110	56	57	110
58	59	115	58	59	115	58	59	115	58	59	115	58	59	115	58	59	115
61	62	120	61	62	120	61	62	120	61	62	120	61	62	120	61	62	120
64	65	123	64	65	123	64	65	123	63	65	124	64	65	123	64	65	123
66	67	130	66	67	130	66	67	130	66	67	130	66	67	130	66	67	130
68	69	135	68	69	135	68	69	135	68	69	135	68	69	135	68	69	135
71	72	140	71	72	140	71	72	140	71	72	140	71	72	140	71	72	140
73	74	145	73	74	145	73	74	145	73	74	145	73	74	145	73	74	145
76	77	150	76	77	150	76	77	150	76	77	150	76	77	150	76	77	150
78	79	155	78	79	155	78	79	155	78	79	155	78	79	155	78	79	155
81	82	160	81	82	160	81	82	160	81	82	160	81	82	160	81	82	160
83	84	165	83	84	165	83	84	165	83	84	165	83	84	165	83	84	165
86	87	170	86	87	170	86	87	170	86	87	170	86	87	170	86	87	170
88	89	175	88	89	175	88	89	175	88	89	175	88	89	175	88	89	175
91	92	180	91	92	180	91	92	180	91	92	180	91	92	180	91	92	180
93	94	185	93	94	185	93	94	185	93	94	185	93	94	185	93	94	185
96	97	190	96	97	190	96	97	190	96	97	190	96	97	190	96	97	190
98	99	195	98	99	195	98	99	195	98	99	195	98	99	195	98	99	195
100	101	202	100	101	202	100	101	202	100	102	201	100	101	202	100	101	202
103	104	205	103	104	205	103	104	205	103	104	205	103	104	205	103	104	205
106	107	210	106	107	210	106	107	210	106	107	210	106	107	210	106	107	210
108	109	215	108	109	215	108	109	215	108	109	215	108	109	215	108	109	215
111	112	220	111	112	220	111	112	220	111	112	220	111	112	220	111	112	220
113	114	225	113	114	225	113	114	225	113	114	225	113	114	225	113	114	225

Figure 6: Partitions fifteen through twenty

When we use the algorithm from the proof of Proposition 1.5 and we get to position $2^k \cdot 20+1$, where we need to put in the numbers $2^{k+1} \cdot 25+1 = 2^k \cdot 50+1$ and $2^{k+1} \cdot 25+2 = 2^k \cdot 50+2$, we notice that the element $2^k \cdot 50+1$ was already used on position $2^k \cdot 10+1$, but $2^k \cdot 50$ was not used. Therefore, we need to use $2^k \cdot 50$ instead of $2^k \cdot 50+1$, along with $2^k \cdot 50+2$. The sum of the three elements on position $2^k \cdot 20+1$ is

$$11(2^k \cdot 20+1) - 2\left\lfloor \frac{2^k \cdot 20+1}{2} \right\rfloor - 8 = 2^k \cdot 200+3,$$

so the element in the third set is $2^k \cdot 200+3-2^k \cdot 50-2-2^k \cdot 50 = 2^k \cdot 100+1 = 2^{k+1} \cdot 50+1$, and the claim is established.

On positions $2^k \cdot 12+2$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+3, 2^k \cdot 30+4, \text{ and } 2^k \cdot 60+5,$$

while a partition in Class 1 has elements

$$2^k \cdot 30+3, 2^k \cdot 30+5, \text{ and } 2^k \cdot 60+4.$$

A simple even/odd argument can be used to get that $2^k \cdot 10+1 \neq 2^l \cdot 12+2$ for all $k, l \geq 0$. The same argument holds for all equivalence classes except class 5, which has three types of exceptions.

Finally, we note that the two types of exceptions are interlaced. This follows from the following chain of inequalities, which is true for all k:

$$2^k \cdot 50+1 < 2^k \cdot 60+5 < 2^k \cdot 100+1 < 2^k \cdot 120+4.$$

Class 2: A partition in this class coincides with the standard partition after the 11th elements, with the following exceptions:

On positions $2^k \cdot 10+1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 25+1, 2^k \cdot 25+2, \text{ and } 2^k \cdot 50,$$

while a partition in Class 2 has elements

$$2^k \cdot 25, 2^k \cdot 25+1, \text{ and } 2^k \cdot 50+2.$$

On positions $2^k \cdot 12+2$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+3, 2^k \cdot 30+4, \text{ and } 2^k \cdot 60+5,$$

while a partition in Class 2 has elements

$$2^k \cdot 30+4, 2^k \cdot 30+5, \text{ and } 2^k \cdot 60+3.$$

Class 3: Partition 6 coincides with the standard partition after the 9th elements, with the following exceptions:

On positions 2^k+1, $k \geq 3$, the standard partition has elements

$$2^{k-1} \cdot 5+1, 2^{k-1} \cdot 5+2, \text{ and } 2^k \cdot 5,$$

while partition 6 has elements

$$2^{k-1} \cdot 5, 2^{k-1} \cdot 5+1, \text{ and } 2^k \cdot 5+2.$$

On positions $2^k + 2$, $k \geq 3$, the standard partition has elements

$$2^{k-1} \cdot 5 + 3, 2^{k-1} \cdot 5 + 4, \text{ and } 5(2^k + 1),$$

while partition 6 has elements

$$2^{k-1} \cdot 5 + 4, 5(2^{k-1} + 1), \text{ and } 2^k \cdot 5 + 3.$$

Class 4: Partition 7 coincides with the standard partition starting with the 7th elements.

Class 5: Partition 8 coincides with the standard partition after the 11th elements, with the following exceptions:

On positions $2^k \cdot 10 + 1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 25 + 1, 2^k \cdot 25 + 2, \text{ and } 2^k \cdot 50,$$

while partition 8 has elements

$$2^k \cdot 25, 2^k \cdot 25 + 2, \text{ and } 2^k \cdot 50 + 1.$$

On positions $2^k \cdot 12 + 1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30 + 1, 2^k \cdot 30 + 2, \text{ and } 2^k \cdot 60,$$

while partition 8 has elements

$$2^k \cdot 30, 2^k \cdot 30 + 2, \text{ and } 2^k \cdot 60 + 1.$$

On positions $2^k \cdot 12 + 2$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30 + 3, 2^k \cdot 30 + 4, \text{ and } 2^k \cdot 60 + 5,$$

while partition 8 has elements

$$2^k \cdot 30 + 4, 2^k \cdot 30 + 5, \text{ and } 2^k \cdot 60 + 3.$$

We also need to check that $2^k \cdot 10 + 1 \neq 2^l \cdot 12 + 1$ for all $k, l \geq 0$. This is true because the left hand side is congruent to 1 modulo 5, and the right hand side is not.

Class 6: Partition 11 coincides with the standard partition after the 9th elements, with the following exceptions:

On positions $2^k + 1$, $k \geq 3$, the standard partition has elements

$$2^{k-1} \cdot 5 + 1, 2^{k-1} \cdot 5 + 2, \text{ and } 2^k \cdot 5,$$

while partition 11 has elements

$$2^{k-1} \cdot 5, 2^{k-1} \cdot 5 + 2, \text{ and } 2^k \cdot 5 + 1.$$

On positions 2^k+2, $k \geq 3$, the standard partition has elements

$$2^{k-1} \cdot 5+3, 2^{k-1} \cdot 5+4, \text{ and } 5(2^k+1),$$

while partition 11 has elements

$$2^{k-1} \cdot 5+3, 5(2^{k-1}+1), \text{ and } 2^k \cdot 5+4.$$

Class 7: Partition 12 coincides with the standard partition after the 11th elements, with the following exceptions:

On positions $2^k \cdot 12+1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+1, 2^k \cdot 30+2, \text{ and } 2^k \cdot 60,$$

while partition 12 has elements

$$2^k \cdot 30, 2^k \cdot 30+2, \text{ and } 2^k \cdot 60+1.$$

On positions $2^k \cdot 12+2$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+3, 2^k \cdot 30+4, \text{ and } 2^k \cdot 60+5,$$

while a partition in Class 1 has elements

$$2^k \cdot 30+3, 2^k \cdot 30+5, \text{ and } 2^k \cdot 60+4.$$

Class 8: Partition 13 coincides with the standard partition after the 13th elements, with the following exceptions:

On positions $2^k \cdot 12+1$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+1, 2^k \cdot 30+2, \text{ and } 2^k \cdot 60,$$

while partition 13 has elements

$$2^k \cdot 30, 2^k \cdot 30+1, \text{ and } 2^k \cdot 60+2.$$

On positions $2^k \cdot 12+2$, $k \geq 0$, the standard partition has elements

$$2^k \cdot 30+3, 2^k \cdot 30+4, \text{ and } 2^k \cdot 60+5,$$

while partition 13 has elements

$$2^k \cdot 30+4, 2^k \cdot 30+5, \text{ and } 2^k \cdot 60+3.$$

This completes the proof of the proposition. □

Among the partitions obtained from the 36 statements (different ways to fill in the first five elements in the three sets preserving the pattern of the vertical sums) using the algorithm in the proof of Proposition 1.5, there are exactly five that are equivalent to the standard partition:

- the standard partition itself
- the one obtained from the standard partition by switching 8 and 11 in the first set, and 15 in the third set with 12 in the second (this comes from number 15 in the list of 36 possible statements, and is equivalent with the standard partition if we take $N = 5$ in Definition 1.4)
- the one obtained from the standard partition by switching 7 in the second set with 8 in the first, and 10 in the third set with 9 in the second (this comes from number 19 in the list of 36 possible statements, and is equivalent with the standard partition if we take $N = 4$ in Definition 1.4)
- partition 7 in Figure 4 (this comes from number 8 in the list, and is equivalent with the standard partition if we take $N = 6$ in Definition 1.4)
- the partition obtained from the standard partition by swapping the following pairs, in order: 7 and 8, 9 and 10, 10 and 11, 14 and 15, 12 and 13 (this comes from number 26 in the list, and is equivalent with the standard partition if we take $N = 6$ in Definition 1.4).

We end this section by investigating new ways to obtain partitions equivalent to the standard partition by starting with the standard partition and reshuffling elements such that the pattern of the sums is preserved. The way they work is to switch elements in blocks of two: replace two elements on a certain position with sum s with two elements on a different position whose sum is also s. This is summarized in the following proposition.

Proposition 2.2 *There are infinitely many ways to obtain partitions equivalent to the standard partition. These may be obtained in the following two ways:*
i) Replace for each $k \geq 1$ *the elements on position* $4k$ *in the first and last set by the elements on position* $6k - 1$ *in the first two sets.*
ii) Replace for each $k \geq 0$ *the elements on position* $4k + 3$ *in the last two sets by the elements on position* $6k + 4$ *in the first two sets.*
These two techniques consist of starting with the standard partition and reshuffling elements such that the pattern of the vertical sums is preserved.

Proof. *i)* We use Definition 1.2, and we compute the sum of the elements on position $4k$ in the first and last set of the standard partition. We get $8 \cdot 4k - 2k - 2 - 5 = 30k - 7$. The sum of the elements on position $6k - 1$ in the first two sets is the same: $6(6k - 1) - 2(3k - 1) - 3 = 30k - 7$.
ii) If $k \geq 0$, the elements on position $4k + 3$ in the last two sets in the standard partition have sum $8(4k + 3) - 2k - 1 - 6 = 30k + 17$, while the elements on position $6k + 4$ in the first two sets have sum $6(6k + 4) - 2(3k + 2) - 3 = 30k + 17$. □

3 A generalization

In this section we generalize the problem studied in the first two sections.

Definition 3.1 *We assume that $m = 2t+1$ is an odd number. The m-***standard partition** *of the non-negative integers is given by*

$$\begin{aligned} SP_1 &= \{(t+1)(n-1) - \left\lfloor \frac{n}{2} \right\rfloor + 1 \mid n \geq 1\} \\ SP_2 &= \{(t+1)(n-1) - \left\lfloor \frac{n}{2} \right\rfloor + 2 \mid n \geq 1\} \\ &\vdots \\ SP_t &= \{(t+1)(n-1) - \left\lfloor \frac{n}{2} \right\rfloor + t \mid n \geq 1\} \\ SP_{t+1} &= \{(2t+1)(n-1) \mid n \geq 1\} \end{aligned}$$

Note that the sum of the n-th elements is $S(n) = (t+1)^2(n-1) + t\left\lfloor \frac{n-1}{2} \right\rfloor + \frac{t(t+1)}{2}$ (because $n-1 = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$). It is easy to see that SP_i consists of the numbers with remainder i or $t+i$ (mod m) for all $1 \leq i \leq t$, while SP_{t+1} is the set of multiples of $m = 2t+1$.

Equivalent partitions are defined similar to Definition 1.4. We can now construct more partition problems similar to the problem studied in the first two sections. In order to make sure that the solution is equivalent to the m-standard partition, when we design the problem we only use elements from the first five columns of the m-standard partition, because we saw in Proposition 2.1 that using numbers from outside the first five columns can lead to partitions that are not equivalent to the standard one. The following example is a problem in the case $m = 7$.

Problem The non-negative integers are divided into four groups as follows:

$$\begin{array}{lcllllll} A &=& \{1, & 6, & 9, & 21, & 28, & \ldots\} \\ B &=& \{0, & 5, & 8, & 14, & 15, & \ldots\} \\ C &=& \{3, & 7, & 13, & 10, & 17, & \ldots\}. \\ D &=& \{2, & 4, & 11, & 12, & & \ldots\}. \end{array}$$

Explain.

Solution. The first vertical sum is 6. The second vertical sum is 22, which is 16 more than the first one. The third vertical sum is 41, which is 19 more than the second one. The fourth vertical sum is 57, whch is again 16 more than the third one. Asssuming that the fifth vertical sum is $57 + 19 = 76$, we find that the fifth element of D is $76 - 28 - 15 - 17 = 16$. Now we notice that the set of the first five elements in the four sets is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 28\}$. Therefore, we can set the fifth elements of the first three sets to be 18, 19, and 20, and we find that the fifth element of D is $76 + 16 - 18 - 19 - 20 = 35$. We continue as above with the first unused numbers (i.e. 22, 23, and 24) as the sixth elements of the first three sets, and the next multiple of 7 (i.e. 42) as the sixth element of D. The solution is a partition equivalent to the 7-standard partition, because after the sixth column we get exactly the columns of the 7-standard partition.

Remark 3.2 We designed the statement of the problem as follows: we started with the 7-standard partition, we permuted the elements on the same position, and we swapped the pairs of numbers 13 and 14, and 11 and 10 (we can do this because $10+14 = 11+13$).

Results similar to those in Section 2 may be obtained for generalized partitions. For the case $m = 5$ we saw in Proposition 2.1 that there are 8 equivalence classes in the partitions obtained from all possible ways of choosing the first five elements of the sets in the partition. We used a computer program to determine that for $m = 7$ there are 13 equivalence classes, for $m = 9$ there are 19, and for $m = 11$ the number of equivalence classes is 26.

Acknowledgments

We thank Constantin Manoil for helpful conversations, and the referee for many valuable suggestions leading to a substantially improved exposition.

References

[1] P. Zeitz, *The art and craft of problem solving*, John Wiley & Sons, Inc., New York, second ed., 2007.

[2] P. Zeitz, Private communication.

Eunice Krinsky
California State University, Dominguez Hills
Mathematics Department
1000 E Victoria St
Carson, CA 90747
E-mail: ekrinsky@csudh.edu

Serban Raianu
California State University, Dominguez Hills
Mathematics Department
1000 E Victoria St
Carson, CA 90747
E-mail: sraianu@csudh.edu

Alexander Wittmond
California State University, Dominguez Hills
1000 E Victoria St
Carson, CA 90747
E-mail: awittmond1@toromail.csudh.edu

Received: June 30, 2017 **Accepted**: October 23, 2017
Communicated by Oscar Vega